FCS
Math: Functions
Exercises

Massimo Caboara

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Exercise 1. *Solve the following equations*

1. $\sqrt{2x-1} = \frac{1}{2}x$. *[*$x = \frac{4\pm2\sqrt{3}}{2}$*]*
2. $\sqrt{x^2+1} = x+1$. *[*$x = 0$*]*
3. $\sqrt{x^4-x} = x^2$. *[*$x = 0$*]*
4. $2^{x+1} + 4^x = 8$. *[*$x = 1$*]*
5. $3^{\sqrt{2x}} = 1$. *[*$\nexists$ *solution]*

Exercise 2. *Are the following functions invertible? If not, restrict the domain/codomain to build an invertible function. Give the explicit formula for the inverse.*

1. $\begin{array}{cccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 3^{x+2} \end{array}$ *Not invertible. Restriction and inverse*

$$\begin{array}{cccc} F': & \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ & x & \mapsto & 3^{x+2} \end{array} \qquad \begin{array}{cccc} F'^{-1}: & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_3(x) - 2 \end{array}$$

2. $\begin{array}{cccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 3^{2x-1} \end{array}$ *Not invertible. Restriction and inverse*

$$\begin{array}{cccc} F': & \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ & x & \mapsto & 3^{x+2} \end{array} \qquad \begin{array}{cccc} F'^{-1}: & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{\log_3(x)-1}{2} \end{array}$$

3. $\begin{array}{cccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_5(\mid x \mid) \end{array}$ *This is not really a function, but a quasi function. The associated function is*

$$\begin{array}{cccc} F': & \mathbb{R} - \{0\} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_5(\mid x \mid) \end{array}$$

This is not invertible. Restriction and inverse:

$$F' : \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & \log_5(x) \end{array} \qquad F'^{-1} : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \mapsto & 5^x \end{array}$$

4. $F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan(x) \end{array}$ *This is not invertible. Restriction and inverse:*

$$F' : \begin{array}{ccc} \mathbb{R} & \longrightarrow & (-\pi/2, \pi/2) \\ x & \mapsto & \arctan(x) \end{array} \qquad F'^{-1} : \begin{array}{ccc} (-\pi/2, \pi/2) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \tan(x) \end{array}$$

5. $F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan(x^2) \end{array}$ *This is not invertible. Restriction and inverse:*

$$F' : \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & [0, \pi/2) \\ x & \mapsto & \arctan(x^2) \end{array} \qquad F'^{-1} : \begin{array}{ccc} [0, \pi/2) & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & \sqrt{\tan(x)} \end{array}$$

6. $F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan(x-2)+1 \end{array}$ *This is not invertible. Restriction and inverse:*

$$F' : \begin{array}{ccc} \mathbb{R} & \longrightarrow & (\frac{2-\pi}{2}, \frac{2+\pi}{2}) \\ x & \mapsto & \arctan(x-2)+1 \end{array} \qquad F'^{-1} : \begin{array}{ccc} (\frac{2-\pi}{2}, \frac{2+\pi}{2}) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \tan(x-1)+2 \end{array}$$

7. $F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan(3^x) \end{array}$ *This is not invertible. Restriction and inverse:*

$$F' : \begin{array}{ccc} \mathbb{R} & \longrightarrow & (0, \pi/2) \\ x & \mapsto & \arctan(3^x) \end{array} \qquad F'^{-1} : \begin{array}{ccc} (0, \pi/2) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \log_3(\tan(x)) \end{array}$$

8. $F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan(\sqrt{x}) \end{array}$

This is not really a function, but a quasi function. The associated function is

$$F' : \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan(\sqrt{x}) \end{array}$$

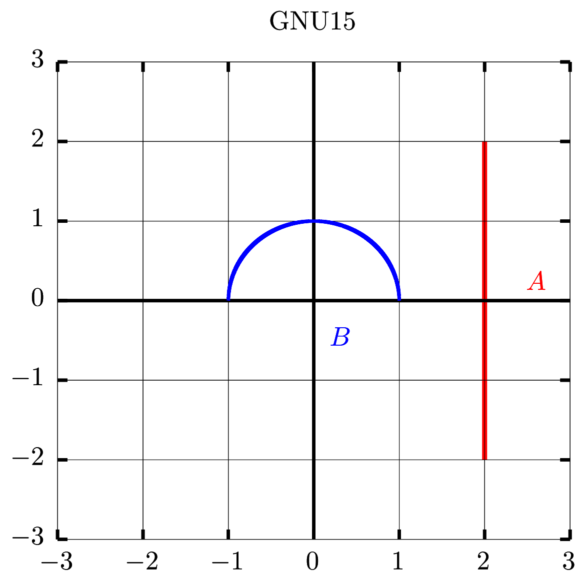
This is not invertible. Restriction and inverse:

$$F' : \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & [0, \pi/2) \\ x & \mapsto & \arctan(\sqrt{x}) \end{array} \qquad F'^{-1} : \begin{array}{ccc} [0, \pi/2) & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & \tan(x)^2 \end{array}$$

Exercise 3. *Is there an invertible function between the following sets? If the answer is affermative, give the explicit function, by formula or an explict description.*

1. $A = \{5n - 2 \mid n \in \mathbb{N}\}$, $B = \{k - 4 \mid k \in \mathbb{N}\}$. *[YES]*
2. $A = \{n^2 \mid n \in \mathbb{N}\}$, $B = \{k^3 \mid k \in \mathbb{N}\}$. *[YES]*
3. $A = \{n^2 \mid n \in \mathbb{N}\}$, $B = \{k^3 \mid k \in \mathbb{Z}\}$. *[YES]*
4. $A = \{n^2 \mid n \in \mathbb{Z}\}$, $B = \{k^3 \mid k \in \mathbb{Z}\}$. *[YES]*
5. $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$. *[YES]*

Exercise 4. *Do the subsets A, B in $\mathbb{R}^2$ have the same cardinality? [YES]*



Exercise 5. *Do the subsets* $A = (0,1) \times \{1\}$, $B = \mathbb{R} \times \{2\}$ *in* $\mathbb{R}^2$ *have the same cardinality? [YES]*

Remark that $A = \{(t,1) \in \mathbb{R}^2 \ | \ t \in (0,1)\}$ *and* $B = \{(t,2) \in \mathbb{R}^2 \ | \ t \in \mathbb{R}\}$. *Graphically,*

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