FCS
Math: Functions
Definitions

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Definition 1. *Given the function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

and $C \subset A$, the function

$$\begin{array}{rccc} G: & C & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

is called the restriction *of F to C and is written $F_{|C}$*

Remark 1. *The graph of* id_A *is* $\{(a,a) \mid a \in A\} \subset A \times A$

Definition 2. *The function*

$$\begin{array}{rccc} F: & A & \longrightarrow & A \\ & a & \mapsto & a \end{array}$$

is called the identity of A *and we write* $F \equiv \mathrm{id}_A$

Remark 2. *The graph of* id_A *is* $\{(a,a) \mid a \in A\} \subset A \times A$

Definition 3. *Given a function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

if there is a function

$$\begin{array}{rccc} G: & B & \longrightarrow & A \\ & b & \mapsto & G(b) \end{array}$$

such that

$$F \circ G \equiv \mathrm{id}_B \text{ and } G \circ F \equiv id_A$$

we say that F is invertible, *G* is the inverse of *G and we write $G \equiv F^{-1}$*

Proposition 1. *A function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is invertible if and only if

$$\forall\ b \in B\ \exists! a \in A \text{ suche that } F(a) = b$$

or, rephrased, for every $b \in B$ the equation $F(a) = b$ has exactly one solution in A.

Proposition 2. *A function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

for which the graph $GR(F)$ is known is invertible if and only if for every horizontal line $y = b,\ b \in B$ there is exactly one intersection between the line and $GR(F)$.

Proposition 3. *A function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is invertible if and only if F is a one-to-one (biunivocal) correspondence between the sets A and B.

Definition 4. *We have the functions*

$$f: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & f(a) \end{array} \quad g: \begin{array}{ccc} A & \longrightarrow & B \\ a' & \mapsto & g(a') \end{array}$$

and, for $x \in A$ the equations

$$f(x) = 0 \text{ and } g(x) = 0$$

The equation

$$f(x) = 0$$

is equivalent *to the equation*

$$g(x) = 0$$

if and only if they have the same solutions

$$\{a \in A \ |\ f(a) = 0\} = \{a \in A \ |\ g(a) = 0\}$$

Proposition 4. *We have the functions*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & f(a) \end{array} \quad G: \begin{array}{ccc} A & \longrightarrow & B \\ a' & \mapsto & g(a') \end{array}$$

and, for $x \in A$ the equation

$$F(x) = G(x) \quad (\text{ or } F(x) - G(x) = 0)$$

If the function

$$\begin{array}{rccc} H: & B & \longrightarrow & C \\ & b & \mapsto & F(b) \end{array}$$

is injective, then the equation

$$H \circ F(x) = H \circ G(x)$$

is equivalent to the equation

$$F(x) = G(x)$$

Proposition 5. *Let A, B be sets. We say that A* has the same cardinalty of *B if and only if there is a one-to-one correspondence between A and B. We write $| A |=| B |$.*

Example 1. *The sets $\mathbb{N} = \{0, 1, 2, 3, 4, \dots, n, \dots\}$, $EVEN = \{0, 2, 4, 6, 8, \dots, 2n, \dots\}_{n\in\mathbb{N}}$ have the same cardinality, because the function*

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \text{EVEN} \\ & n & \mapsto & 2n \end{array}$$

is invertible (and hence a one-to-one correspondence), since the function

$$\begin{array}{rccc} G: & \text{EVEN} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & n/2 \end{array}$$

is its inverse, as we can see from

$$\forall\ n \in \mathbb{N}\ G \circ F(n) = G(F(n)) = G(2n) = \frac{1}{2}2n = n \Longleftrightarrow G \circ F \equiv \text{id}_{\mathbb{N}}$$

and

$$\forall\ n \in \text{EVEN}\ F \circ G(n) = F(G(n)) = F(\frac{n}{2}) = 2\frac{n}{2} = n \Longleftrightarrow F \circ G \equiv \text{id}_{\text{EVEN}}$$

Example 2 (The salary paradox)**.** *we have two employees, Joe and Jane. Each get 1000 euros a month.*

- *Joe every month puts one euro on a pile on his desk and spends the others 999. He never draw any money from his pile.*
- *Jane every month puts 1000 euros on top of a pile of money on his desk and draws 500 euro* from the bottom og the pile.

I argue that "at infinity", Joe has an infinite amount of money, and Jane has nothing, because every single euro that she puts on the pile is, at some moment, spent.