FY - FCS - Homework 2

Massimo Caboara

May 20, 2021

Exercise 1. *We have the function*

$$\begin{array}{cccc} F: & [-1, +\infty] & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_e(x+1)+2 \end{array}$$

1. *Draw the graph of F. Mark the intersections with the axis and the vertex.*
2. *Determine $F([1, 4])$.*
3. *Determine $F^{-1}(1)$.*
4. *Determine $F^{-1}(1/2)$.*
5. *Determine $F^{-1}([-1, 1])$.*
6. *Solve, with approximation if necessary, the equation $F(x) = x^2$*
7. *Is F invertible? If it is, determine the formula for its inverse and draw its graph.*

Solutions: we draw the graph F, remembering the graph of $\log_e : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and shifting it up by 2 and to the left by 1.

1. *Since $F(1) = \log_e(2)+2 \simeq 0.7+2 \simeq 2.7$ and $F(4) = \log_e(5)+2 \simeq 1.6+2 = 3.6$, looking at the graph we have $F([1, 4]) = [F(1), F(4)] \simeq [2.7, 3.6]$.*

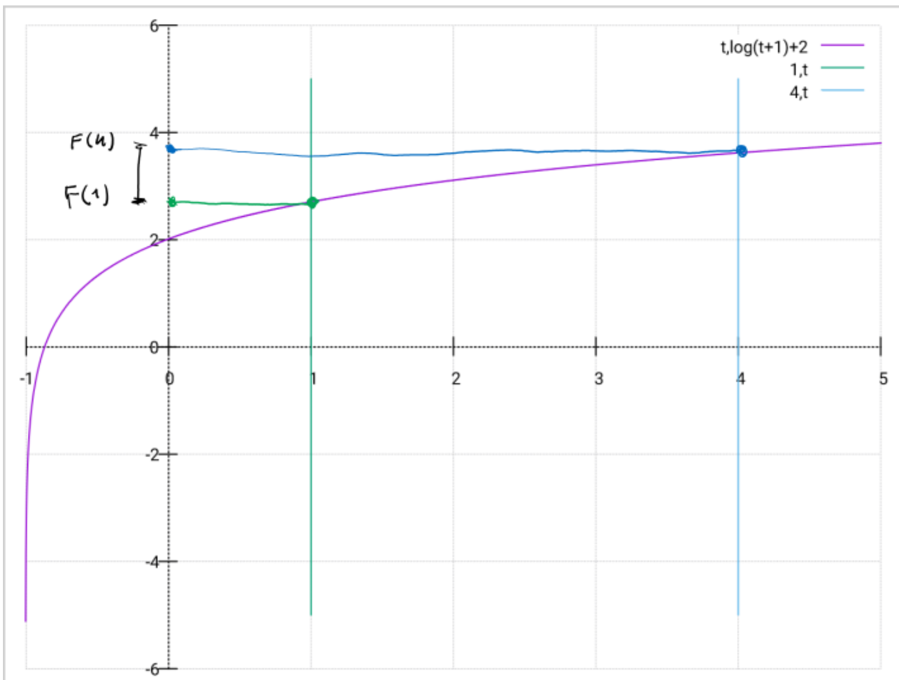


Figure 1: $F([1,2])$

2. Determine $F^{-1}(1)$.

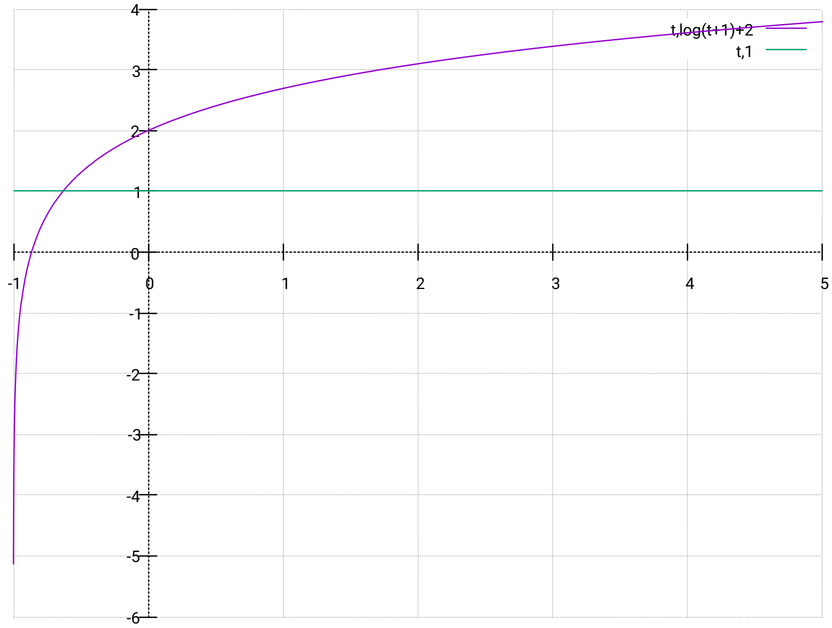


Figure 2: Intersection of the graph of F with the line $y = 1$

We have to solve

$$\begin{aligned}
\log_e(x+1)+2 &= 1 \\
\log_e(x+1) &= -1 \\
e^{\log_e(x+1)} &= e^{-1} \\
x+1 &= 1/e \\
x &= 1/e - 1 \\
x &\simeq 0.36 - 1 \\
x &\simeq -0.64
\end{aligned}$$

3. Determine $F^{-1}(1/2)$.

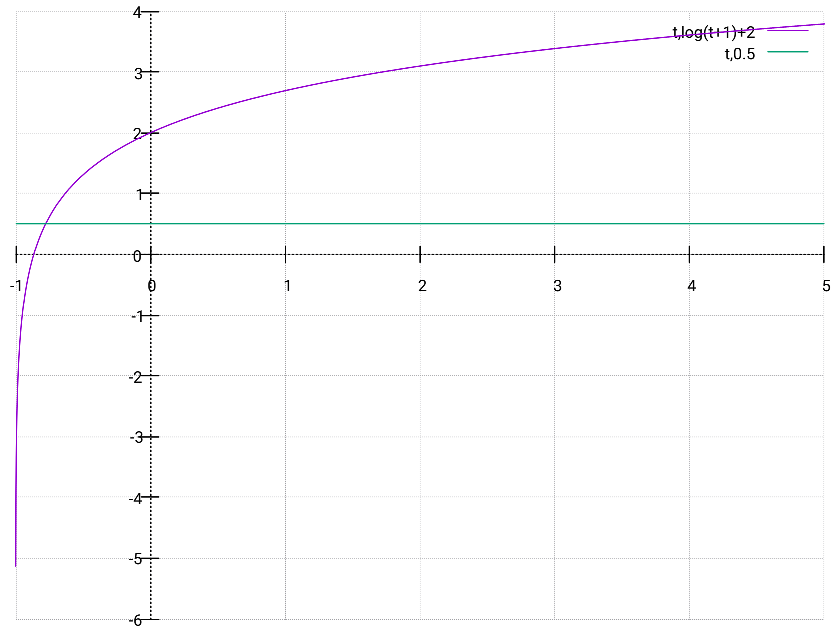


Figure 3: Intersection of the graph of F with the line $y = 1/2$

We solve

$$\begin{aligned}
\log_e(x+1) + 2 &= 1/2 \\
\log_e(x+1) &= -3/2 \\
e^{\log_e(x+1)} &= e^{-3/2} \\
x + 1 &= \frac{1}{e^{3/2}} \\
x + 1 &= \frac{1}{e\sqrt{e}} \\
x + 1 &= \frac{1}{e\sqrt{e}} - 1 \\
x + 1 &= 0.2 - 1 \\
x &\simeq -0.8
\end{aligned}$$

4. Determine $F^{-1}([-1,1])$.

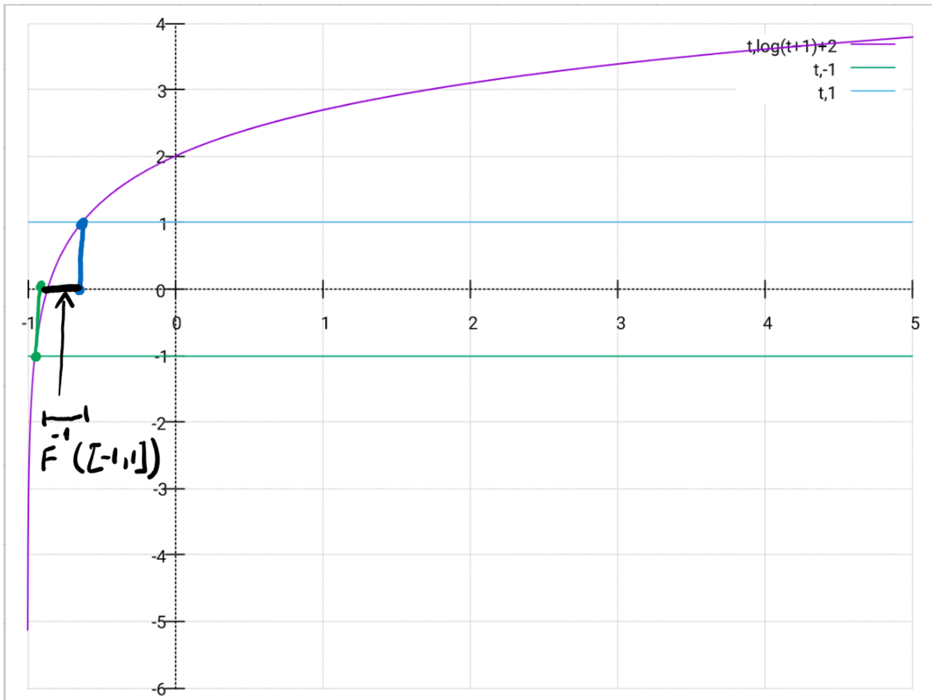


Figure 4: Intersection of the graph of F with the lines $y = -1, y = 1$

It is immediate from the graph that $F^{-1}([-1,1]) = [a,b]$.

We have already computed b, *that is* $F^{-1}(1) \simeq -0.64$. *To compute* $a = F^{-1}(-1)$ *We solve*

$$
\begin{aligned}
\log_e(x+1) + 2 &= -1 \\
\log_e(x+1) &= -3 \\
e^{\log_e(x+1)} &= e^{-3} \\
x+1 &= \frac{1}{e^3} \\
x &= \frac{1}{e^3} - 1 \\
x &= 0.05 - 1 \\
x &\simeq -0.95
\end{aligned}
$$

and so $a = -0.95$ *and*

$$F^{-1}([-1,1]) \simeq [-0.95, -0.64]$$

5. *Solve, with approximation if necessary, the equation* $F(x) = x^2$.

We draw the graph of the function and the parabola $y = x^2$.

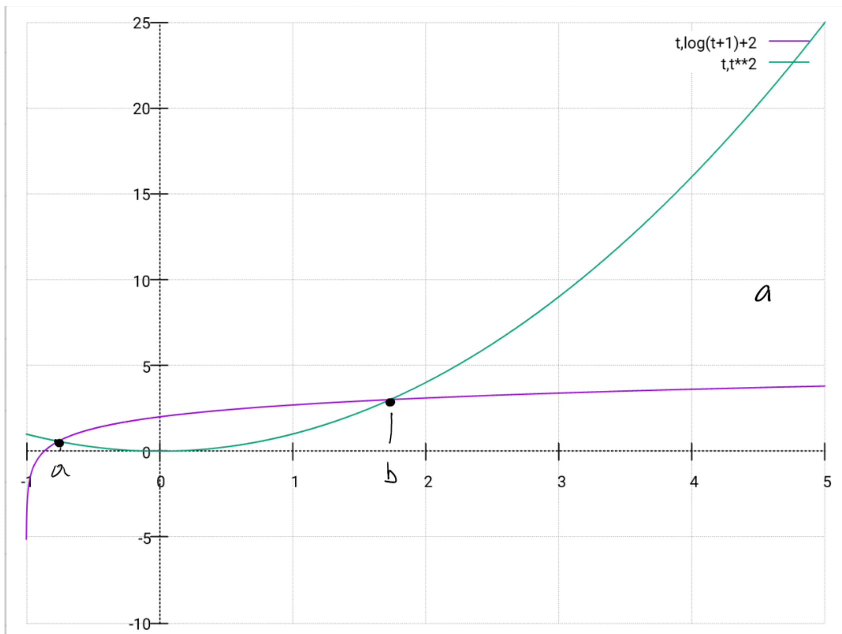


Figure 5: Intersection of the graph of F with the parabola $y = x^2$

It is clear that there are two intersections, a, b, *with* $a \in (-1, -0.5)$ *and* $b \in (1.5, 2)$

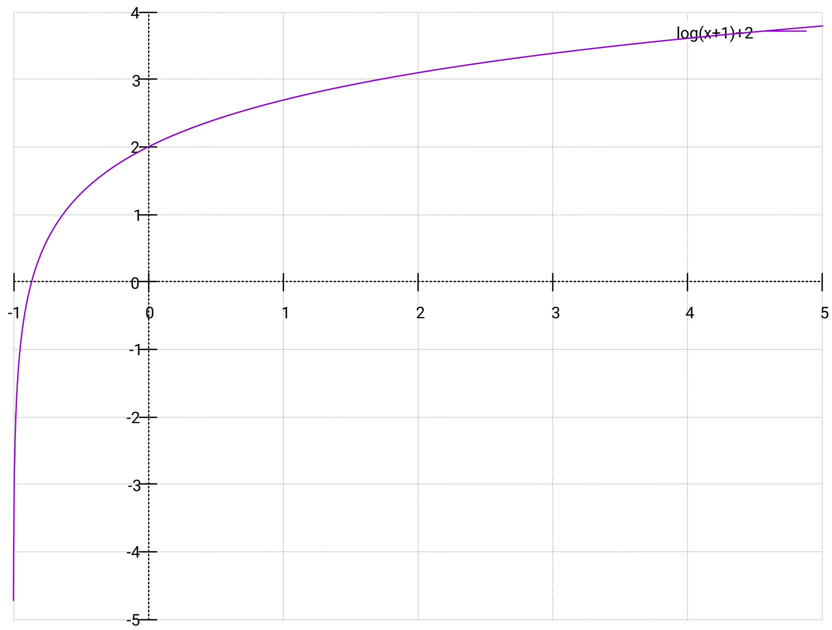


Figure 6: Graph of $Fx) = \log_e(x+1) + 2$

6. *By looking at the graph of F it is obvious that any horizontal line $y = a$, with $a \in \mathbb{R}$ intersects F once and only once. The function F is hence invertible, and there exists $F^{-1} : \mathbb{R} \longrightarrow (-1, +\infty)$. The get the inverse formula, we solve for x the equation*

$$\begin{aligned} \log_e(x+1) + 2 &= y \\ \log_e(x+1) &= y - 2 \\ e^{\log_e(x+1)} &= e^{y-2} \\ x + 1 &= e^{y-2} \\ x &= e^{y-2} - 1 \end{aligned}$$

The third step is OK becasuse the function e^x has as domain the whole $\mathbb{R}$, and we can hence apply it to both sides of the equation without regard to the values $\log_e(x+1)$, $y - 2$. The fourth step is OK because the functions e^x and $\log_e(x)$ are inverses.

The inverse is so

$$\begin{array}{rccc} F^{-1}: & \mathbb{R} & \longrightarrow & (-1, +\infty) \\ & y & \mapsto & e^{y-2} - 1 \end{array}$$

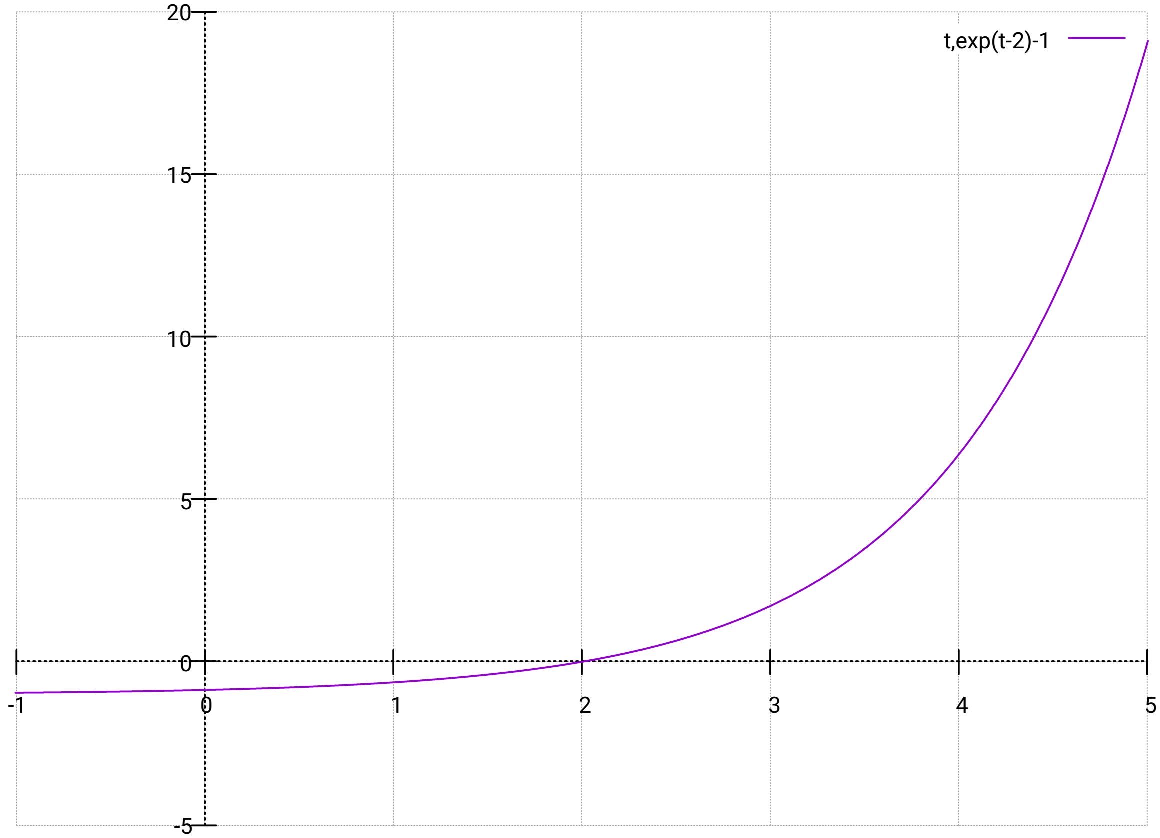


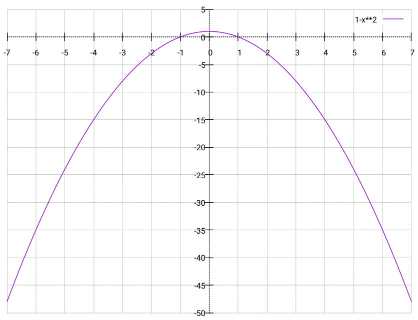
Figure 7: The graph of $F^{-1} : \mathbb{R} \longrightarrow (-1, +\infty)$, $F^{-1}(x) = e^{y-2} - 1$

Exercise 2. *Find the definition field for the formula*

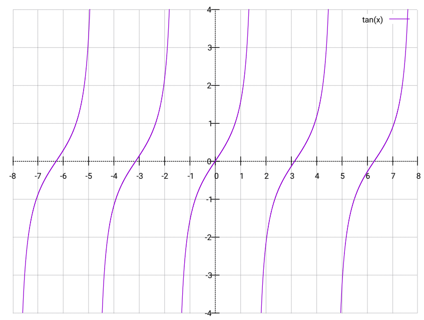
$$\begin{array}{rccc} h: & [-2\pi, 2\pi] & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{\sqrt{1-x^2}\tan(x)}{(1-x^2)\sin(x)} \end{array}$$

We have the following constraints

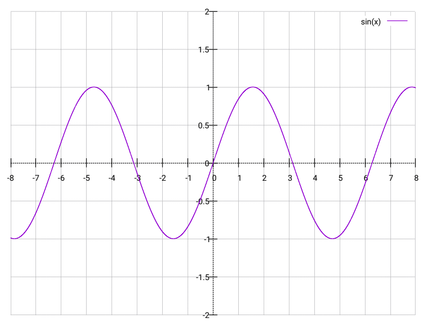
$$\begin{cases} 1 - x^2 \geq 0 & \textit{root existence} \\ x \neq k\pi/2,\ k \in \mathbb{Z} & \textit{tangent existence} \\ 1 - x^2 \neq 0 & \textit{denominator } \neq 0 \\ sin(x) \neq 0 & \textit{denominator } \neq 0 \end{cases}$$



(a) $y = 1 - x^2$



(b) $y = \tan(x)$



(c) $y = \sin(x)$

Figure 8: The elementary function graphs

Looking at the graphs we easily see that

1. $1 - x^2 \geq 0 \Longrightarrow x \in [-1, 1]$.
2. $x \neq k\pi/2,\ k \in \mathbb{Z} \Longrightarrow x \notin \{\ldots, -5/2\pi, -3/2\pi, -\pi/2, \pi/2, 3/2\pi, 5/2\pi, \ldots\}$. *And since,* $x \in [-2\pi, 2\pi]$ *this is equivalent to* $x \notin \{-3/2\pi, -\pi/2, \pi/2, 3/2\pi\}$.
3. $1 - x^2 \neq 0 \Longrightarrow x^2 \neq 1 \Longrightarrow x \neq \pm 1$.
4. $sin(x) \neq 0 \Longrightarrow x \neq \pi + k\pi,\ k \in \mathbb{Z} \Longrightarrow x \notin \{\ldots, -3\pi, -2\pi, \pi, 0, \pi, 2\pi, 3\pi, \ldots\}$. *And since,* $x \in [-2\pi, 2\pi]$ *this is equivalent to* $x \notin \{-2\pi, \pi, 0, \pi, 2\pi\}$

Since $\pi \simeq 3,14$ *we have* $\pi/2 \simeq 1.57$ *and the four conditions, that have to held at the same time, give as final constraint*

$$x \in (-1, 0) \cup (0, 1)$$

And this is the definition set of the formula h. The associated function is

$$\begin{array}{cccc} h': & (-1,0) \cup (0,1) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{\sqrt{1-x^2}\tan(x)}{(1-x^2)\sin(x)} \end{array}$$

and since, for $x \in (-1,0) \cup (0,1)$ we have that

$$\begin{aligned} \frac{\sqrt{1-x^2}\tan(x)}{(1-x^2)\sin(x)} &= \frac{\sqrt{1-x^2}\frac{\sin(x)}{cos(x)}}{(1-x^2)\sin(x)} \\ &= \frac{\sqrt{1-x^2}\frac{1}{cos(x)}}{(1-x^2)} \\ &= \frac{\sqrt{1-x^2}}{(1-x^2)\cos(x)} \end{aligned}$$

we have that h' is equal, e.g., to the simpler function

$$\begin{aligned} h'' : \quad (-1,0) \cup (0,1) &\longrightarrow \mathbb{R} \\ x &\mapsto \tfrac{\sqrt{1-x^2}}{(1-x^2)\cos(x)} \end{aligned}$$

Exercise 3. *Find the number of solutions of the equation*

$$4sin(x) + 2 = x^2 - 5$$

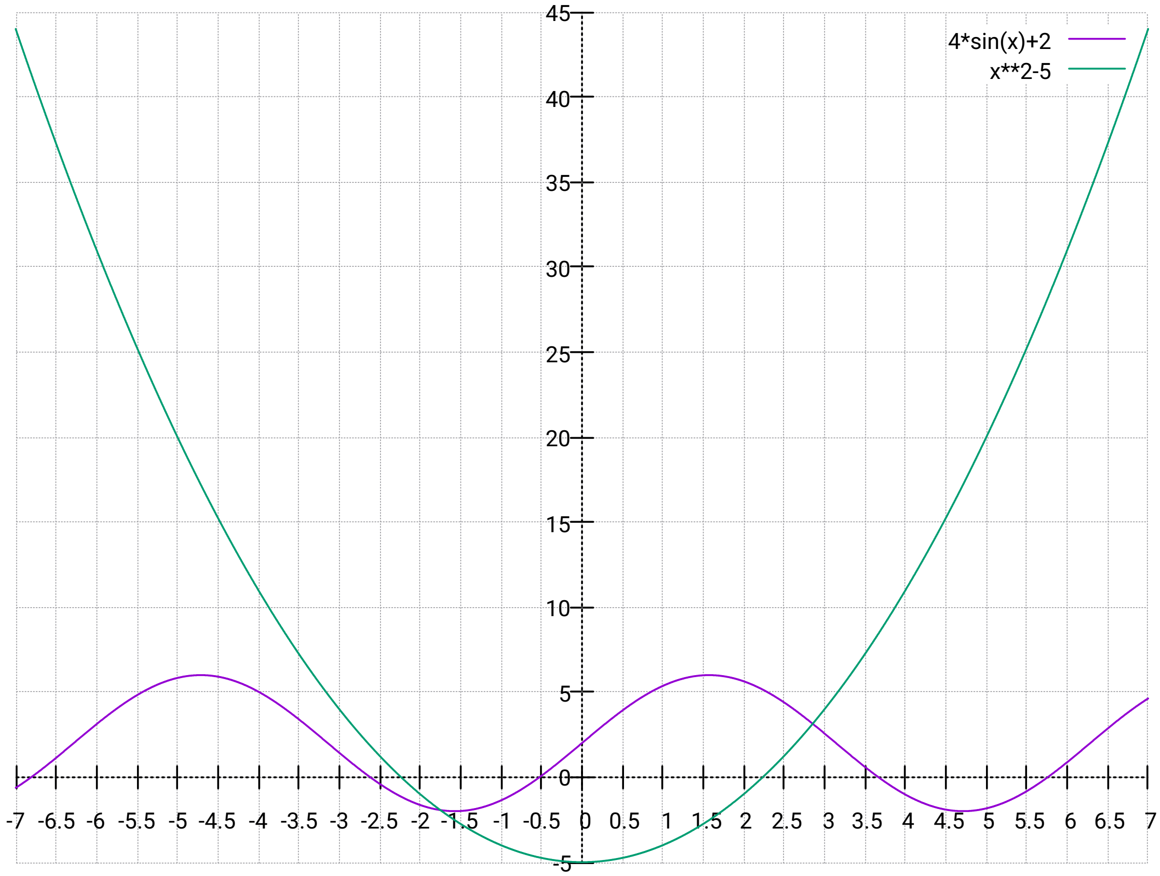


Figure 9: Intersections of $y = 4sin(x) + 2$ and $x^2 - 5$

From the graph it is immediate that there are two solutions $x = a, b$ and $a \in (-2, -1.5)$, $b \in (2.5, 3)$.

Exercise 4. *Draw the graph of the function*

$$F: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} \dfrac{2^{x+1}+1}{2} & x \in [-5,1) \\[2ex] \dfrac{x^2+2}{2} & x \in [1,2] \\[2ex] \dfrac{\sqrt{x}+8}{2} & x \in (2,8) \end{cases}$$

Is the function

1. *Injective?*
2. *Surjective?*
3. *Invertible?*

Restrict the domain, codomain of F to obtain an invertible function.
The purple curve is for $\frac{2^{x+1}+1}{2}$, the green curve is for $\frac{x^2+2}{2}$ and the blu curve is for $\frac{\sqrt{x}+8}{2}$.

Solution:
From the graph of the function it is clear that

1. *F is not injective because, for example, the red horizontal line $y = 2$ intersects the graph of F twice.*
2. *F is not surjective, because the magenta horizontal line $y = 4$ does not intersect the graph of F.*
3. *F is thus not invertible.*

If, for example, we restrict the domain of F to $(-5, 1]$ and the codomain to

$$[F(-5), F(1)] = \left[\frac{2^{-4}+1}{2}, \frac{2^2+1}{2}\right] = \left[\frac{1/16+1}{2}, \frac{5}{2}\right] = \left[\frac{1\frac{17}{16}}{2}, \frac{5}{2}\right] = \left[\frac{17}{32}, \frac{5}{2}\right] = [17/32, 4] = [0.5125, 2.5]$$

we get the function

$$F_{|[-5,1]}: [-5,1] \longrightarrow [0.5125, \tfrac{5}{2}]$$
$$x \mapsto F(x)$$

that is clearly invertible