FCS
Math: Functions
Exercises

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Remark 1. *If we have an invertible function* $f : A \longrightarrow B$*, it is immediate that* $\left(f^{-1}\right)^{-1} = f$*, since, by definition, if* f^{-1} *is the inverse of* f*, then* f *is the inverse of* f^{-1}*.*

Remark 2. *Let us consider the invertible functions* $F : A \longrightarrow B$ *and* $G : B \longrightarrow C$ *and the function*

$$\begin{array}{rccc} G \circ F : & A & \longrightarrow & C \\ & a & \mapsto & G \circ F(a) = G(F(a)) \end{array}$$

It is easy to see that $G \circ F$ *is invertible and the function*

$$\begin{array}{rccc} H : & C & \longrightarrow & A \\ & c & \mapsto & (F^{-1} \circ G^{-1})(c) = F^{-1}(G^{-1}(c)) \end{array}$$

is its inverse.

Since $F : A \longrightarrow B$ *is invertible, the function* $F^{-1} : B \longrightarrow A$ *exists and* $F^{-1} \circ F \equiv \mathrm{id}_A$, $F \circ F^{-1} \equiv \mathrm{id}_B$.

Since $G : B \longrightarrow C$ *is invertible, the function* $G^{-1} : C \longrightarrow B$ *exists and* $G \circ G^{-1} \equiv \mathrm{id}_C$, $G^{-1} \circ G \equiv \mathrm{id}_B$.

Hence, for all $a \in A$,

$$(F^{-1} \circ G^{-1}) \circ (G \circ F)(a) = (F^{-1} \circ G^{-1} \circ G \circ F)(a) = (F^{-1} \circ \mathrm{id}_C \circ F)(a) = (F^{-1} \circ F)(a) = \mathrm{id}_A(a) = a$$

so $(F^{-1} \circ G^{-1}) \circ (G \circ F) \equiv \mathrm{id}_A$.

For all $c \in C$,

$$(G \circ F) \circ (F^{-1} \circ G^{-1})(c) = (G \circ F \circ F^{-1} \circ G^{-1})(c) = (G \circ \mathrm{id}_A \circ G^{-1})(c) = (G \circ G^{-1})(c) = c$$

so $(G \circ F) \circ (F^{-1} \circ G^{-1}) \equiv \mathrm{id}_C$.

Exercise 1. *Solve the equation* $\log_3(x-1) = 3$.

The existence field of $\log_3(x-1)$ *is* $(1, +\infty)$*, as we can easily see from the graph of*

$$\begin{array}{rccc} \log_3(\cdot): & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_3(x) \end{array}$$

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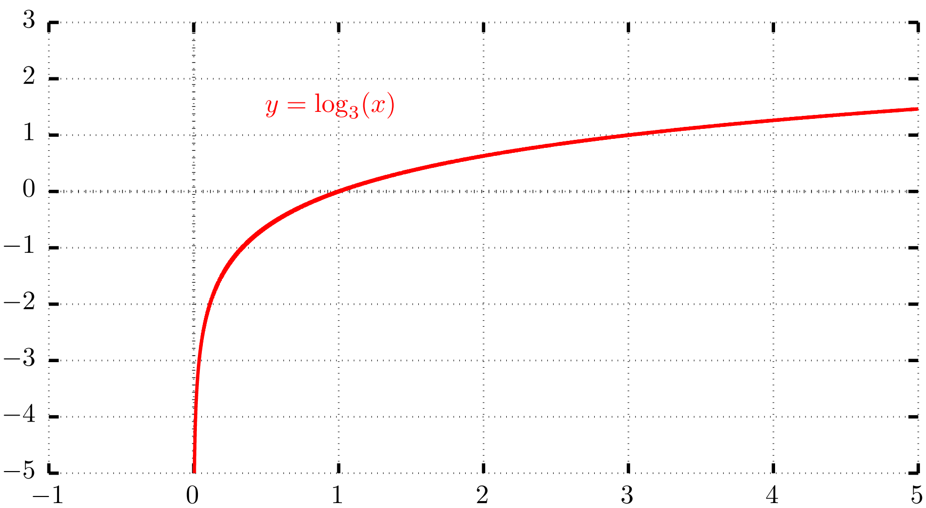
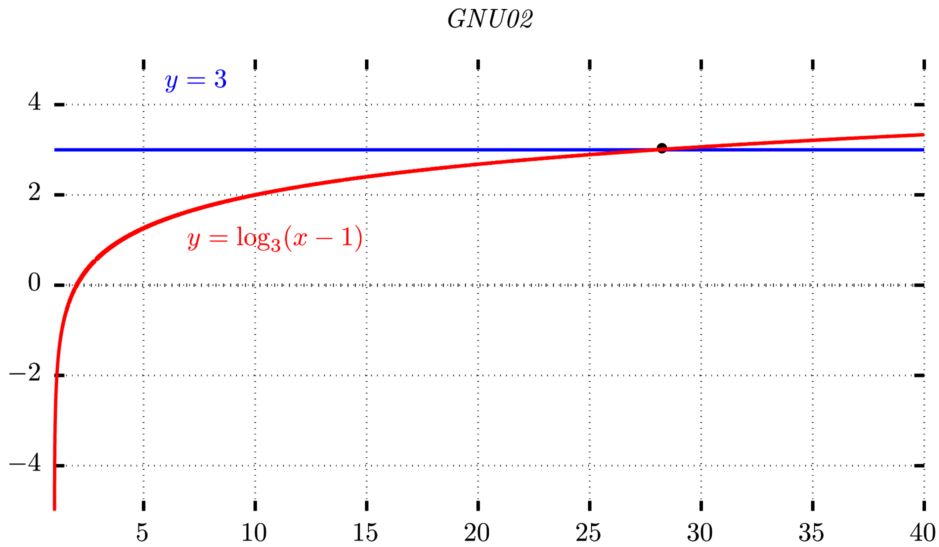


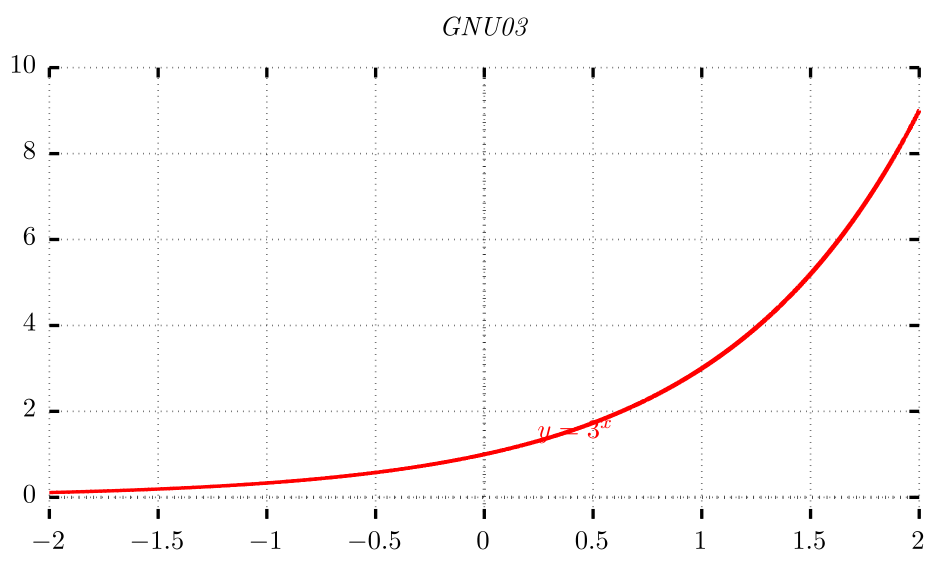
We draw the graphs of the functions

$$\begin{array}{rccc} F: & (1,+\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_3(x-1) \end{array} \quad \textit{and} \quad \begin{array}{rccc} G: & (1,+\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 3 \end{array}$$



It is easy to see that there is one intersection. To solve the equation, and to determine this intersection, we find convenient to apply to both sides the function

$$3^{(\cdot)} : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \mapsto & 3^x \end{array}$$



the inverse of $\log_3(\cdot)$*. Since the domain of* $3^{(\cdot)}$ *is* $\mathbb{R}$ *there is no problem with application. Since the function* $3^{(\cdot)}$ *is injective, the equation's solutions don't* *change and we have that*

$$\log_3(x-1) = 3 \Longleftrightarrow 3^{\log_3(x-1)} = 3^3$$

and, since $3^{(\cdot)}$ *and* $\log_3(\cdot)$ *are inverses,*

$$3^{\log_3(x-1)} = 3^3 \Longleftrightarrow x-1 = 3^3 \Longleftrightarrow x = 28$$

Hence, the solution of $\log_3(x-1) = 3$ *is* $x = 28$*.*

Exercise 2. *Solve the equation* $\mid x \mid = x$. *We draw the graphs of*

$$\begin{array}{rccc} \mid \cdot \mid : & \mathbb{R} & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & \mid x \mid \end{array}$$

and

$$\begin{array}{rccc} id_{\mathbb{R}} : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x \end{array}$$

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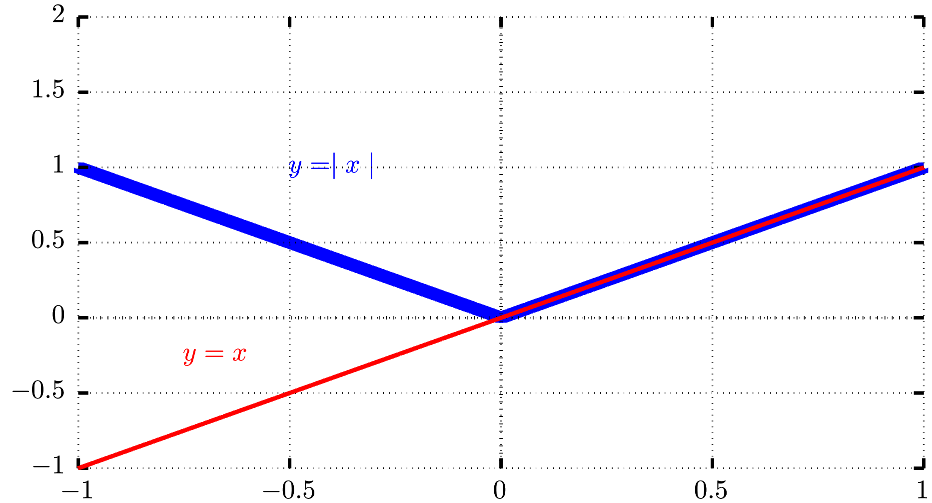

From the graphs it is immediate that x's such that $\mid x \mid = x$ *are* $x \in [0, +\infty)$.

Exercise 3. *Solve the equation* $\sin x = \cos(x)$*, if* $x \in [0, 2\pi)$*.*

We draw the graph of

$$\begin{array}{rccc} \sin(\cdot): & [0,2\pi) & \longrightarrow & [-1,1] \\ & x & \mapsto & \sin(x) \end{array}$$

and

$$\begin{array}{rccc} \cos(\cdot): & [0,2\pi) & \longrightarrow & [-1,1] \\ & x & \mapsto & \cos(x) \end{array}$$

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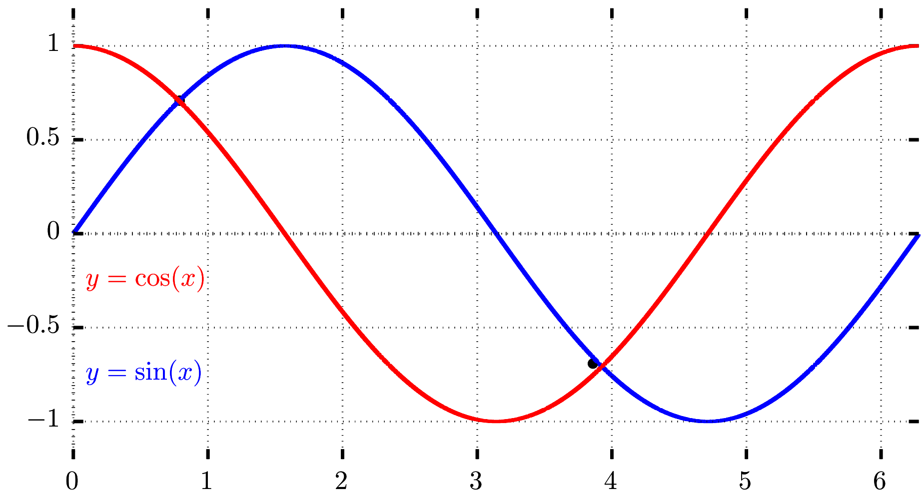
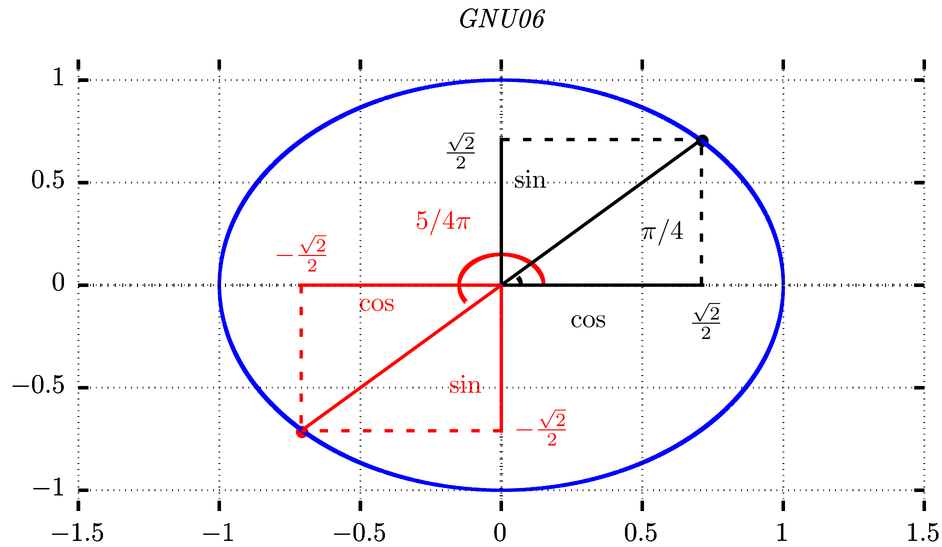


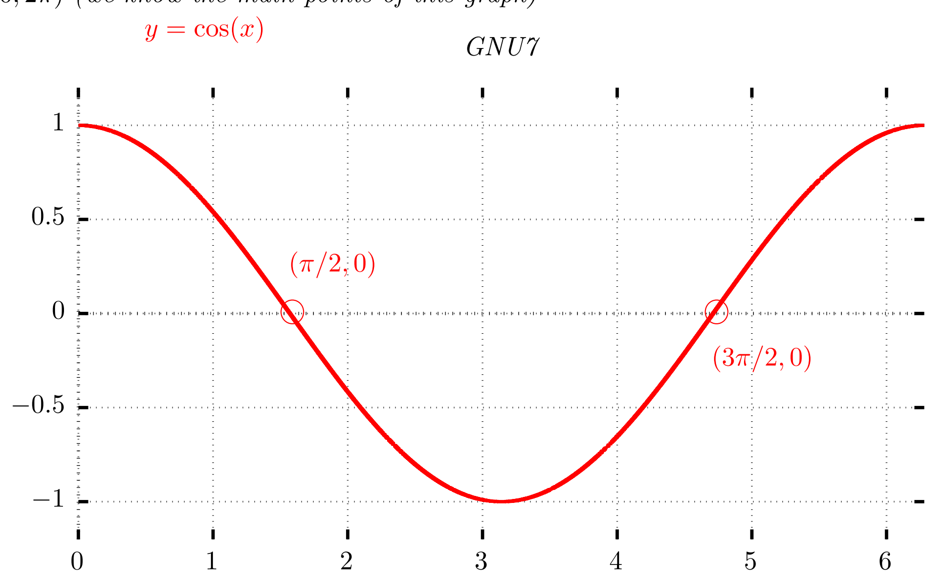
We see that there are two intersections.

Using the goniometric circle we get that the only two angles x *for which* $\sin(x) = \cos(x)$ *are* $x = \frac{\pi}{4}$ *and* $x = \frac{5\pi}{4}$*.*



There is also the standard algebraic method, e.g. dividing by $\cos(x)$ *when possibile.*

We have $\sin(x) = \cos(x)$. *Let us draw the graph of* $\cos(x)$ *to get its zeroes in* $[0, 2\pi)$ *(we know the main points of this graph)*



- *If* $\cos(x) = 0 \iff x = \pi/2, 3\pi/2$, *then the equation is*

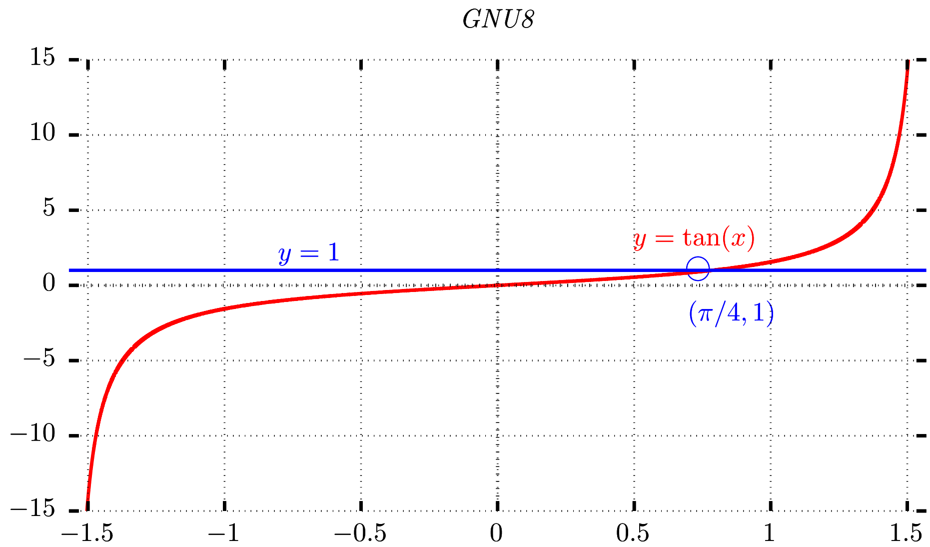
$$\sin(x) = 0 \text{ with solutions} x = 0, \pi$$

Since $\{x = \pi/2, 3\pi/2\} \cap \{0, \pi\} = \emptyset$, *there are no solutions if* $\cos(x) = 0$.

- *If* $\cos(x) \neq 0 \iff x \neq \pi/2, 3\pi/2$ *we can divide both sides of the equation by* $\cos(x)$

$$\sin(x) = \cos(x) \iff \frac{\sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)} \iff \tan(x) = 1$$

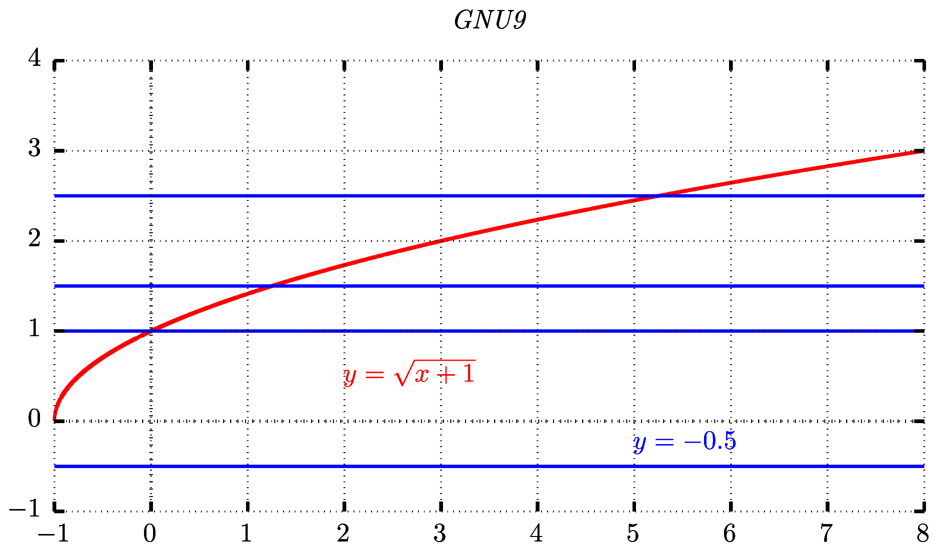
looking at the graph



we see that there is one solution for $x = \pi/4$ *(we know the main intersting points of the graph of the tangent). Since the tangent is periodc with PERIOD* π, *we know that there is another solution for* $x = \pi/4 + \pi = 5\pi/4$

Exercise 4. *Let us consider the function.*

$$F: \begin{array}{ccc} [-1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x+1} \end{array} \quad \textit{whose graph is}$$

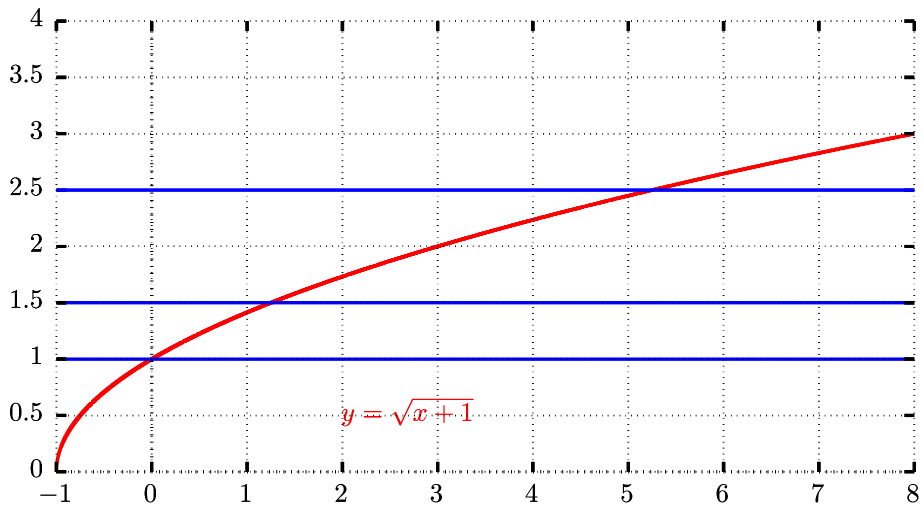


It is clearly injective by the horizontal line rule. It is not invertible because the line $y = -0.5$ does not intersects its graph.

If we restrict the codomain and we consider the function

$$F': \begin{array}{ccc} [-1,+\infty) & \longrightarrow & [0,+\infty) \\ x & \mapsto & \sqrt{x+1} \end{array} \quad \textit{whose graph is}$$

GNU10



The function is clearly invertible by the horizontal line rule. We thus know that there is a function

$$\begin{array}{cccc} G: & [0,+\infty) & \longrightarrow & [-1,+\infty) \\ & x & \mapsto & G(x) \end{array}$$

that is the inverse of F. (Note that the domain of G is the codomain of F and the codomain of G is the domain of F). We want the formula of G.

We remember the algebraic definition of invertibility:

a function $F : A \longrightarrow B$ is invertible if and only if the equation $F(x) = b$ has exactly one solution for the unknown $x \in A$ for any parameter $b \in B$.

We are looking to solve the equation

$$F(x) = b \Longleftrightarrow \sqrt{x+1} = b$$

for the unknown $x \in [-1,+\infty)$ for any parameter $b \in [0,+\infty)$.

We would like to apply to both sides of the equation the function

$$\begin{array}{cccc} (\cdot)^2 : & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & x^2 \end{array}$$

the inverse of the function

$$\begin{array}{cccc} \sqrt{\cdot} : & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & \sqrt{x} \end{array}$$

we can apply $(\cdot)^2$ to both sides of the equation because both sides belong to the domain of $(\cdot)^2$ (both $\sqrt{x+1}$ and b are positive). We thus obtain the equivalent equation

$$(\sqrt{x+1})^2 = b^2 \Longleftrightarrow x + 1 = b^2 \Longleftrightarrow x = b^2 - 1$$

The first equivalence holds because the two functions $(\cdot)^2$ *and* $\sqrt{\cdot}$ *are inverses [Remark: they are inverses when condidered with their specific domains and codomains]. That means that the solution for* x *is* b^2-1 *and so* b *goes to* b^2-1 *and the function we are looking for, the inverse of* F*, is*

$$G: \begin{array}{ccc} [0,+\infty) & \longrightarrow & [-1,+\infty) \\ b & \mapsto & b^2-1 \end{array}$$

or if we prefer

$$G: \begin{array}{ccc} [0,+\infty) & \longrightarrow & [-1,+\infty) \\ x & \mapsto & x^2-1 \end{array}$$

We can check

$$G \circ F(x) = G(F(x)) = G(\sqrt{x+1}) = (\sqrt{x+1})^2 - 1 = x$$

and

$$F \circ G(x) = F(G(x)) = G(x^2-1) = \sqrt{(x^2-1)+1} = \sqrt{x^2} = x \text{ since } x \geq 0$$

Exercise 5. *Is the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2-4 \end{array}$$

invertible? If the answer is no, determine a restriction of the domain and/or codomain that produces an invertible function with the same formula.

Let us draw the graph of F

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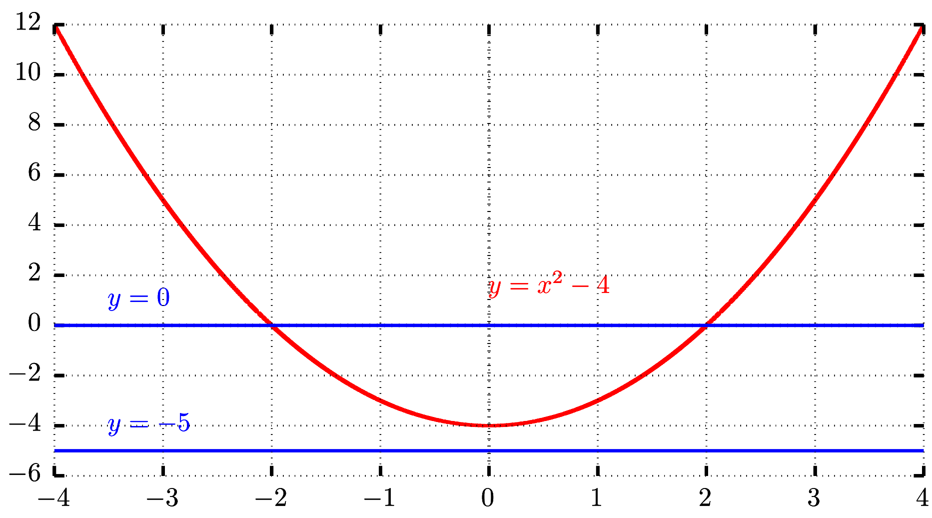


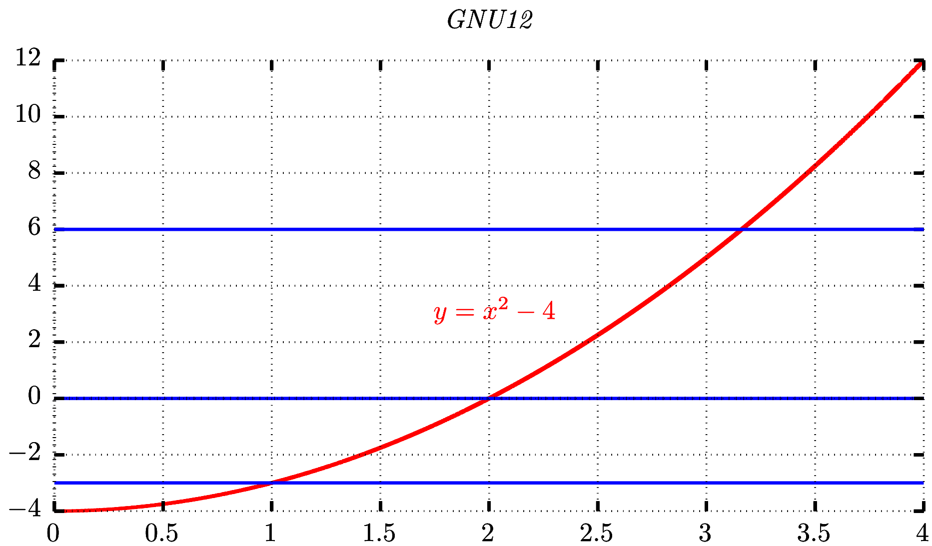
We see that F *is not invertible using the horizontal line rule, since there are lines, like the line* $y=-5$ *that don't intersect the graph of* F*, while there are other lines, like the line* $y=0$*, that itersect it twice.*

We try to restrict the codomain to avoid the first problem, and the domain to avoid the second.

The function

$$F: \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & [-4,+\infty) \\ x & \mapsto & x^2-4 \end{array}$$

is invertible by the horizontal line rule



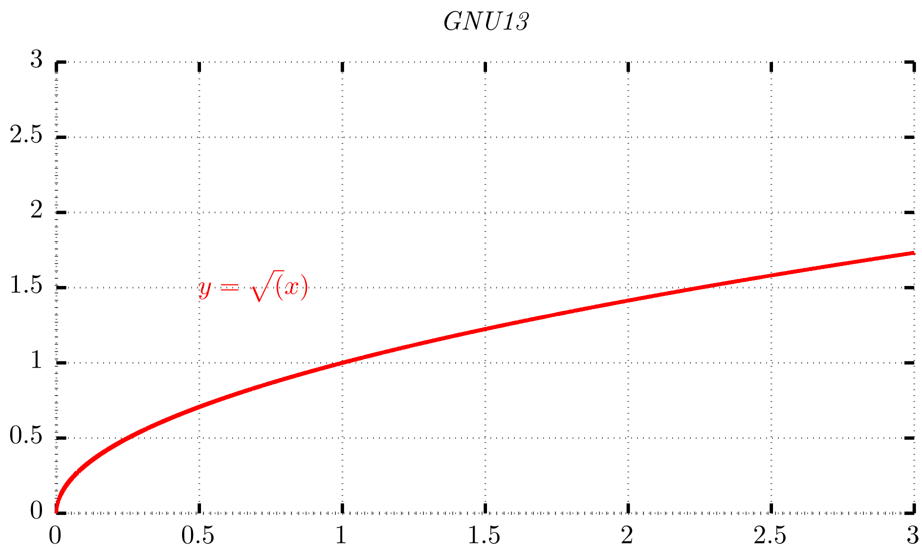
We want to determine the explicit formula for the inverse of F, $F^{-1} : [-4,+\infty) \longrightarrow \mathbb{R}_0^+$. *Notice again that the domain/codomain of* F^{-1} *are the codomain/domain of* F.

As we did in the exercise aboce, we solve the equation

$$F(x) = b \Longleftrightarrow x^2 - 4 = b \Longleftrightarrow x^2 = b + 4$$

for unnknown $x \in \mathbb{R}_0^+$ *and parameter* $b \in [-4,+\infty)$. *Since* $b \in [-4,+\infty)$, *we have always* $b+4 \geq 0$, *and we can apply to both sides of the equation the function*

$$\sqrt{\cdot}: \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & \sqrt{x} \end{array}$$



inverse of

$$(\cdot)^2 : \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & x^2 \end{array}$$

we get

$$x^2 = b + 4 \iff \sqrt{x^2} = \sqrt{b+4} \iff x = \sqrt{b+4}$$

and the second implication holds because $(\cdot)^2$, $\sqrt{\cdot}$ *are inverses. The inverse is then*

$$F^{-1} : \begin{array}{ccc} [-4, +\infty) & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & \sqrt{x+4} \end{array}$$

As an exercise, restrict the domain to the negatives and find the explicit formula for the inverse.

Exercise 6. *Solve the equation*

$$\sqrt{x-1} = x-2$$

First, we determine the existence fields: for the left side it is $[1,+\infty)$*, for the second side* $\mathbb{R}$*. The existence field for the equation in hence* $[1,+\infty)$*, the intersection.*

the we draw the graphs of

$$F: \begin{array}{ccc} [1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x-1} \end{array} \quad \text{and} \quad G: \begin{array}{ccc} [1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \mapsto & x-2 \end{array}$$

that we get easily from the graphs of $\sqrt{x}$*, shifted right by* 1*, and of* $y = x$*, shifted right by* 2*.*

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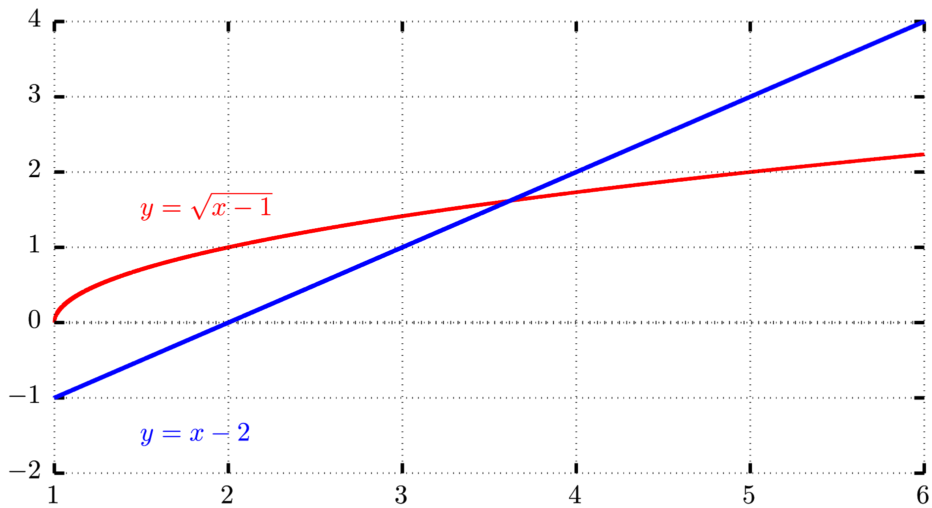


there is an intersection between 3 *and* 4*. The determine it, we have to solve the equation*

$$\sqrt{x-1} = x-2$$

we want to apply the function

$$(\cdot)^2: \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & x^2 \end{array}$$

inverse of

$$\sqrt{\cdot}: \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & \sqrt{x} \end{array}$$

To do that we have to be sure that, for all $x \in [1,+\infty)$*, both sides of the equations are positive. The first side is always positive, being a square root. The second is positive only if* $x \geq 2$*. So we have to consider both cases and to join the solutions.*

- *If* $1 \leq x < 2$, *the first side is positive and the second is striclty negative. The equality never holds.*
- *If* $x \geq 2$, *we can apply* $(\cdot)^2$ *to both sides,*

  $$\sqrt{x-1} = x-2 \Longleftrightarrow (\sqrt{x-1})^2 = (x-2)^2 \Longleftrightarrow x-1 = x^2-4x+4 \Longleftrightarrow x^2-5x+5 = 0$$

  where the first implication holds because $(\cdot)^2$ *is injective, and the second because* $\sqrt{\cdot}$, $(\cdot)^2$ *are inverses. We have*

  $$x^2 - 5x + 6 = 0 \Longleftrightarrow x = \frac{5 \pm \sqrt{25-20}}{2} = x = \frac{5 \pm \sqrt{5}}{2}$$

  but

  $$\frac{5+\sqrt{5}}{2} > 2 \textit{ and } \frac{5-\sqrt{5}}{2} < 2$$

  hence only the first solution is acceptable. The solution of $\sqrt{x-1} = x-2$ *is thus* $x = \frac{5+\sqrt{5}}{2}$.

Exercise 7. *Is it possibile to find a one-to-one correspondence between the sets* $A = \{n \in \mathbb{N} \mid n \text{ is multiple of } 3\}$ *and* $B = \{n \in \mathbb{N} \mid n \text{ is multiple of } 4\}$*?*

We consider the function

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ n & \mapsto & \frac{4}{3}n \end{array}$$

that is well defined because $n \in A$ *and hence* 3 *is a multiple of* n*, and* $n/3 \in \mathbb{N}$*.*

The function

$$G: \begin{array}{ccc} B & \longrightarrow & A \\ n & \mapsto & \frac{3}{4}n \end{array}$$

is well defined because $n \in B$ *and hence* 4 *is a multiple od* n*, and* $n/4 \in \mathbb{N}$*.*

The function G *is clearly the inverse of* F*, since*

$$G \circ F(n) = G\left(\frac{4}{3}n\right) = \frac{3}{4}\left(\frac{4}{3}n\right) = n \text{ and } F \circ G(n) = F\left(\frac{3}{4}n\right) = \frac{4}{3}\left(\frac{3}{4}n\right) = n$$

and so F *is invertible, a one-to-one correspondence and* $\mid A \mid = \mid B \mid$*.*

Exercise 8. *Is it possibile to find a one-to-one correspondence between the sets $A = \{2n+2 \mid n \in \mathbb{N}\}$ and $B = \{n^2 \mid n \in \mathbb{N}\}$.*

We have $\mid A \mid = \mid \mathbb{N} \mid$ and $\mid B \mid = \mid \mathbb{N} \mid$, so our guess is that $\mid A \mid = \mid B \mid$ and thus there is a one-to-one correspondence between A and B, but we are not sure that the rules for equality apply to the cardinality, so we need to find an explicit invertible function between A and B. This could be difficult, we know the one-to-one correspondences (invertible functions)

$$F: \begin{array}{ccc} \mathbb{N} & \longrightarrow & A \\ n & \mapsto & 2n+2 \end{array} \quad \text{and} \quad G: \begin{array}{ccc} \mathbb{N} & \longrightarrow & B \\ n & \mapsto & n^2 \end{array}$$

we find the inverses solving the equations

$$2n+2 = a \text{ and } n^2 = b$$

for $n \in \mathbb{N}$, $a \in$ and $b \in B$. We get

$$n = \frac{a-2}{2} \text{ and } n = \sqrt{b}$$

The inverses are

$$F^{-1}: \begin{array}{ccc} A & \longrightarrow & \mathbb{N} \\ n & \mapsto & \frac{n-2}{2} \end{array} \quad \text{and} \quad G^{-1}: \begin{array}{ccc} B & \longrightarrow & \mathbb{N} \\ n & \mapsto & \sqrt{n} \end{array}$$

well defined because in the first case, since $n \in A$ we have $\frac{n-2}{2} \in \mathbb{N}$ and in the second case, since $b \in B$, b is a perfect square and $\sqrt{b} \in \mathbb{N}$.

So we need an invertible function

$$A \xrightarrow{H} B$$

while we have

$$\mathbb{N} \underset{F^{-1}}{\overset{F}{\rightleftarrows}} A \quad \text{and} \quad \mathbb{N} \underset{G^{-1}}{\overset{G}{\rightleftarrows}} B$$

The idea is to build the function $A \xrightarrow{H} B$ using the functions we have

$$A \xrightarrow{F^{-1}} \mathbb{N} \xrightarrow{G} B$$

so $H \equiv F^{-1} \circ G$ and so for any $n \in A$

$$F^{-1} \circ G(n) = F^{-1}(G(n)) = F^{-1}(n^2) = \frac{n^2-2}{2}$$

and

$$H: \begin{array}{ccc} A & \longrightarrow & B \\ n & \mapsto & \frac{n^2-2}{2} \end{array}$$

is invertible because composition if invertible functions. If we want its explicit inverse,

$$H^{-1}: \begin{array}{ccc} A & \longrightarrow & B \\ n & \mapsto & G^{-1} \circ F(n) = \sqrt{2n+2} \end{array}$$

since

$$H \equiv F^{-1} \circ G \Longrightarrow H^{-1} \equiv (F^{-1} \circ G)^{-1} \equiv G^{-1} \circ F$$

Exercise 9. *Do the sets $\mathbb{N}, \mathbb{Z}$ have the same cardinality?*

The question is, by definition, equivalent to : there is an invertible function (one-to-one correspondence) between A and B? We build one such function.

If we rearrange the elements of $\mathbb{N}$ setting the even numbers before 0 and the odd numbers after 0 like that $\mathbb{N} = \{\cdots, 6, 4, 2, 0, 1, 3, 5, \cdots, \}$, it is natural to define a function like

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ 6 & \mapsto & -3 \\ 4 & \mapsto & -2 \\ 2 & \mapsto & -1 \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ 3 & \mapsto & 2 \\ 5 & \mapsto & 3 \\ \vdots & \vdots & \vdots \end{array}$$

If we write down the formula for this function we have

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & \begin{cases} -\frac{n}{2} & \textit{if } n \textit{ is even} \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd} \end{cases} \end{array}$$

that is well defined and invertible. It is well defined because if n is even, $-\frac{n}{2} \in \mathbb{Z}$ and if n is odd, $\frac{n+1}{2} \in \mathbb{Z}$, so in any case n goes to an integer number. It is invertibe because the function

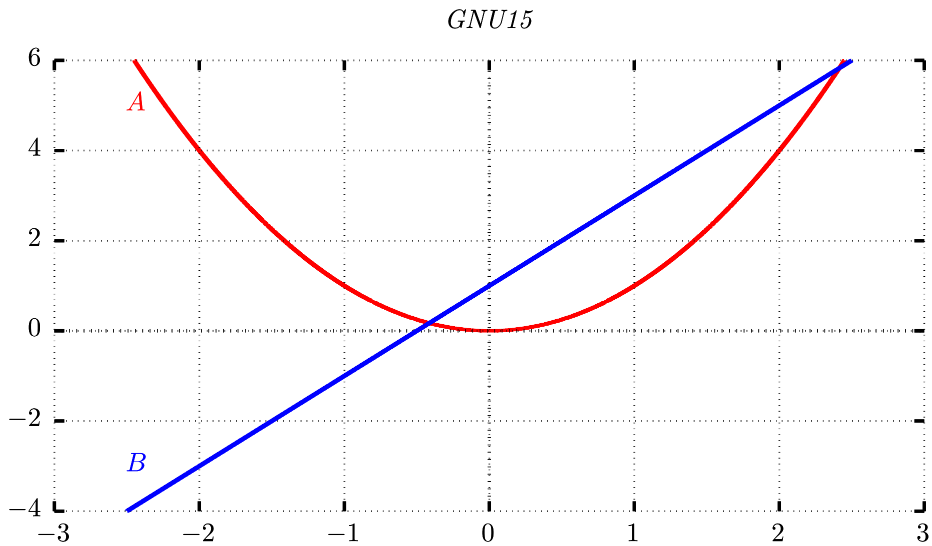
$$\begin{array}{rccc} G: & \mathbb{Z} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \begin{cases} -2n & \textit{if } n \geq 0 \\ 2n-1 & \textit{if } n < 0 \end{cases} \end{array}$$

is its inverse, as we can check. For all $n \in \mathbb{N}$

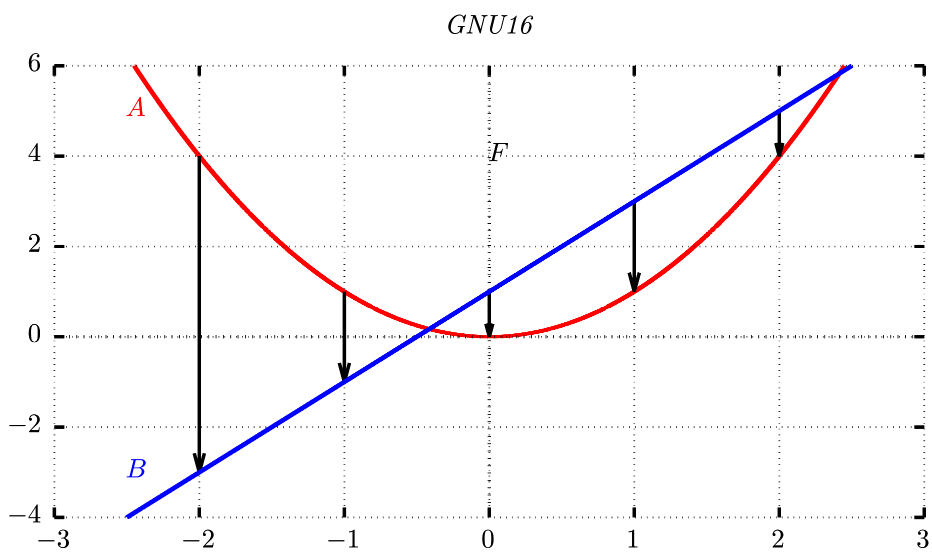
$$G \circ F(n) = G\left(\begin{cases} -\frac{n}{2} & \textit{if } n \textit{ is even} \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd} \end{cases}\right) = \begin{cases} -2(-\frac{n}{2}) & \textit{if } n \geq 0 \\ 2(\frac{n+1}{2}) - 1 & \textit{if } n < 0 \end{cases} = \begin{cases} n & \textit{if } n \geq 0 \\ n & \textit{if } n < 0 \end{cases} = n$$

The check $\forall\ n \in \mathbb{Z}\ F \circ G(n) = n$ is left as an exercise to the reader.

Exercise 10. *Do the subsets A, B in $\mathbb{R}^2$ have the same cardinality?*



Yes, because the function $F : A \longrightarrow B$ detailed below (the projection of A ounto B) is a one-to-one correspondence (is invertible).



Exercise 11. *Do the subsets* A, B *in* $\mathbb{R}^2$ *have the same cardinality?*

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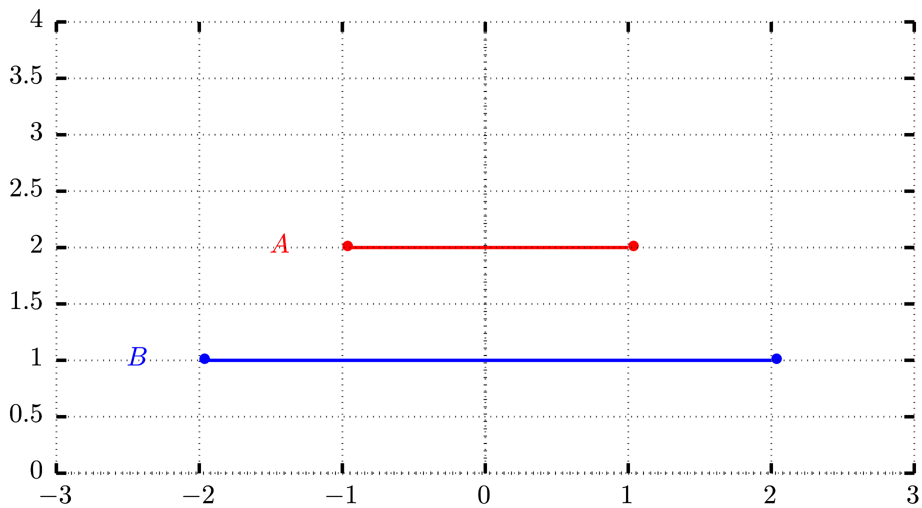

Yes, because the function $F : A \longrightarrow B$ *detailed below is a one-to-one correspondence (is invertible).*

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