FY - FCS - Final

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Instructions: You have to keep an audio/video conncetion open all the time. The camera has to show you writing. Solve all the exercises. At 16:00, stop writing and send me photos of your exam. Exams received after 16:15 will receive a substantial downgraded evaluation. **Write down your step by step reasoning, NOT ONLY THE FINAL RESULTS**

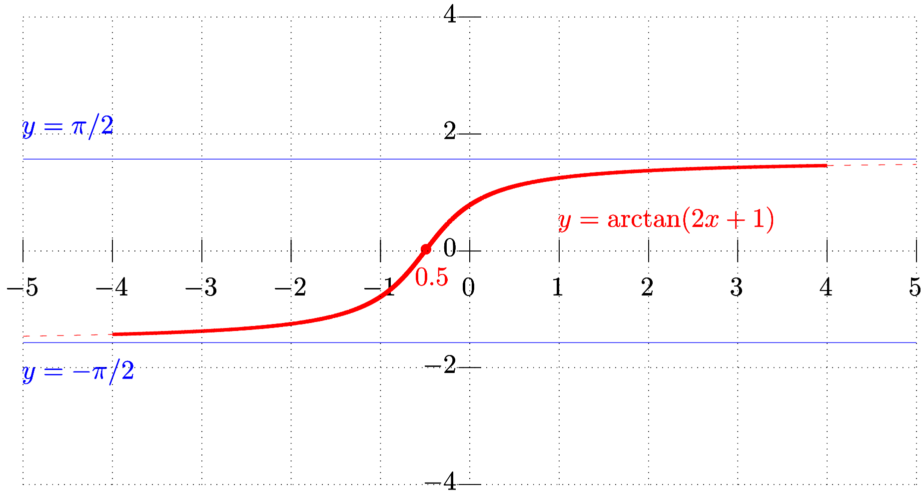
Exercise 1. *Given the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \arctan(2x+1) \end{array}$$

1. *Draw the graph of F.*
2. *Is F even, odd?*
3. *Is F increasing, decreasing?*
4. *Has F a maximum? A minumum?*
5. *Determine $Im(F)$, the image of F.*
6. *Find $F(0)$, $F(-1/2)$, $F([0,+\infty))$.*
7. *Find $F^{-1}(0)$, $F^{-1}([0,\pi/2])$.*
8. *Is F injective, surjective, invertible?*
9. *If F is not invertible, restrict the domain/codomain of F to get an invertible function G.*
10. *Find the explicit inverse of G or F if it is invertible, and draw its graph.*

Solution:

GNU1: The graph of F



1

2

$$0 = \arctan(0) = F((-1/2) \neq F(1/2) = \arctan(2) > 0 \Longrightarrow \boxed{F \ \textit{ is not even}}$$

$$0 = \arctan(0) = F((-1/2) \neq -F(1/2) = -\arctan(2) < 0 \Longrightarrow \boxed{F \ \textit{ is not odd}}$$

3 From the graph, it is evident that $\boxed{F \textit{ is increasing}}$.

4 From the graph, it is evident that $\boxed{F \textit{ has no a maximum or minumum}}$.

5 Since

$$F(x) = \arctan(2x+1) \ \textit{and} \ \forall \clubsuit \in \mathbb{R} \arctan(\clubsuit) \in (-\pi/2, \pi/2)$$

we have

$$\forall \ x \in \mathbb{R} \arctan(2x+1) \in (-\pi/2, \pi/2)$$

and so $\boxed{Im(F) = (-\pi/2, \pi/2)}$

6 $\boxed{F(0) = \arctan(2 \cdot 0 + 1) = \arctan(1) = \pi/4}$. $\boxed{F(-1/2) = \arctan(2 \cdot (-1/2) + 1) = \arctan(0) = 0}$. *Since F is increasing and continuous,*

$$\boxed{F([0, +\infty)) = [\pi/4, \pi/2]}$$

7 Looking at the graph, we see that $\boxed{F^{-1}(0) = 0}$ *and that*

$$\boxed{F^{-1}([0, \pi/2]) = [F^{-1}(0), F^{-1}(\pi/2)] = [0, +\infty)}$$

8 *Looking at the graph we see that every hotizontal line $y = a,\ a \in \mathbb{R}$ intersects the graph of F once or not at all. Thus* $\boxed{F \text{ is injective}}$. *Looking at the graph we see that, for example, the hotizontal line $y = 2$ doesn't intersect the graph of F. Thus* $\boxed{F \text{ is not surjective}}$, *and hence* $\boxed{F \text{ is not invertible}}$.

9 *F is injective, so the domain stays the same. We restrict the codomain to the image of F to get the invertible function*

$$\boxed{\begin{array}{cccc} G: & \mathbb{R} & \longrightarrow & (-\pi/2, \pi/2) \\ & x & \mapsto & \arctan(2x+1) \end{array}}$$

10 *The inverse is $G^{-1} : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$. To have the inverse formula, we solve the equation*

$$\arctan(2x+1) = y$$

for the variable x and the parameter y, where $\arctan(2x+1), y \in (-\pi/2, \pi/2)$

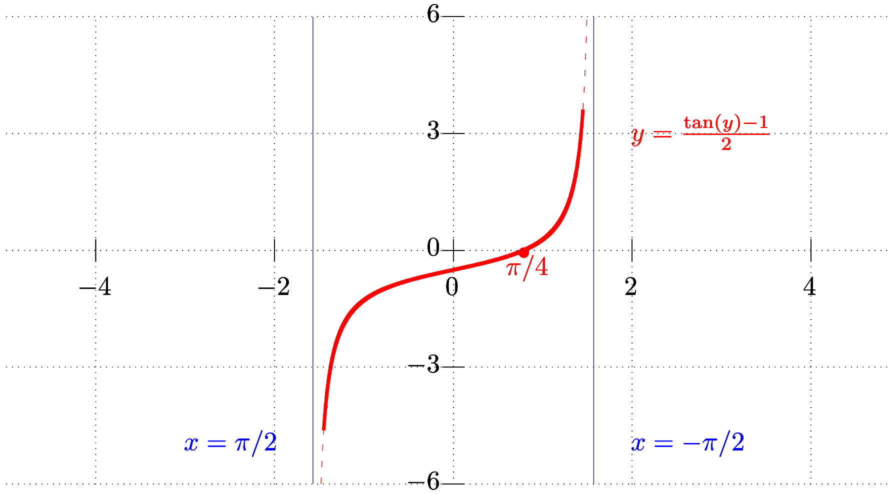
$$\begin{array}{rcl} \arctan(2x+1) & = & y \\ \tan(\arctan(2x+1)) & = & \tan(y) \\ 2x+1 & = & \tan(y) \\ x & = & \dfrac{\tan(y)-1}{2} \end{array}$$

For the second setp, we can apply the function $\tan : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ *to the equation since* $\arctan(2x+1), y \in (-\pi/2, \pi/2)$, *the domain of* tan. *Since* tan *is inverible we don' change the equation solutions (we get an ecquivalent equation). For the third step we remember that* $\tan(\arctan) = id$.

So we have

$$\boxed{\begin{array}{cccc} G^{-1}: & (-\pi/2, \pi/2) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{\tan(y)-1}{2} \end{array}}$$

GNU2: The graph of G^{-1}



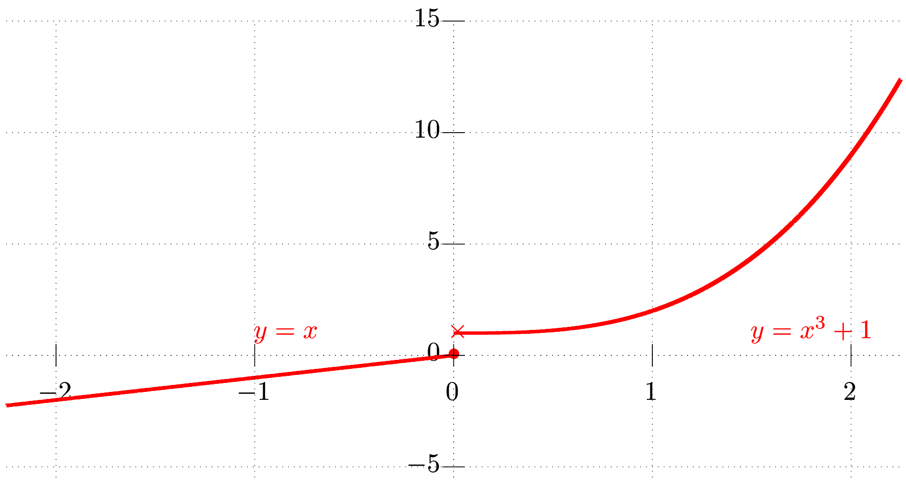
Exercise 2. *Given the function*

$$F: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} x & x \leq 0 \\ x^3+1 & x > 0 \end{cases}$$

1. *Draw the graph of F.*
2. *Is F even, odd?*
3. *Is F increasing, decreasing?*
4. *Has F a maximum? A minumum?*
5. *Determine $Im(F)$, the image of F.*
6. *Find $F(-1)$, $F(2)$, $F([-1,2])$.*
7. *Find $F^{-1}([-1,0])$, $F^{-1}([0,9])$, $F^{-1}([-1,9])$.*
8. *Is F injective, surjective, invertible?*
9. *If F is not invertible, restrict the domain/codomain of F to get an invertible function G.*
10. *Find the explicit inverse of G, or F if it is invertible, and draw its graph.*

Solution:

GNU3: The graph of F



2 We have

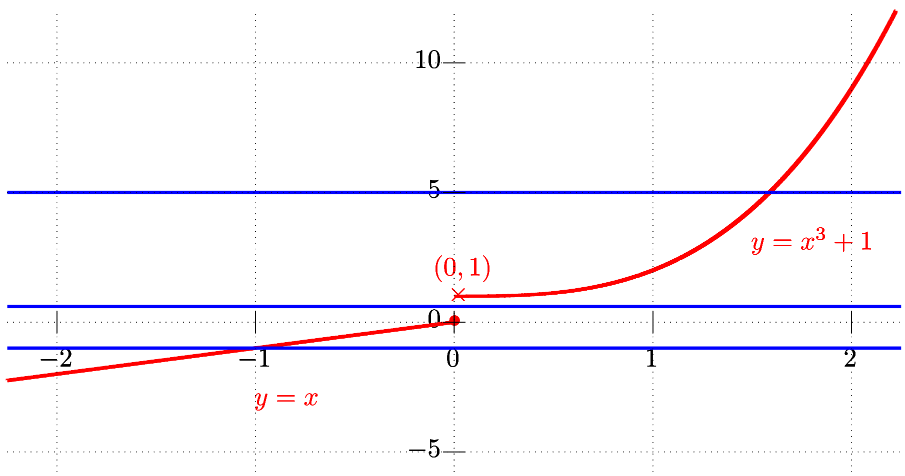
$$F(-1) = -1 \neq F(1) = 2 \Longrightarrow \boxed{F \text{ is not even}}.$$

$$F(-1) = -1 \neq -F(1) = -2 \Longrightarrow \boxed{F \text{ is not odd}}.$$

3 Looking at the graph, $\boxed{F \text{ is increasing}}$.

4 Looking at the graph, $\boxed{F \text{ has no maximum or minimum}}$.

GNU4: The graph of F



5:

Looking at the graph, we see that all the horizontal lines $y = a,\ a \in \mathbb{R}$ *intersects the graph of* F*, except for the lines* $y = a,\ a \in (0, 1]$

$$\boxed{Im(F) = \mathbb{R} - (0, 1] = (\infty, 0] \cup (1, +\infty}$$

6 $\boxed{F(-1) = -1}$. $\boxed{F(2) = 10}$.

$$\boxed{F([-1, 2]) = F((-1, 0] \cup (0, 2]) = (-1, 0] \cup (0, 2]}$$

7

$$\boxed{F^{-1}([-1, 0]) = [-1, 0]}$$

$$\boxed{F^{-1}([0, 9]) = F^{-1}(\{0\} \cup (0, 9]) = \{0\} \cup (1, 2]}$$

$$\boxed{F^{-1}([-1, 9]) = F^{-1}([-1, 0] \cup (0, 9]) = [-1, 0] \cup (1, 2]}$$

8 Looking at the graph we see that every hotizontal line $y = a,\ a \in \mathbb{R}$ *intersects the graph of* F *once or not at all. Thus* $\boxed{F \text{ is injective}}$. *Looking at the graph we see that, for example, the hotizontal line* $y = 0.5$ *doesn't intersect the graph of* F*. Thus* $\boxed{F \text{ is not surjective}}$, *and hence* $\boxed{F \text{ is not invertible}}$.

9 F is injective, so the domain stays the same. We restrict the codomain to the image of F to get the invertible function

$$\boxed{\begin{array}{llcc} G: & \mathbb{R} & \longrightarrow & \mathbb{R}-(0,1] \\ & x & \mapsto & \begin{cases} x & x \leq 0 \\ x^3+1 & x > 0 \end{cases} \end{array}}$$

10 We have $G^{-1} : \mathbb{R}-(0,1] \longrightarrow \mathbb{R}$. To find the formula for the inverse we solve

$$F(x) = y \textit{ for } y \in (-\infty, 0] \textit{ and } y \in (1, +\infty)$$

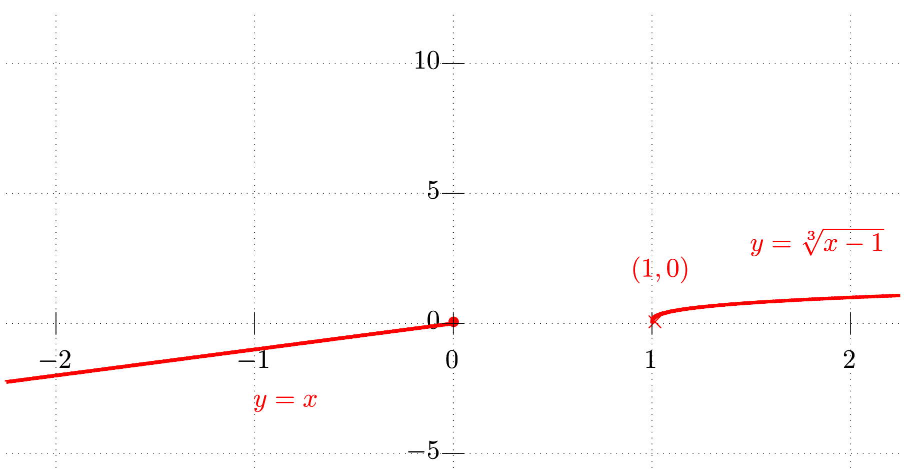
the firse is trival $x = y$, for the second,

$$\begin{array}{rcl} x^3+1 & = & y \\ x^3 & = & y-1 \\ \sqrt[3]{x} & = & \sqrt[3]{y-1} \\ x & = & \sqrt[3]{y-1} \end{array}$$

In the third step, we can apply to the equation the invertible function $\sqrt[3]{\cdot}$, whose domain is $\mathbb{R}$, and whose inverse is $(\cdot)^3$

$$\boxed{\begin{array}{llcc} G^{-1}: & \mathbb{R}-(0,1] & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \begin{cases} x & x \leq 0 \\ \sqrt{x-1} & x > 1 \end{cases} \end{array}}$$

GNU5: The graph of G^{-1}



Exercise 3. *We have the equation*

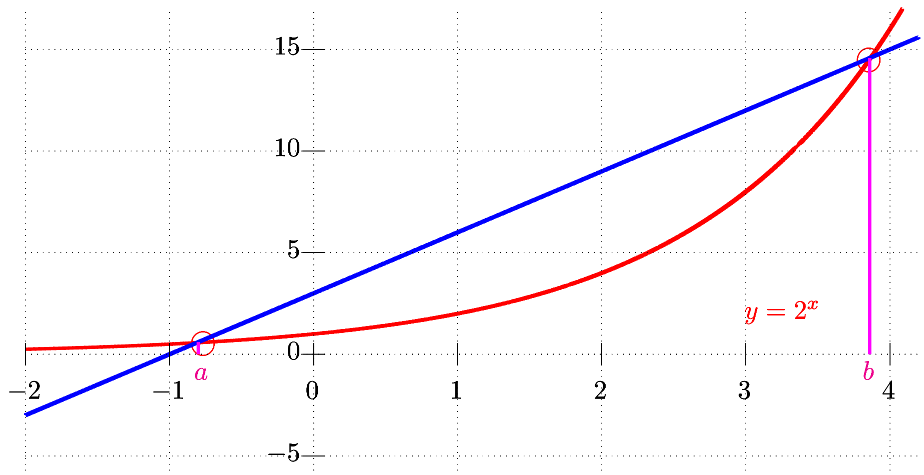
$$2^x = 3x + 3$$

1. *How many solutions are there?*
2. *Find the solutions, exactly if possible, with approximation of* 0.1 *if necessary. Motivate your answer.*

We solve the equation graphically, drawing the graph of

$$f : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 2^x \end{array} \quad \textit{and} \quad g : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 3x+3 \end{array}$$

GNU6: The graphs of f, g



1 $\boxed{\textit{There are two solutions, } a \textit{ and } b}$, *with* $-1 < a < -0.5$ *and* $3.5 < b < 4$.

2 *We find the zeros of the function*

$$F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 3x+3-2^x \end{array}$$

when the function F *is negative,* $f(x) > g(x)$ *and when* F *is positive,* $f(x) < g(x)$

a) *The first (left) root: we have* $-1 < a < -0.5$, *and* $F(-1) < 0$, $F(-0.5) > 0$.

$$\textit{We try } x = \frac{-1-0.5}{2} = -0.75 \textit{ and } F(-0.75) > 0 \Longrightarrow -1 < a < -0.75$$

$$We\ try\ x = \frac{-1-0.75}{2} = -0.875\ and\ F(-0.875) < 0 \Longrightarrow -0.875 < a < -0.75$$

$$We\ try\ x = \frac{-0.875-0.75}{2} = -0.8125\ and\ F(-0.8125) < 0 \Longrightarrow -0.8125 < a < -0.75$$

The width of the interval is $-0.8125 + 0.75 = 0.0925$, *within the required precision of* 0.01, *so the solution is*

$$x = \frac{-0.8125-0.75}{2} \pm \frac{0.0925}{2} = -0.78125 \pm 0.04625$$

b) The second (right) root: we have $3.5 < b < 4$, *and* $F(3.5) > 0$, $F(4) < 0$.

$$We\ try\ \frac{3.5+4}{2} = 3.75\ and\ F(3.75) > 0 \Longrightarrow 3.75 < b < 4$$

$$We\ try\ \frac{3.75+4}{2} = 3.875\ and\ F(3.875) < 0 \Longrightarrow 3.75 < a < 3,875$$

$$We\ try\ \frac{3.75+3.875}{2} = 3.8125\ and\ F(3.8125) < 0 \Longrightarrow 3.75 < b < 3.8125$$

The width of the interval is $3.8125 - 3.75 = 0.0625$, *within the required precision of* 0.01, *so the solution is*

$$x = \frac{3.8125+3.75}{2} \pm \frac{0.0625}{2} = 3.78125 \pm 0.03125$$

Exercise 4. *We have the two sets*

$$A = \{\sqrt[5]{k+1} - 2 \ \mid \ k \in \mathbb{N}\}\ and\ B = \{2t^3 + 2 \ \mid \ t \in \mathbb{N}\}$$

1. $\mid A \mid = \mid B \mid$? *Motivate your answer.*
2. *Exhibit, if one exists, an invertible function* $F : B \longrightarrow A$.
3. *Exhibit, if one exists, an invertible function* $G : A \longrightarrow B$.

1 To show that $\mid A \mid = \mid B \mid$ *we can to show that there exists an invertible function* $h : A \longrightarrow B$. *That can be difficult, so we proceed by showing that* $\mid A \mid = \mid \mathbb{Z} \mid$ *and* $\mid \mathbb{Z} \mid = \mid B \mid$, *deducing thus that* $\mid A \mid = \mid B \mid$.

- *To show that* $\mid A \mid = \mid \mathbb{Z} \mid$*, we have to find an invertible function* $f : \mathbb{Z} \longrightarrow A$ *or* $\psi : A \longrightarrow \mathbb{Z}$*. Let's do the first. We define the function*

$$\begin{array}{rccc} f: & \mathbb{Z} & \longrightarrow & A \\ & n & \mapsto & \sqrt[5]{n+1} - 2 \end{array}$$

to show that this function is inverible, we build a function $f' : A \longrightarrow \mathbb{Z}$ *and we show that* f' *is the inverse of* f*.*

$$\begin{array}{rcl} \sqrt[5]{n+1} - 2 & = & y \\ \sqrt[5]{n+1} & = & y + 2 \\ \left(\sqrt[5]{n+1}\right)^5 & = & (y+2)^5 \\ n + 1 & = & (y+2)^5 \\ n & = & (y+2)^5 - 1 \end{array}$$

In the third step we aplly to the equation the invertible function $\sqrt[5]{\cdot} : \mathbb{R} \longrightarrow \mathbb{R}$*, getting an equivalent equation. The inverse of* $\sqrt[5]{\cdot}$ *is* $(\cdot)^5$*, and so we get fourth step. By construction, the function*

$$\begin{array}{rccc} f': & A & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & (n+2)^5 - 1 \end{array}$$

is the inverse of f*, that is hence invertible.*

- *To show that* $\mid B \mid = \mid \mathbb{Z} \mid$*, we have to find an invertible function* $g : \mathbb{Z} \longrightarrow B$ *or* $\phi : B \longrightarrow \mathbb{Z}$*. Let's do the first. We define the function*

$$\begin{array}{rccc} g: & \mathbb{Z} & \longrightarrow & B \\ & n & \mapsto & 2n^3 + 1 \end{array}$$

to show that this function is inverible, we build a function $g' : B \longrightarrow \mathbb{Z}$ *and we show that* g' *is the inverse of* g*.*

$$\begin{array}{rcl} 2n^3 + 1 & = & y \\ n^3 & = & \dfrac{y-1}{2} \\ \sqrt[3]{n^3} & = & \sqrt[3]{\dfrac{y-1}{2}} \\ n & = & \sqrt[3]{\dfrac{y-1}{2}} \end{array}$$

In the third step we aplly to the equation the invertible function $\sqrt[3]{\cdot} : \mathbb{R} \longrightarrow \mathbb{R}$*, getting an equivalent equation. The inverse of* $\sqrt[3]{\cdot}$ *is* $(\cdot)^3$*, and so we get fourth step. By construction, the function*

$$\begin{array}{rccc} g': & A & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & \sqrt[3]{\frac{y-1}{2}} \end{array}$$

is the inverse of g, that is hence invertible.

Since have built an invertible function $f : \mathbb{Z} \longrightarrow A$ *and an invertible function* $g : \mathbb{Z} \longrightarrow B$*. So* $\mid A \mid = \mid \mathbb{Z} \mid$ *and* $\mid \mathbb{Z} \mid = \mid B \mid$*, hence* $\boxed{\mid A \mid = \mid B \mid}$.

2 To buind an invertible function $F : B \longrightarrow A$ *we compose* $g^{-1} : B \longrightarrow \mathbb{Z}$ *and* $f : \mathbb{Z} \longrightarrow A$*, getting the composition function* $B \xrightarrow{g^{-1}} \mathbb{Z} \xrightarrow{f} B$

$$\boxed{\begin{array}{cccc} f \circ g^{-1} : & B & \longrightarrow & A \\ & n & \mapsto & f(g^{-1}(n)) \end{array}}$$

where

$$\boxed{f(g^{-1}(n)) = f(\sqrt[3]{\frac{y-1}{2}}) = \sqrt[5]{\sqrt[3]{\frac{y-1}{2}} + 1} - 2}$$

This function is invertible because it is the composition of two invertible functions (they are one-to-one correspondences, and the composition of one-to-one correspondences is trivially a one-to-one correspondence).

3 To buind an invertible function $G : A \longrightarrow B$ *we compose* $f^{-1} : A \longrightarrow \mathbb{Z}$ *and* $g : \mathbb{Z} \longrightarrow B$*, getting the composition function* $A \xrightarrow{f^{-1}} \mathbb{Z} \xrightarrow{g} B$

$$\boxed{\begin{array}{cccc} g \circ f^{-1} : & B & \longrightarrow & A \\ & n & \mapsto & g(f^{-1}(n)) \end{array}}$$

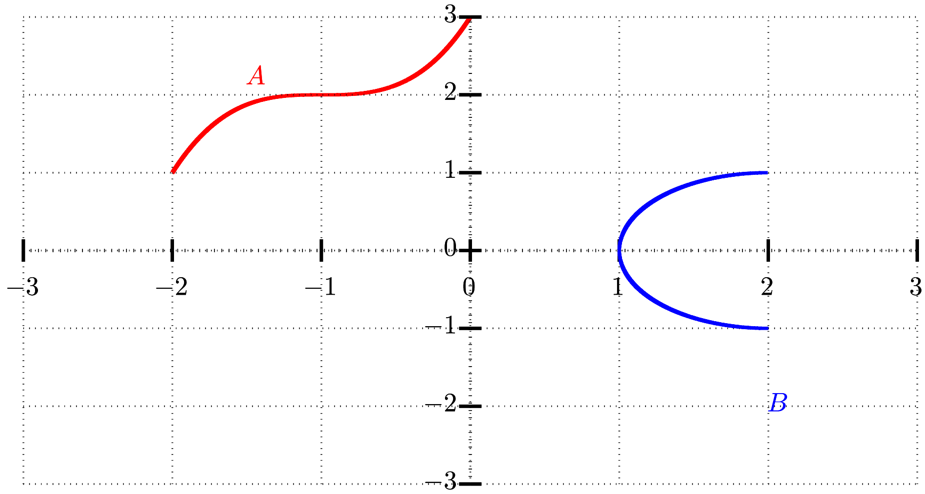
where

$$\boxed{g(f^{-1}(n)) = g\left((n+2)^5 - 1\right) = 2\left((n+2)^5 - 1\right)^3 + 1}$$

This function is invertible because it is the composition of two invertible functions (they are one-to-one correspondences, and the composition of one-to-one correspondences is trivially a one-to-one correspondence).

Exercise 5. *We have the following sets A, B in $\mathbb{R}^2$, the real plane:*

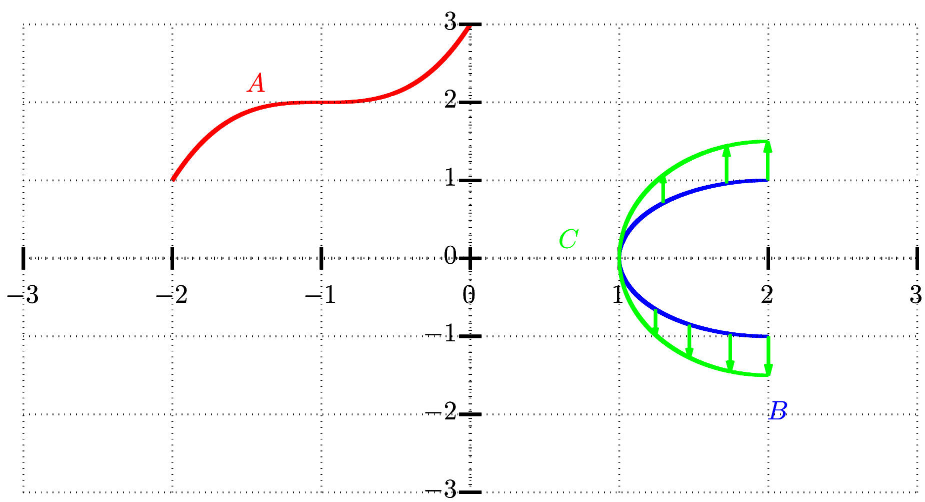
GNU6



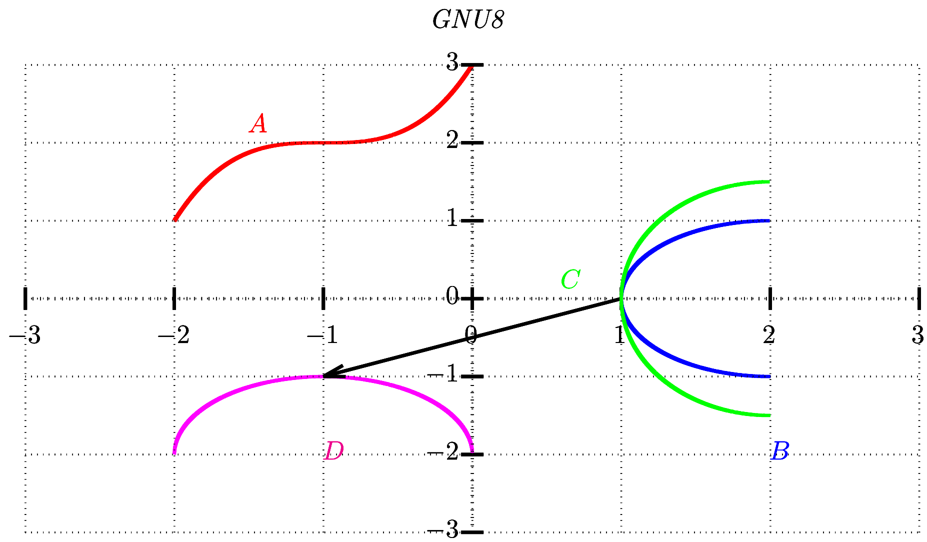
1. *Do we have $|A| = |B|$?*
2. *Show, if one exists, an one-to-one correspondence between A and B, directly or through some intermediate step.*

Solution:
We enlarge B to C, and $|B| = |C|$

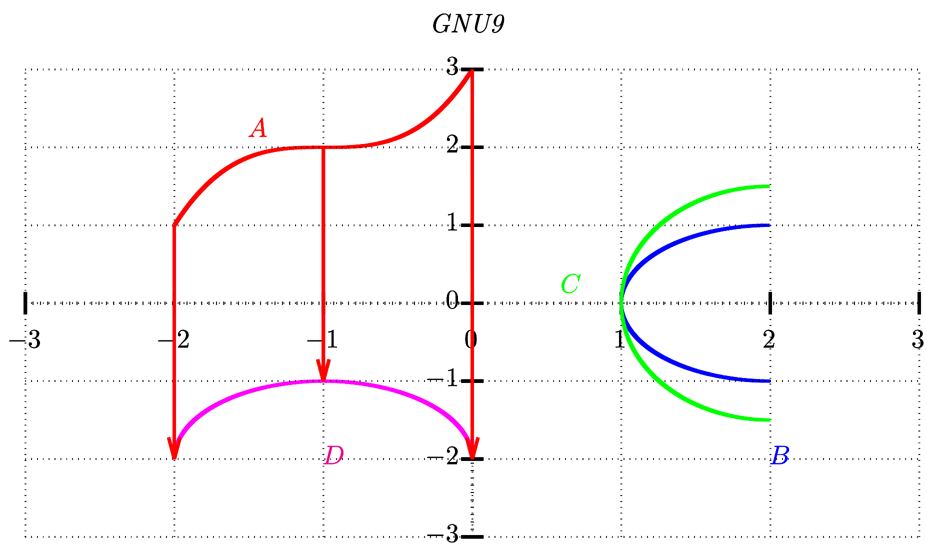
GNU7



We translate C to D, and $|C| = |D|$



We project A onto D, and $|A| = |D|$



1 $\boxed{\textit{We have } |A| = |B|}$.

2 The combination of the above functions gives an one-to-one correspondence between A and B.

Exercise 6. *Given the sets* $A = \{0, 1, 2, 3\}$ *and* $B = \{9, 7, 6, 5, 4\}$. *We have*

$$\boxed{|A| = 4 \neq 5 = |B|}$$