FCS
Math: Functions
First partial

Massimo Caboara

April 2^{st}, 2021

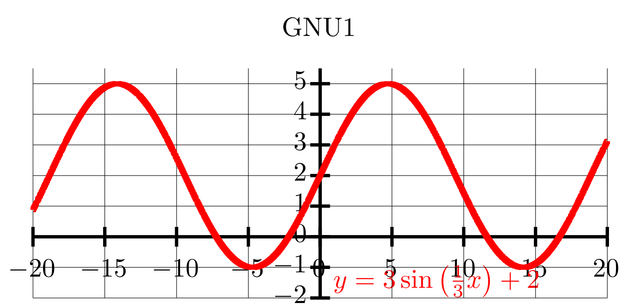
Exercise 1. *Given the function*

$$T: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 3\sin\left(\frac{1}{3}x\right)+2 \end{array}$$

1. *Draw the graph of T.*
2. *What is the PERIOD of T?*

Solution:

1. *The graph is the graph of* $\sin(x)$ *multiplied by* 3*, shifted up by* 2 *and stretched by* 3*.*



2. *: the PERIOD of T is* 6π*, since we stretch by* 3 *a function whose PERIOD is* 2π*.*

Exercise 2. *Solve the equation*

$$\sqrt{x+2}+x=2$$

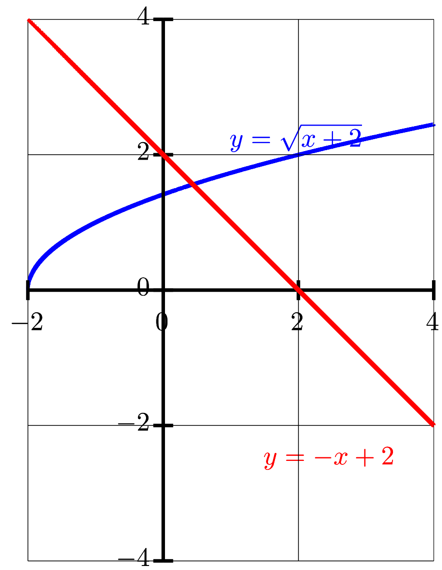
Solution

The existence field is $\mathrm{EF}=[-2,+\infty)$. *We consider the equivalent equation*

$$\sqrt{x+2}=-x+2$$

The graphical solution

GNU3



$$(\sqrt{x+2})^2=(-x+2)^2 \iff x+2=x^2-4x+4 \iff x^2-5x+2$$

$$x=\frac{5\pm\sqrt{25-8}}{2}=\frac{5\pm\sqrt{17}}{2}$$

since $\sqrt{17}\sim 4$, *looking at the graph we can say that*

$\dfrac{5-\sqrt{17}}{2}\sim\dfrac{1}{2}\in[-2,2]$ *is acceptable* $\dfrac{5+\sqrt{17}}{2}\sim\dfrac{9}{2}\notin[-2,2]$ *is not acceptable*

Exercise 3. *Given the sets* $A=\{3n+5 \ \mid \ n\in\mathbb{N}\}$ *and* $B=\{k^2+2 \ \mid \ k\in\mathbb{Z}\}$

1. *Do we have* $\mid A \mid = \mid B \mid$*?*
2. *If the answer is positive, determine explicitly an invertible function* $H: A\longrightarrow B$.

3. *If such H exists, determine its inverse.*

Solution We notice that $B = \{k^2 + 2 \mid k \in \mathbb{Z}\} = \{k^2 + 2 \mid k \in \mathbb{N}\}$ *since* $\forall\, k \in \mathbb{Z}\ k^2 + 2 > 0$*. We remember that in a set there are no repetitions.*

1. *We have* $| A |=| \mathbb{N} |$ *and* $| B |=| \mathbb{N} |$*, since the two functions*

$$F: \begin{array}{ccc} \mathbb{N} & \longrightarrow & A \\ n & \mapsto & 3n+5 \end{array} \qquad G: \begin{array}{ccc} \mathbb{N} & \longrightarrow & B \\ n & \mapsto & n^2+2 \end{array}$$

are invertible with inverses

$$F^{-1}: \begin{array}{ccc} A & \longrightarrow & \mathbb{N} \\ n & \mapsto & \frac{n-5}{3} \end{array} \qquad G^{-1}: \begin{array}{ccc} B & \longrightarrow & \mathbb{N} \\ n & \mapsto & \sqrt{n-2} \end{array}$$

determined by solving

$$3n + 5 = y \Longleftrightarrow n = \frac{y-5}{3}$$

and

$$n^2 + 2 = y \Longleftrightarrow n^2 = y - 2 \Longleftrightarrow n = \sqrt{y-2}$$

in the last case we get the positive solution looking at the domains/codomains. Since

$$| A |=| \mathbb{N} | \ \ \textit{and} \ \ | B |=| \mathbb{N} |$$

we get

$$| A |=| B |$$

2. *We build an invertible function H from A to B by composition,*

$$A \xrightarrow{F^{-1}} \mathbb{N} \xrightarrow{G} B$$

$$H \equiv \begin{array}{cccc} G \circ F^{-1}: & A & \longrightarrow & B \\ & n & \mapsto & F^{-1}(G(n)) = \left(\frac{n-5}{3}\right)^2 + 2 = \frac{n^2-10n+43}{9} \end{array}$$

This is an invertible function since it is the composition of invertible functions

3. *Since $H \equiv G \circ F^{-1}$ is invertible, its inverse is*

$$H^{-1} \equiv (G \circ F^{-1})^{-1}) \equiv (F^{-1})^{-1} \circ G^{-1} \equiv F \circ G^{-1}$$

and since

$$F \circ G^{-1}(n) = F(G^{-1}(n)) = F(\sqrt{n-2}) = 3(\sqrt{n-2}) + 5$$

we have

$$H^{-1}: \begin{array}{ccc} B & \longrightarrow & A \\ n & \mapsto & 3(\sqrt{n-2}) + 5 \end{array}$$

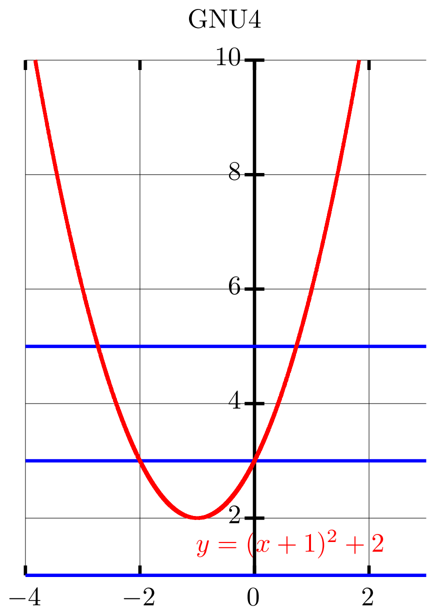
Exercise 4. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & (x+1)^2+2 \end{array}$$

1. *Is F invertible?*
2. *If the answer is no, determine a domain/codomain restriction of F that makes it invertible. Proceed with the new F.*
3. *Determine explicitly the inverse of F*

Solution

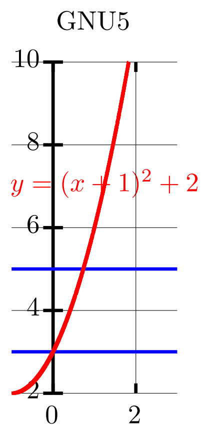
The graph of F is just the graph of the standard parabola shifted by 2 up and by 1 to the left.



1. *So, F is not invertible by the horizontal line rule.*
2. *We restrict to domain and codomain of F to get the new function*

$$F': \begin{array}{ccc} [-1,+\infty) & \longrightarrow & [2,+\infty) \\ x & \mapsto & (x+1)^2+2 \end{array}$$

whose graph is



F' is clearly invertible by the horizontal line rule

3. *The inverse of*

$$F' : [-1, +\infty) \longrightarrow [2, +\infty)$$

is

$$F'^{-1} : [2, +\infty) \longrightarrow [-1, +\infty)$$

$$\begin{aligned} (x+1)^2 + 2 &= y \\ x^2 + 2x + 3 - y^2 &= 0 \\ x &= \frac{-2 \pm \sqrt{4 - 4(3 - y^2)}}{2} \\ x &= -1 \pm \sqrt{y^2 - 2} \end{aligned}$$

Since $x \in [-1, +\infty)$ we take the positive root only.

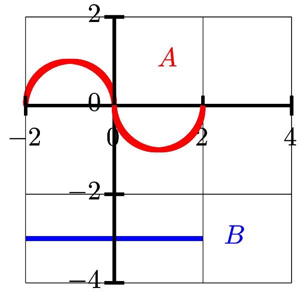
The explicit inverse is

$$\begin{array}{rccc} F'^{-1} : & [2, +\infty) & \longrightarrow & [-1, +\infty) \\ & x & \mapsto & -1 + \sqrt{x^2 - 2} \end{array}$$

Exercise 5.

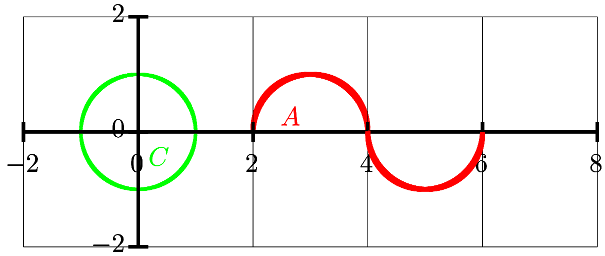
1. *Do the subsets* A, B *of* $\mathbb{R}^2$ *have the same cardinality? Detail the answer.*

GNU6



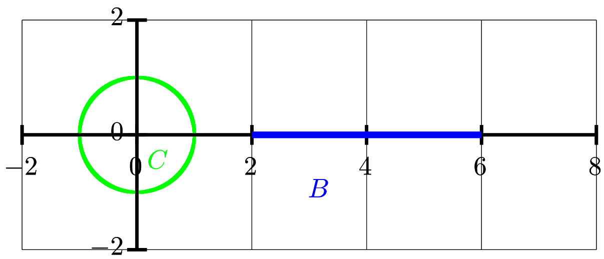
2. *Do the subsets* A, C *of* $\mathbb{R}^2$ *have the same cardinality? Detail the answer.*

GNU7



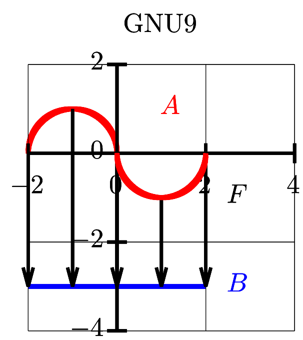
Do the subsets B, C *of* $\mathbb{R}^2$ *have the same cardinality? Detail the answer.*

GNU8

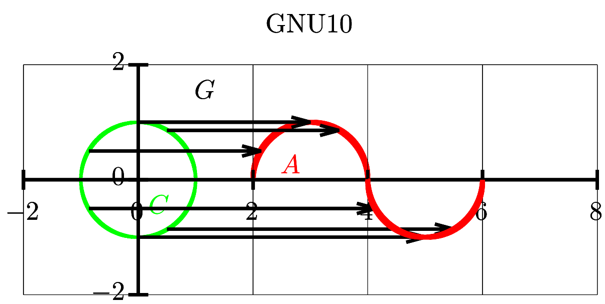


Solution

1. *We have* $\mid A \mid=\mid B \mid$ *since the function* F *detailed below is a one-to-one correspondence.*



2. *We have* $\mid A \mid=\mid C \mid$ *since the function* G *detailed below is a one-to-one correspondence.*



3. *Since* $\mid A \mid=\mid B \mid$ *and* $\mid A \mid=\mid C \mid$*, we have* $\mid B \mid=\mid C \mid$.

Exercise 6.

1. *Draw the function*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sin\left(\frac{1}{3}x\right) \end{array}$$

2. *Draw the function*

$$g: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sin\left(\frac{2}{3}x\right) \end{array}$$

3. *Determine the PERIOD of f.*

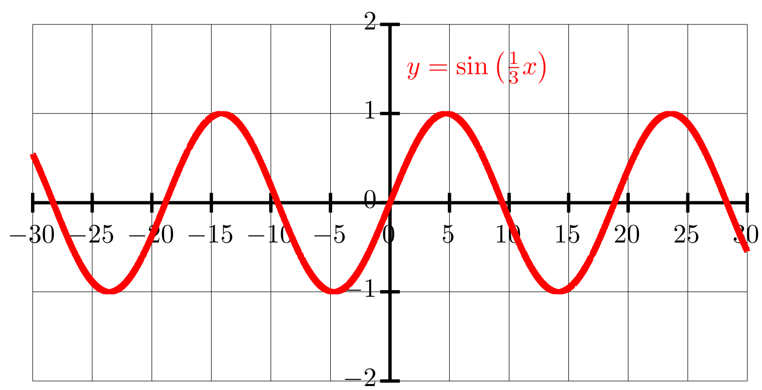
4. *Determine the PERIOD of g.*

5. *Determine a period of $f+g$.*

Solution

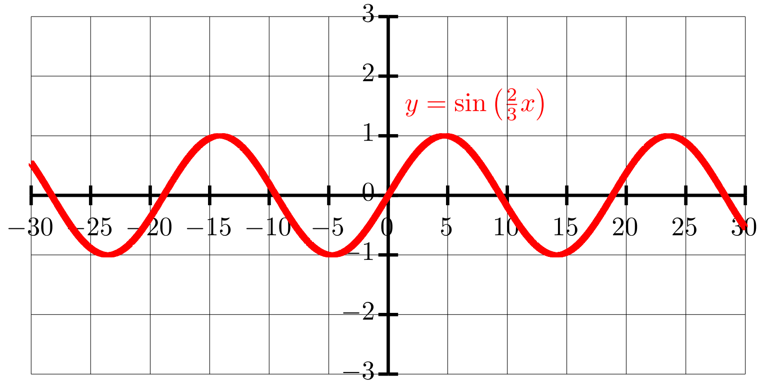
1. *The graph of* $\sin\left(\frac{1}{3}x\right)$ *is the graph of* $\sin(x)$, *but stretched by* 3.

GNU11



2. *The graph of* $\sin\left(\frac{2}{3}x\right)$ *is the graph of* $\sin(x)$, *but stretched by* $3/2$.

GNU12



3. *The PERIOD of f is, for the same resons as in exercise* 1, 6π.

4. *The PERIOD of g is, for the same resons as in exercise* 1, 3π.

5. *A period of $f+g$ is thus* $lcm(3\pi, 6\pi) = 6\pi$.