FCS
Math: Functions
Second Partial: solutions

Massimo Caboara

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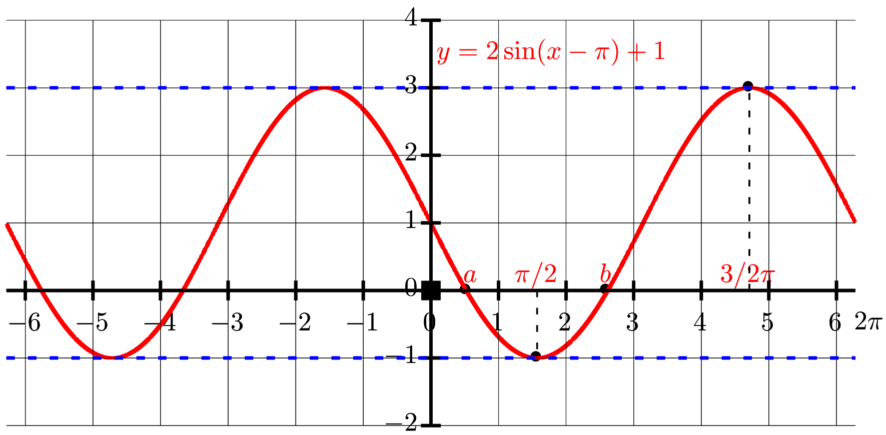
Exercise 1. *Draw the graph of function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 2\sin(x-\pi)+1 \end{array}$$

Determine the zeroes, positivity, where the function is increasing/decreasing, maximum and minimums.

The graph may be drawn directly from the graph of $\sin(x)$*, shifted to the left by* π*, amplified by* 2 *and raised by* 1*. All the required informations can by derived from the graph, or computed algebraically as done below.*

GNU1: The Graph of F



We remark that since $\sin(x - \pi) = \sin(x)$

$$2\sin(x - \pi) + 1 = -2\sin(x) + 1$$

and since $-1 \leq \sin(x) \leq 2$ *we have* $-1 \leq 2\sin(x - \pi) + 1 \leq 3$. *Also, the function is periodic of period* 2π. *We detail the salient points in* $[0, 2\pi)$.

- *The functions has zeroes* $a, b \in \mathbb{R}$, *that we do not detail further.*
- *The function in negative only for* $x \in (a, b)$.
- *By considering the graph of* $\sin(x)$, *the function has a minimum in* $\pi/2$, *for which* $f(\pi/2) = -1$ *and a maximum in* $3/2\pi$, *for which* $f(3/2\pi) = 3$. *Also, we could have got this points algebraically, since we know that*
  - *the minimum is* -1, *and the maximum is* 3

$$\begin{aligned} 2\sin(x - \pi) + 1 &= -1 \\ -2\sin(x) + 1 &= -1 \\ -2\sin(x) &= -2 \\ \sin(x) &= 1 \\ x &= \frac{1}{2}\pi \end{aligned} \qquad \begin{aligned} 2\sin(x - \pi) + 1 &= 3 \\ -2\sin(x) + 1 &= 3 \\ -2\sin(x) &= 2 \\ \sin(x) &= -1 \\ x &= \frac{3}{2}\pi \end{aligned}$$

- *The function increasing for* $x \in [\pi/2, 3/2\pi)$.

Exercise 2. *Are the following function ODD or EVEN?*

1. $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=x^2\cos(x)+\cos^2(x)+\sin^2(x)$. *The function is EVEN since for all* $x\in\mathbb{R}$

$$\begin{aligned} f(-x) &= (-x)^2\cos(-x)+\cos^2(-x)+\sin^2(-x) \\ &= x^2\cos(x)+\cos^2(x)+(-\sin(x))^2 \\ &= x^2\cos(x)+1 \\ &= f(x) \end{aligned}$$

2. $f:(\pi/2,\pi/2)\longrightarrow\mathbb{R},\ f(x)=\tan(\sin(\sqrt{\mid x\mid}))$. *The function is EVEN since it is the composition of four functions,* $\tan(\cdot),\sin(\cdot),\sqrt{\cdot},\mid\cdot\mid$, *the last of which is EVEN. Alternatively, for all* $x\in\mathbb{R}$

$$\begin{aligned} f(-x) &= \tan(\sin(\sqrt{\mid -x\mid})) \\ &= \tan(\sin(\sqrt{\mid x\mid})) \\ &= f(x) \end{aligned}$$

3. $f:(-\pi/2,\pi/2)\longrightarrow\mathbb{R},\ f(x)=\tan(\sin(\sqrt[3]{x}))$. *The function is ODD since it is the composition of ODD functions. Alternatively, for all* $x\in\mathbb{R}$

$$\begin{aligned} f(-x) &= \tan(\sin(\sqrt[3]{-x})) \\ &= \tan(\sin(-\sqrt[3]{x})) \\ &= \tan(-\sin(\sqrt[3]{x})) \\ &= -\tan(\sin(\sqrt[3]{x})) \\ &= -f(x) \end{aligned}$$

4. $f:(-\pi/2,\pi/2)\longrightarrow\mathbb{R},\ f(x)=\mid x+1\mid$. *For* $x=1$ *we have*

$$f(1)=\mid 1+1\mid=2 \text{ and } f(-1)=\mid -1+1\mid=\mid 0\mid=0$$

So it is false that for all $x\in\mathbb{R}$ $f(-x)=f(x)$ *or* $f(-x)=-f(x)$ *and so the function* f *is neither ODD nor EVEN.*

5. $f:(-\pi/2,\pi/2)\longrightarrow\mathbb{R},\ f(x)=1/x^2$. *The function is EVEN since for all* $x\in\mathbb{R}$

$$\begin{aligned} f(-x) &= \frac{1}{(-x)^2} \\ &= \frac{1}{(x)^2} \\ &= f(x) \end{aligned}$$

Exercise 3. *Given the functions $f\uparrow$, $g\uparrow$, is it possibile to determine a priori if f/g is .*

We examine the two functions

$$f: \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \textit{and} \quad g: \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array}$$

The function

$$f \cdot g: \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is $\uparrow$.

We examine the two functions

$$f: \begin{array}{ccc} \mathbb{R}^- & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \textit{and} \quad g: \begin{array}{ccc} \mathbb{R}^- & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array}$$

The function

$$f \cdot g: \begin{array}{ccc} \mathbb{R}^- & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is $\downarrow$.

We examine the two functions

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \textit{and} \quad g: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array}$$

The function

$$f \cdot g: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is neither $\uparrow$ *nor* $\downarrow$.

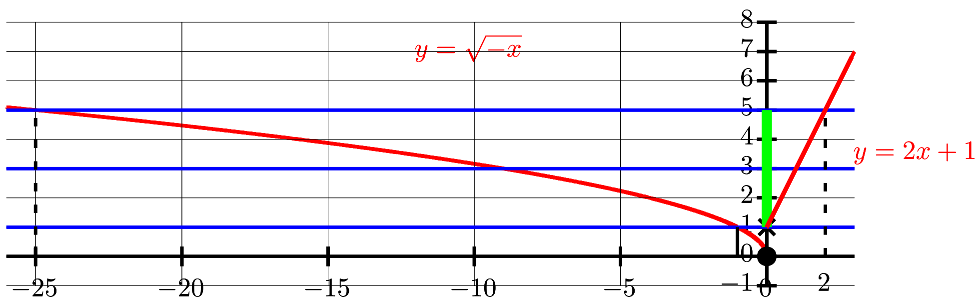
Is thus not possible to determine a priori if f/g is increasing, decreasing or neither.

Exercise 4. *Given the function*

$$F : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} \sqrt{-x} & x \leq 0 \\ 2x+1 & x > 0 \end{cases}$$

1. *Draw the graph of F. Mark the intersections with the axis and the interesting points.*
2. *Is F injective, surjective, invertible? What is the image of F?*
3. *Determine $F(0)$, $F(-1)$, $F(1)$, $F(2)$, $F([1,2])$, $F([-2,-1])$.*
4. *Determine $F^{-1}(0)$, $F^{-1}(1)$, $F^{-1}(5)$, $F^{-1}([1,5])$, $F^{-1}([-3,-2])$.*
5. *Build an invertible function from F by restricting its domain and/or codomain.*
6. *Determine the formula for this inverse.*

GNU2: The Graph of F



1. *F is not injective or surjective, and thus not invertible. The image (range) of F is $\mathbb{R}_0^+$.*

2. *Determine $F(0) = 0$, $F(-1) = \sqrt{-(-1)}1$, $F(1) = 3$, $F(2) = 5$.*

   - *Since F is increasing in $(0, +\infty$ we have that*

   $$F([1,2]) = [F(1), F(2)] = [3,5]$$

   - *Since F is decreasing in $(-\infty, 0]$ we have that*

   $$F([-2,-1]) = [F(-1), F(-2)] = [1, \sqrt{2}]$$

3. *We have*

   $$F^{-1}(0) = 0, \quad F^{-1}(1) = \{1\}, \quad F^{-1}(5) = \{-25, 2\}, \quad F^{-1}([-3,-2]) = \emptyset$$

   if

   $$G: \begin{array}{ccc} \mathbb{R}_0^- & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x} \end{array} \quad \textit{and} \quad H: \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & 2x+1 \end{array}$$

   with $G \quad \downarrow$ and $H \quad \uparrow$

   $$\begin{aligned} F^{-1}([1,5]) &= G^{-1}([1,5]) \cup H^{-1}([1,5]) \\ &= [G^{-1}(5), G^{-1}(1)] \cup [H^{-1}(1), H^{-1}(5)] \\ &= [-25,-1] \cup [0,2] \end{aligned}$$

4. *We restrict F to, for example, the negative reals*

   $$F_{|\mathbb{R}_0^-}: \begin{array}{ccc} \mathbb{R}_0^- & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & \sqrt{-x} \end{array}$$

   Here F is invertible, as we can see from the graph of F.

5. *The inverse of $F_{|\mathbb{R}_0^-}$ is*

   $$(F_{|\mathbb{R}_0^-})^{-1}: \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^- \\ x & \mapsto & -x^2 \end{array}$$

   since we can invert the formula with the usual method

   $$y = \sqrt{-x} \Longleftrightarrow y^2 = -x \Longleftrightarrow -y^2 = x \Longrightarrow x = -y^2$$

Exercise 5. *Given the equation*

$$x^2 + 2x = 2^x - 1$$

1. *Find the number of solutions graphically .*
2. *Find the value of the solutions by approximation up to a precision of* 0.1.

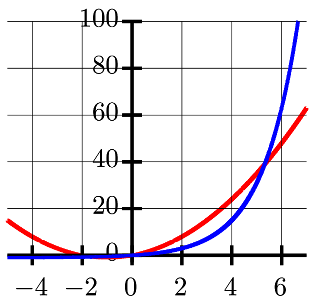
We define the functions

$$F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 + 2x \end{array} \quad \textit{and} \quad G : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 2^x - 1 \end{array}$$

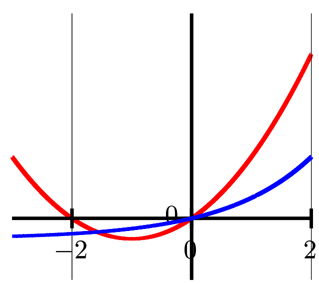
and we want to determine the solutions of $F(x) = G(x)$.

For the graph of the function we see that there are at least two solutions. We also remark that the exponential function grows more quickly than the parabola, hence there is another solution far away on the right

GNU3: F and G



GNU4: closeup near 0



- *The negative solution is* $x = a \sim -1.6$, *since*

$$(1.5)^2 + 2 \cdot (1.5) > 2^{(1.5)} - 1 \quad \textit{and} \quad (1.7)^2 + 2 \cdot (1.7) < 2^{(1.7)} - 1$$

- *There is an exact solution,* $x = 0$, *since*

$$0^2 + 2 \cdot 0 = 2^0 - 1$$

- *The positive solution is* $x = b \sim 5.3$, *since*

$$(5.2)^2 + 2 \cdot (5.2) > 2^{(5.2)} - 1 \quad \textit{and} \quad (5.4)^2 + 2 \cdot (5.4) < 2^{(5.4)} - 1$$

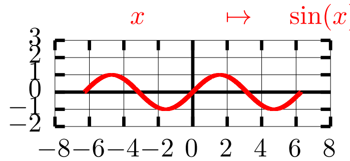
Exercise 6. *Draw the graph of the function*

$$\begin{array}{rccc} F: & [-2\pi, 2\pi] & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 2^{\sin(x)} \end{array}$$

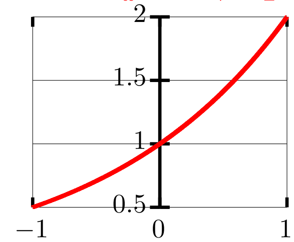
Determine the zeroes, positivity, where the function is increasing/decreasing, maximum and minimums.

We have that $F = H \circ G$, where

$$\begin{array}{rccc} G: & [-2\pi, 2\pi] & \longrightarrow & [-1,1] \\ & x & \mapsto & \sin(x) \end{array}$$



$$\begin{array}{rccc} H: & [-1, 1] & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 2^x \end{array}$$



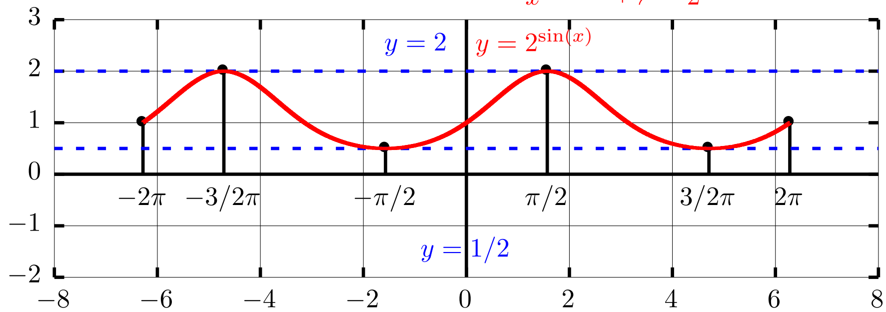
Just looking at the graphs of $\sin(x)$ *and* 2^x, *we can say that* F *has no zeroes and is always positive (because* 2^x *has no zeroes and is always positive).*

The function $\sin(x)$ *has minimums in* $-\frac{1}{2}\pi, -\frac{3}{2}\pi$, *where its value is* -1 *and maximums in* $-\frac{3}{2}\pi, \frac{1}{2}\pi$, *where its value is* 1. *The function* $2^{\sin(x)}$ *has minimums and maximums in the same points, with values* $2^{-1} = \frac{1}{2}$ *and* $2^1 = 2$ *respectivley.*

Also, at the existence field extremes, there is a minimum in -2π, *with value* $2^{\sin(-2\pi)} = 2^0 = 1$ *and a maximum in* 2π, *with value* $2^{\sin(2\pi)} = 2^0 = 1$.

The function is increasing in $\left[-2\pi, -\frac{3}{2}\pi\right]$, $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ *and* $\left[\frac{3}{2}\pi, 2\pi\right]$.

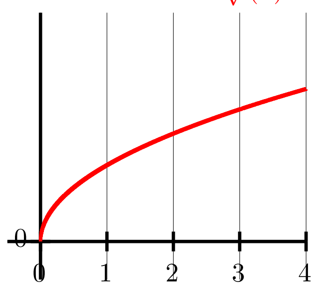
GNU4: Graph of $\begin{array}{rccc} 2^{\sin(\cdot)}: & [-2\pi, 2\pi] & \longrightarrow & [1/2, 2] \\ & x & \mapsto & 2^{\sin(x)} \end{array}$



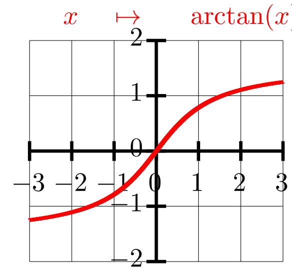
Exercise 7 (Optional)**.** *Draw the graph of the function with formula* $F(x) = \sqrt{\arctan(x^2-1)}$. *Determine the existence field. Determine the zeroes, positivity, where the function is increasing/decreasing, maximum and minimums. Is the function ODD/EVEN?*

We have that, skipping the domain/codomain part, $F = G \circ H \circ T$, *where*

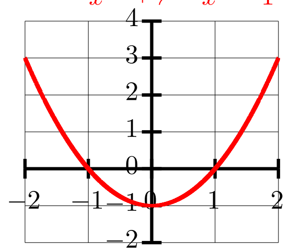
$$G: \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R}^+ \\ x & \mapsto & \sqrt{(x)} \end{array}$$



$$H: \begin{array}{ccc} \mathbb{R} & \longrightarrow & (-\pi/2, \pi/2) \\ x & \mapsto & \arctan(x) \end{array}$$



$$T: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2-1 \end{array}$$



Looking at the arctan *graph, that means that* $x^2 - 1 \geq 0$. *We draw the parabola, and it is immediata that this means* $x \notin (-1, 1)$. *The existence field is hence* $(-\infty, -1] \cup [1, +\infty)$. *The function* F *is thus*

$$F: \begin{array}{ccc} (-\infty, -1] \cup [1, +\infty) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{\arctan(x^2-1)} \end{array}$$

Since $\forall\ x \in \mathrm{EF}(F)$ *we have* $\pi/2 \leq \arctan(x) \leq \pi/2$, *we have that* $\forall\ x \in \mathrm{EF}(F)$, $-\sqrt{\pi/2} \leq f(x) \leq \sqrt{\pi/2}$.

Looking at the graphs for $\sqrt{\cdot}$ *and* $\arctan(\cdot)$, *and the parabola, it is immediate that there are two zeroes,* ± 1. *Alternatively,*

$$\begin{aligned} \sqrt{\arctan(x^2-1)} &= 0 \\ \arctan(x^2-1) &= 0 \\ x^2 - 1 &= 0 \\ x &= \pm 1 \end{aligned}$$

The $\sqrt{\cdot}$ *is always positive, and so* F *is always positive.*

Theere are no maximum, and the minimums are 0 *for* $x - \pm 1$.

The function is EVEN, since the graph is symmetric w.r.t. the y-axis.

The function is not monotone (neither increasing nor decreasing). It is decreasing when restricted to $(-\infty, -1]$ *and increasing when restricted to* $[1, +\infty)$. *This is immdiate by the graph.*

GNU5: Graph of F