FCS
Math: Functions
First partial

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Instructions

You have two hours, plus ten minuts to download your work. You cant' download your files before the two hours limit. Your work is to be downloaded on a folder whose title is your name, on the exam folder in Files, on Teams. The allowed format are pdf and jpg/jpeg/gif. THE FORMAT HEIC IS NOT ALLOWED. If possible, write on white paper. After you have checked that all your files have been downloaded in the correct folder you can close your connection. You have to keep the video connection on during the partial.

If you have serious problems with the Teams download, you can send your files to `caboara@dm.unipi.it`. In this case file size has to be less that 15MB. Wait for me to an acknowledge your submission.

Exercise 1. *Given the function*

$$T: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sin\left(\frac{1}{3}x\right) + 2 \end{array}$$

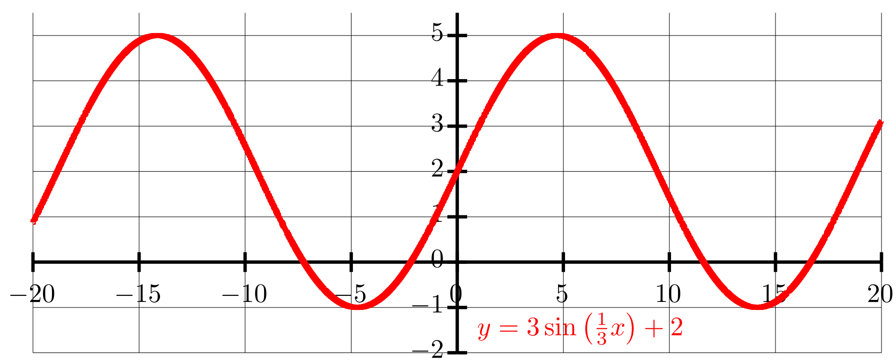
1. *Draw the graph of T.*

2. *What is the PERIOD of T?*

Solution:

1. *The graph is the graph of* $\sin(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$, $\sin(\cdot)(x) = \sin(x)$ *shifted by two up, amplified by a factor of* 3 *and with frequency multipled by* $1/3$. *For the frequency (period) more details below.*

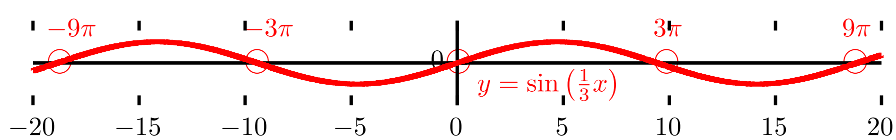
GNU1



2. *: the period of T is the same of the period of*

$$T: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 3\sin\left(\frac{1}{3}x\right) \end{array}$$

GNU2



That is, the graph of $\sin(x)$ *stretched on the x-axis. We find the exact strech by finding zeroes of this function, and that by solving the equation*

$$\sin(1/3x) = 0 \Longleftrightarrow 1/3x = 0{+}2k\pi \text{ and } 1/3x = \pi{+}2k\pi \Longleftrightarrow x = 0{+}6k\pi \text{ and } x = 3\pi{+}6k\pi$$

the PERIOD of this function is thus 6π*, and the PERIOD of* T *is the same,* 6π

Exercise 2. *Solve the equation*

$$\sqrt{x+2}+x=2$$

Solution

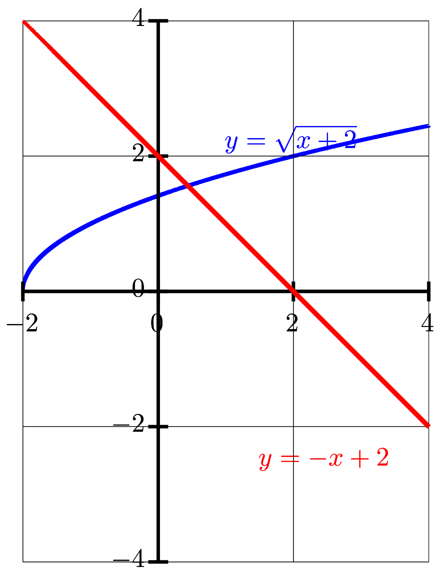
The existence field for the equation is the existence field of $\sqrt{x+2}$*, so we need* $x+2\geq 0$ *and* $\mathrm{EF}=[-2,+\infty)$*.*

To simplify the graph drawing, we transform the equation in the equation

$$\sqrt{x+2}=-x+2$$

The graph of the first side is the graph of $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=\sqrt{x}$ *shifted left by* 2*, the graph of the second side is, likewise, the graph of the line* $g:\mathbb{R}\longrightarrow\mathbb{R},\ g(x)=-x+2$ *shifted left by* 2*. We have*

GNU3



we have one solution. The determine it, we have to solve the equation

$$\sqrt{x+2}=-x+2$$

We want to apply to both sides the function

$$\begin{array}{cccc}(\cdot)^2: & \mathbb{R}+_0 & \longrightarrow & \mathbb{R}+_0\\ & x & \mapsto & x^2\end{array}$$

the inverse of the function

$$\begin{array}{cccc}\sqrt{\cdot}: & \mathbb{R}+_0 & \longrightarrow & \mathbb{R}+_0\\ & x & \mapsto & \sqrt{x}\end{array}$$

To do that, we need both sides to be in the domain of $(\cdot)^2$, *thus to be positive or null. The left side is always positive or null, the second only if* $-x+2 \geq 0 \Longleftrightarrow x \leq 2$. *So we examine the two cases separetly, remembering that* $EF = [-2, +\infty]$

- *If* $x > 2$ *then the left side is positive or null, the right side negative. Equality is impossible and there are no solutions.*
- *If* $x \leq 2$, *and* $x \geq -2$ *by the* EF, *we have* $x \in [-2, 2]$. *In this case we can apply the invertible function* $(\cdot)^2$ *to both sides getting the equivalent equation*

  $$(\sqrt{x+2})^2 = (-x+2)^2 \Longleftrightarrow x+2 = x^2 - 4x + 4 \Longleftrightarrow x^2 - 5x + 2$$

  we solve this with the formula and we get

  $$x = \frac{5 \pm \sqrt{25-8}}{2} = \frac{5 \pm \sqrt{17}}{2}$$

  since $\sqrt{17} \sim 4$

  $$\frac{5-\sqrt{17}}{2} \sim \frac{1}{2} \in [-2, 2] \text{ and } \frac{5+\sqrt{17}}{2} \sim \frac{9}{2} \notin [-2, 2]$$

  we know, by the condition $x \in [-2, 2]$ *or by looking at the graph, that the equation has just the solution*

  $$x = \frac{5-\sqrt{17}}{2}$$

Exercise 3. *Given the sets* $A = \{3n+5 \mid n \in \mathbb{N}\}$ *and* $B = \{k^2+2 \mid k \in \mathbb{Z}\}$

1. *Do we have* $| A | = | B |$*?*
2. *If the answer is positive, determine explicitly an invertible function* $H : A \longrightarrow B$.
3. *If such* H *exists, determine its inverse.*

Solution We notice that $B = \{k^2+2 \mid k \in \mathbb{Z}\} = \{k^2+2 \mid k \in \mathbb{N}\}$ *since* $\forall\, k \in \mathbb{Z}\ k^2+2 > 0$. *We remember that in a set there are no repetitions.*

1. *We have* $| A | = | \mathbb{N} |$ *and* $| B | = | \mathbb{N} |$, *since the two functions*

   $$F: \begin{array}{ccc} \mathbb{N} & \longrightarrow & A \\ n & \mapsto & 3n+5 \end{array} \qquad G: \begin{array}{ccc} \mathbb{N} & \longrightarrow & B \\ n & \mapsto & n^2+2 \end{array}$$

   are invertible with inverses

   $$F^{-1}: \begin{array}{ccc} A & \longrightarrow & \mathbb{N} \\ n & \mapsto & \frac{n-5}{3} \end{array} \qquad G^{-1}: \begin{array}{ccc} B & \longrightarrow & \mathbb{N} \\ n & \mapsto & \sqrt{n-2} \end{array}$$

determined by solving

$$3n + 5 = y \Longleftrightarrow n = \frac{y-5}{3}$$

and

$$n^2 + 2 = y \Longleftrightarrow n^2 = y - 2 \Longleftrightarrow n = \sqrt{y-2}$$

in the last case we get the positive solution looking at the domains/codomains. Since

$$\mid A \mid = \mid \mathbb{N} \mid \quad \textit{and} \quad \mid B \mid = \mid \mathbb{N} \mid$$

we get

$$\mid A \mid = \mid B \mid$$

2. *We build an invertible function H from A to B by composition,*

$$A \xrightarrow{F^{-1}} \mathbb{N} \xrightarrow{G} B$$

$$H \equiv \begin{array}{cccc} G \circ F^{-1}: & A & \longrightarrow & B \\ & n & \mapsto & F^{-1}(G(n)) = \left(\frac{n-5}{3}\right)^2 + 2 = \frac{n^2-10n+43}{9} \end{array}$$

This is an invertible function since it is the composition of invertible functions

3. *Since $H \equiv G \circ F^{-1}$ is invertible, its inverse is*

$$H^{-1} \equiv (G \circ F^{-1})^{-1}) \equiv (F^{-1})^{-1} \circ G^{-1} \equiv F \circ G^{-1}$$

and since

$$F \circ G^{-1}(n) = F(G^{-1}(n)) = F(\sqrt{n-2}) = 3(\sqrt{n-2}) + 5$$

we have

$$\begin{array}{cccc} H^{-1}: & B & \longrightarrow & A \\ & n & \mapsto & 3(\sqrt{n-2}) + 5 \end{array}$$

Exercise 4. *We have the function*

$$\begin{array}{cccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & (x+1)^2 + 2 \end{array}$$

1. *Is F invertible?*

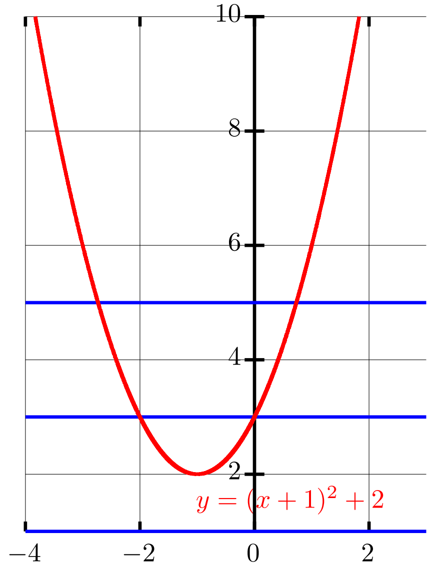
2. *If the answer is no, determine a domain/codomain restriction of F that makes it invertible. Proceed with the new F.*

3. *Determine explicitly the inverse of F*

Solution

We can easily draw the graph of F, since it is just the standard parabola shifted by 2 up and by 1 to the left.

GNU4



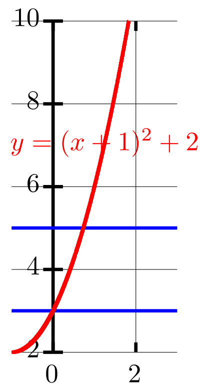
1. *Looking at the graph of F we see that there are horizontal lines that don't intersect it, and there are horiziontal lines that intersect it twice. So, F is not invertible.*

2. *We restrict to domain and codomain of F to get the new function*

$$\begin{array}{cccc} F': & [-1,+\infty) & \longrightarrow & [2,+\infty) \\ & x & \mapsto & (x+1)^2+2 \end{array}$$

whose graph is

GNU5



and F' is clearly invertible by the horizontal line rule

3. *The inverse of*

$$F' : [-1, +\infty) \longrightarrow [2, +\infty)$$

is

$$F'^{-1} : [2, +\infty) \longrightarrow [-1, +\infty)$$

To determine explicitly the formula for the inverse, we solve with respect to the unknown $x \in [-1, +\infty)$ (the domain of F') and the parameter $y \in [2, +\infty]$ (the codomain of F') the equation

$$\begin{aligned} (x+1)^2 + 2 &= y \\ x^2 + 2x + 1 + 2 - y^2 &= 0 \\ x^2 + 2x + 3 - y^2 &= 0 \end{aligned}$$

and by the formula we get

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(3 - y^2)}}{2} \\ x &= \frac{-2 \pm 2\sqrt{1 - (3 - y^2)}}{2} \\ x &= -1 \pm \sqrt{y^2 - 2} \end{aligned}$$

We remark that since $y \in [2, +\infty]$,

- *the root does exist;*
- *that the x has to actually range between -1 and $+\infty$, so we take the positive root.*

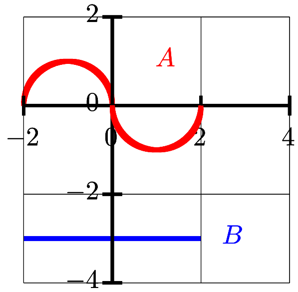
The explicit inverse is

$$\begin{array}{rccc} F'^{-1}: & [2,+\infty) & \longrightarrow & [-1,+\infty) \\ & x & \mapsto & -1+\sqrt{x^2-2} \end{array}$$

Exercise 5.

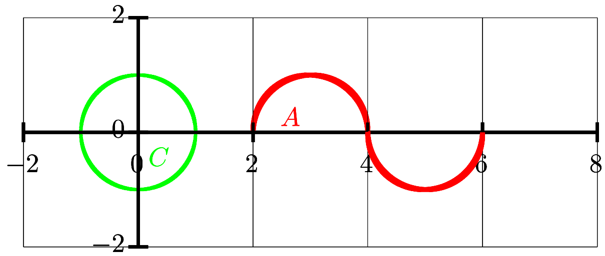
1. *Do the subsets A, B of $\mathbb{R}^2$ have the same cardinality? Detail the answer.*

GNU6



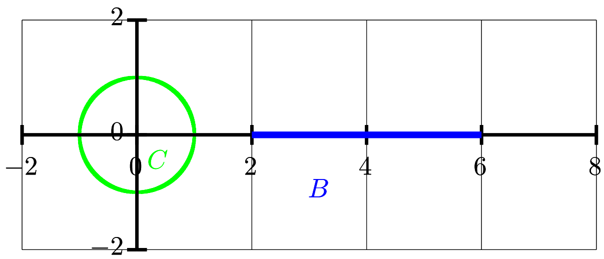
2. *Do the subsets A, C of $\mathbb{R}^2$ have the same cardinality? Detail the answer.*

GNU7



Do the subsets B, C of $\mathbb{R}^2$ have the same cardinality? Detail the answer.

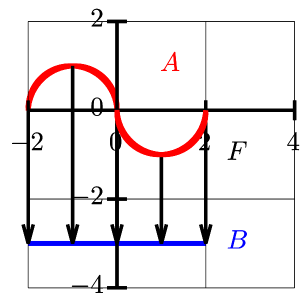
GNU8



Solution

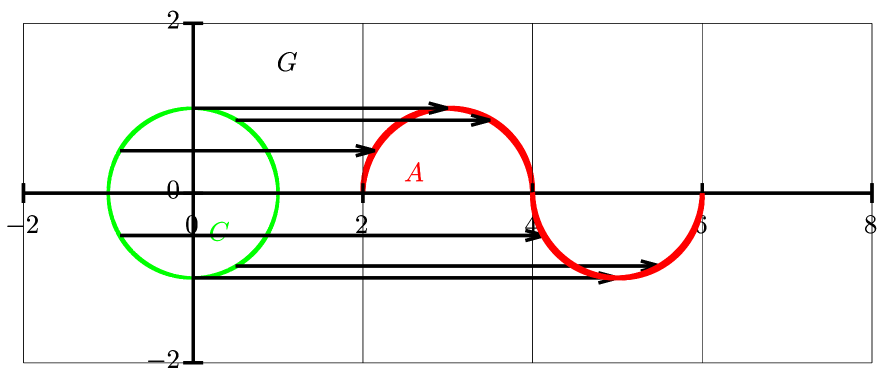
1. *We have* $|A| = |B|$ *since the function* F *detailed below is a one-to-one correspondence.*

GNU9



2. *We have* $|A| = |C|$ *since the function* G *detailed below is a one-to-one correspondence.*

GNU10



3. Since $|A| = |B|$ *and* $|A| = |C|$*, we have* $|B| = |C|$.

Exercise 6.

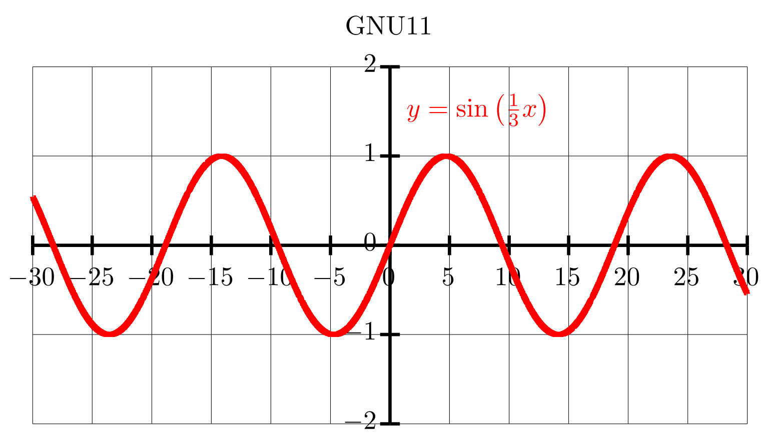
1. *Draw the function*
$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sin\left(\frac{1}{3}x\right) \end{array}$$
2. *Draw the function*
$$\begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sin\left(\frac{2}{3}x\right) \end{array}$$
3. *Determine the PERIOD of f.*
4. *Determine the PERIOD of g.*
5. *Determine a period of $f+g$.*

Solution

1. *The graph of* $\sin\left(\frac{1}{3}x\right)$ *is the graph of* $\sin(x)$*, but stretched. Let's determine the stretch. The zeroes in the solutions of*
$$\sin\left(\frac{1}{3}x\right) = 0$$
Since $\sin(x) = 0 \iff x = 0 + 2\pi, \pi + 2\pi$

$$\sin\left(\frac{1}{3}x\right) = 0 \iff \frac{1}{3}x = 0 + 2k\pi, x = \pi + 2k\pi \iff x = 0 + 6k\pi, 3\pi + 6k\pi$$

for $k \in \mathbb{Z}$



2. *The graph of* $\sin\left(\frac{2}{3}x\right)$ *is the graph of* $\sin(x)$*, but stretched. Let's determine the stretch. The zeroes in the solutions of*

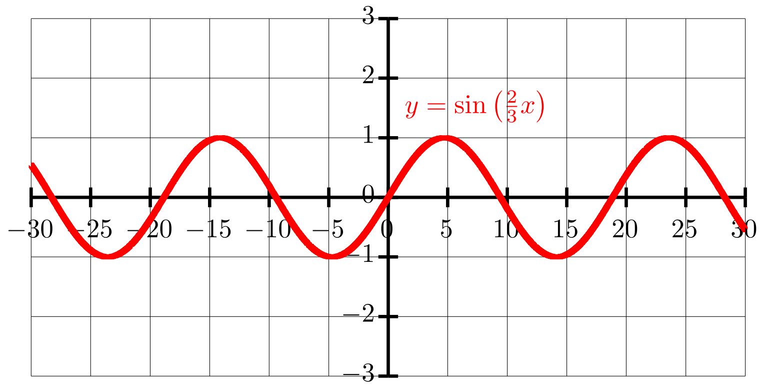
$$\sin\left(\frac{2}{3}x\right) = 0$$

Since $\sin(x) = 0 \iff x = 0 + 2\pi, \pi + 2\pi$

$$\sin\left(\frac{2}{3}x\right) = 0 \iff \frac{2}{3}x = 0 + 2k\pi, x = \pi + 2k\pi \iff x = 0 + 3k\pi, 3\pi + 3k\pi$$

for $k \in \mathbb{Z}$

GNU12



3. *The PERIOD of* f *is, looking at the graph,* 6π.

4. *The PERIOD of* g *is, looking at the graph,* 3π.

5. *A period of* $f + g$ *is thus* $6\pi = lcm(3, 6)\pi$.

Exercise 7 (Optional)**.** *Hint: draw the sets* A_1, A_2, A_3 *in* $\mathbb{N} \times \mathbb{N}$*, the positive integer grid on the plane.*

1. *Find a set* B_1 *in* $\mathbb{N}$ *with the same cardinality of the set*

$$A_1 = [0, 1] \times [0, 1] = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{N} \times \mathbb{N}$$

Find a one-to-one correspondence between A_1 *and* B_1.

2. *Find a set* B_2 *in* $\mathbb{N}$ *with the same cardinality of the set*

$$A_2 = [0, 2]\times[0, 2] = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\} \subset \mathbb{N}\times\mathbb{N}$$

Find a one-to-one correspondence between A_2 *and* B_2.

3. *Find a set* B_3 *in* $\mathbb{N}$ *with the same cardinality of the set* $A_3 = [0,3] \times [0,3] \subset \mathbb{N} \times \mathbb{N}$. *Find a one-to-one correspondence between* A_3 *and* B_3.

4. *Find a set* B_n *in* $\mathbb{N}$ *with the same cardinality of the set* $A_n = [0,n] \times [0,n] \subset \mathbb{N} \times \mathbb{N}$. *Find a one-to-one correspondence between* A_n *and* B_n.

5. *Is it true that* $\mid \mathbb{N} \mid = \mid \mathbb{N} \times \mathbb{N} \mid$? *Detail the answer.*

6. *Is it true that* $\mid \mathbb{N} \mid = \mid \mathbb{Q} \mid$? *Detail the answer.*

This exercise will be fully discussed in class.