FCS
Math: Functions
Exercises

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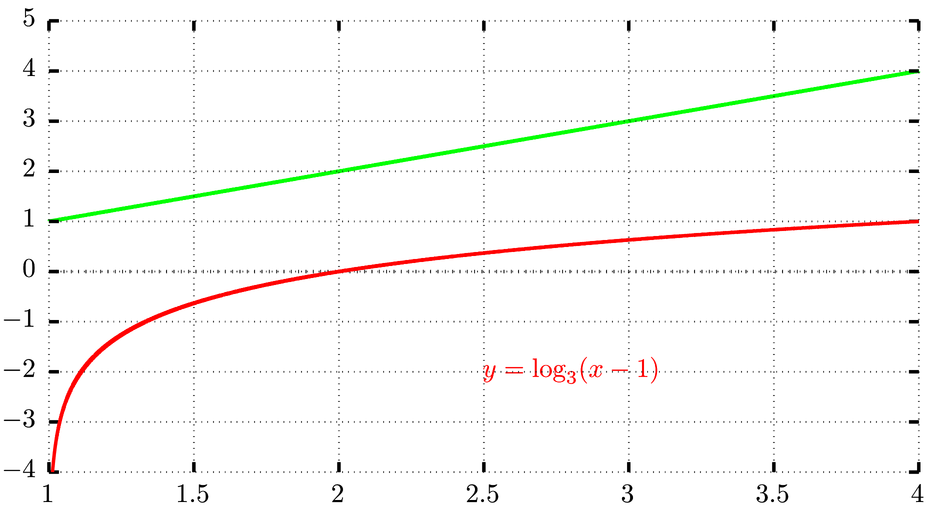
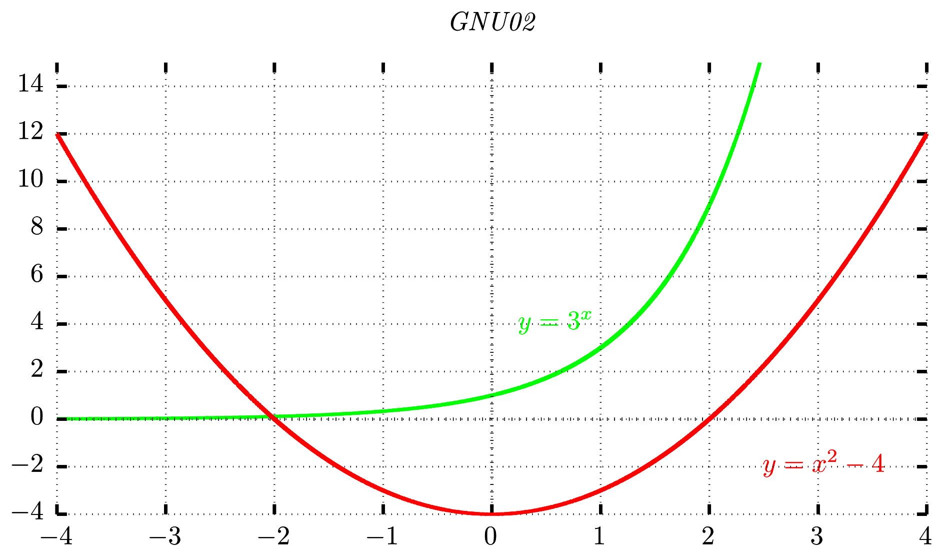
Exercise 1. *Solve the following equations, if possibile. If not, determine as much information you can about the solutions. In every case, determine the x's for which the equation makes sense (the "existence field" for the equation).*

1. $3^x = 3$ *[*$x = 1$*]*
2. $2^x = 2^2$ *[*$x = 2$*]*
3. $5^x = 5^{2x}$ *[*$x = 0$*]*
4. $5^{x^2} = 5^{2x}$ *[*$x = 0, 2$*]*
5. $\log_3(x-1) = 3$ *[*$\mathrm{EF} = (1, +\infty), x = 28$*]*
6. $\log_3(x-1) = x$ *[*$\mathrm{EF} = (1, +\infty)$*, there are no solutions]*

GNU01



7. $3^x = x^2 - 4$ *[One solution]*



8. $\mid x \mid = 3$ *[*$x = \pm 3$*]*

9. $\mid x \mid = x$ *[Solutions* $x \in [0, +\infty)$*]*

10. $\mid x \mid = x^2$ *[*$x = 0, \pm 1$*]*

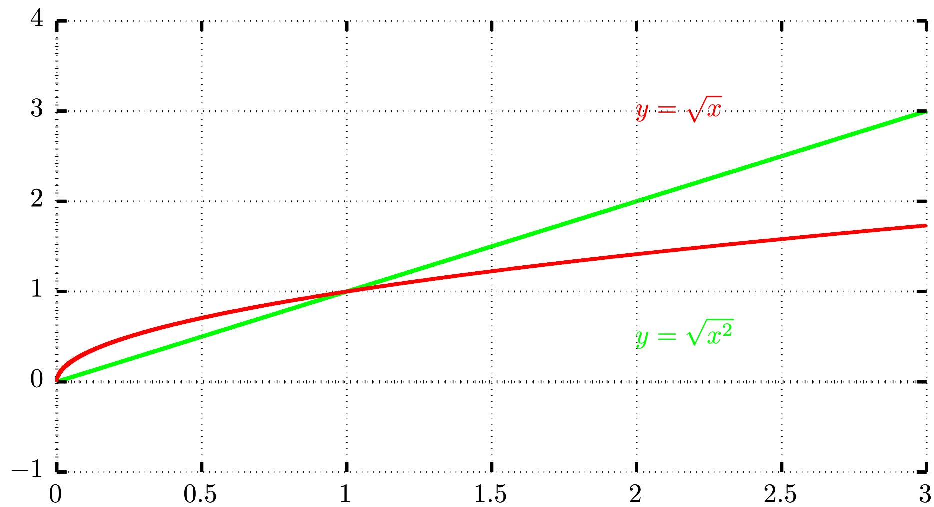
11. $\sin x = \cos(x)$, *if* $x \in [0, 2\pi)$ *[*$x = \frac{\pi}{4}, \frac{5\pi}{4}$*]*

12. $\sin x = \cos(x)$, *if* $x \in \mathbb{R}$ *[*$x = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$*]*

13. $\tan(x^2) = \tan(x)$, *if* $x \in (-\pi/2, \pi/2)$ *[*$x = 0, 1$*]*

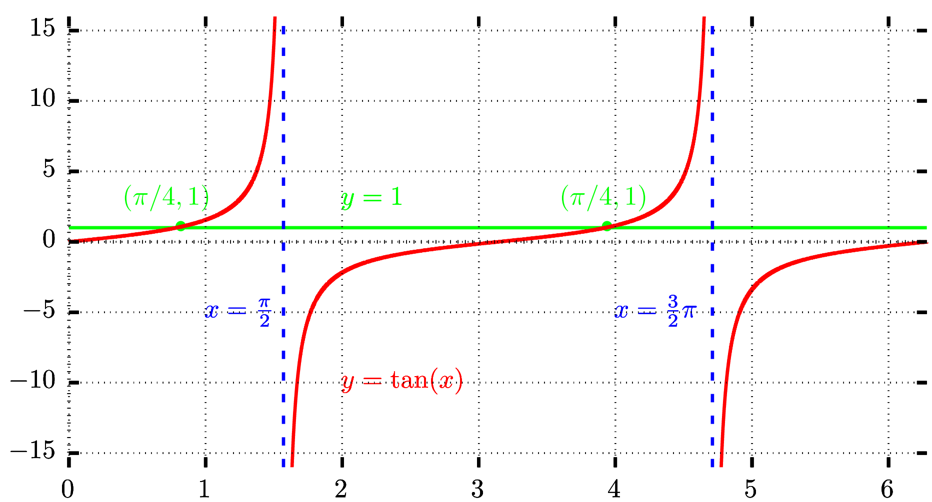
14. $\sqrt{x^2} = \sqrt{x}$ *[*$\mathrm{EF} = [0, +\infty),\ x = 0, 1$*]*

GNU03

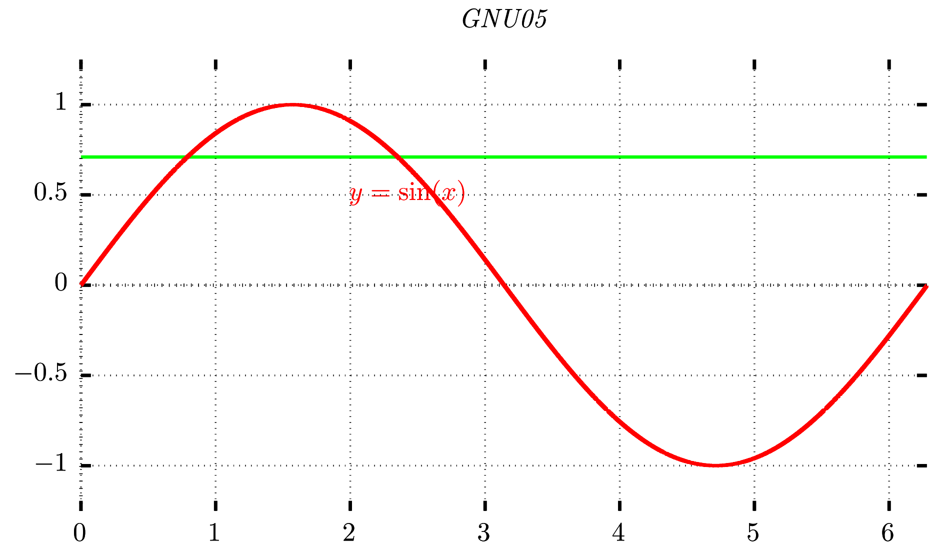


15. $\tan x = 1$*, if* $x \in [0, 2\pi)$ *[*$\mathrm{EF} = [0, 2\pi) - \{\pi/2, 3\pi/2\}$*,* $x = \pi/4, 5\pi/4$*]*

GNU04



16. $\sin x = 5/7$*, if* $x \in [0, 2\pi)$ *[Two solutions]*



17. $\tan x = 3$, *if* $x \in (-\pi/2, \pi/2)$ *[*$x = \arctan(3)$*]*

Exercise 2. *Are to following functions invertible? If the answer is yes, find the inverse if possibile*

1. *if* ODD *is the set of the odd positive numbers* $\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathrm{ODD} \\ & n & \mapsto & 2n+1 \end{array}$

   Function is invertible. Inverse is

   $$\begin{array}{rccc} F^{-1}: & \mathrm{ODD} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \frac{n-1}{2} \end{array}$$

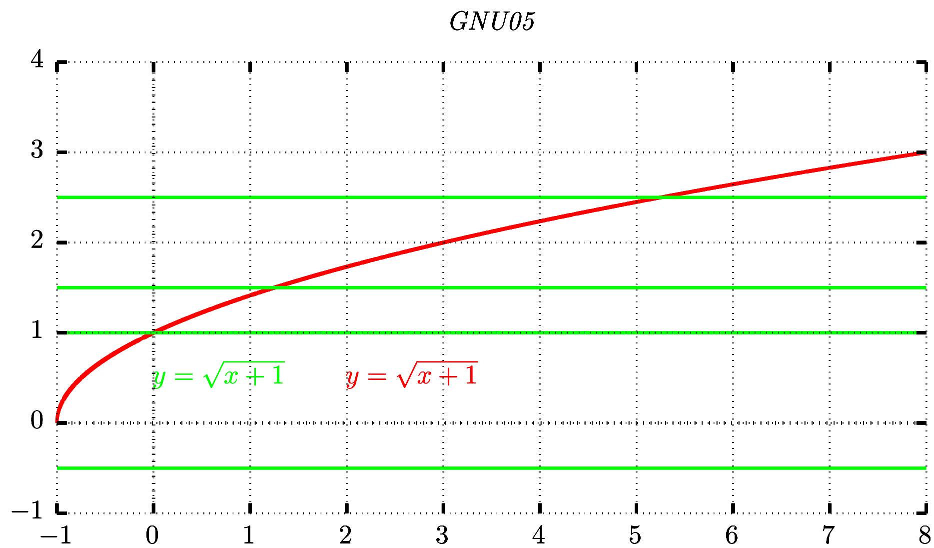
2. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & -3x+4 \end{array}$

   Function is invertible. Inverse is

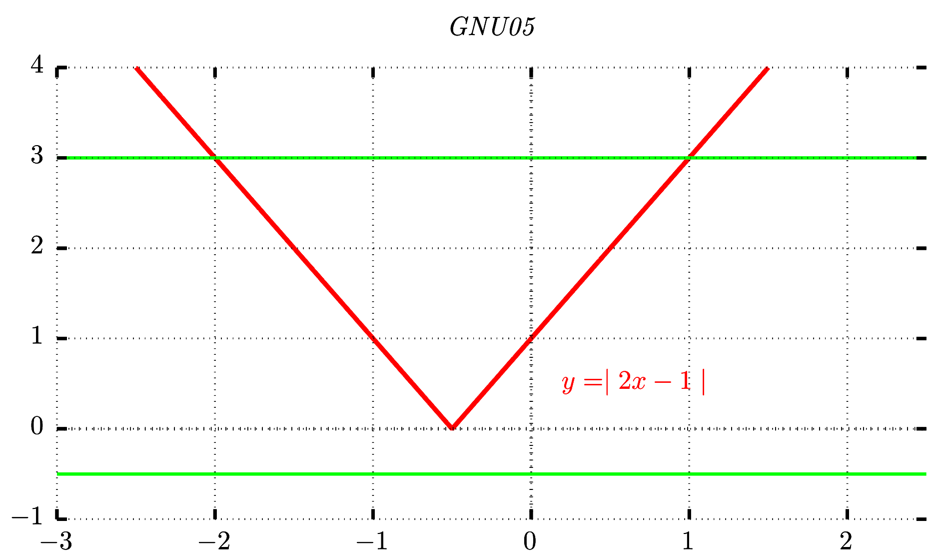
   $$\begin{array}{rccc} F^{-1}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{4-y}{3} \end{array}$$

3. $\begin{array}{rccc} F: & [-1, +\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x+1} \end{array}$

   Function is clearly injective by the horizontal line rule. It is not invertible because the line $y = -0.5$ *does not intersects its graph.*



4. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \mid 2x-1 \mid \end{array}$ *Function is not invertible by the horizontal line rule.*



Exercise 3. *Are to following functions invertible? If the answer is no, determine a restriction of the domain and/or codomain that produces an invertible function with the same formula*

1. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^2-4 \end{array}$

   Not invertible. The function $\begin{array}{rccc} F: & \mathbb{R}_0^+ & \longrightarrow & [-4,+\infty) \\ & x & \mapsto & x^2-4 \end{array}$ *is invertible, as is the function* $\begin{array}{rccc} F: & \mathbb{R}_0^- & \longrightarrow & [-4,+\infty) \\ & x & \mapsto & x^2-4 \end{array}$

2. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^2+x \end{array}$

   Not invertible. The function $\begin{array}{rccc} F: & (-\infty,-1/2] & \longrightarrow & [3/4,+\infty) \\ & x & \mapsto & x^2+x \end{array}$ *is invertible, as is the function* $\begin{array}{rccc} F: & [-1/2,+\infty) & \longrightarrow & [3/4,+\infty) \\ & x & \mapsto & x^2+x \end{array}$

3. $\begin{array}{rccc} F: & [-1,+\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x+1} \end{array}$ *Invertible.*

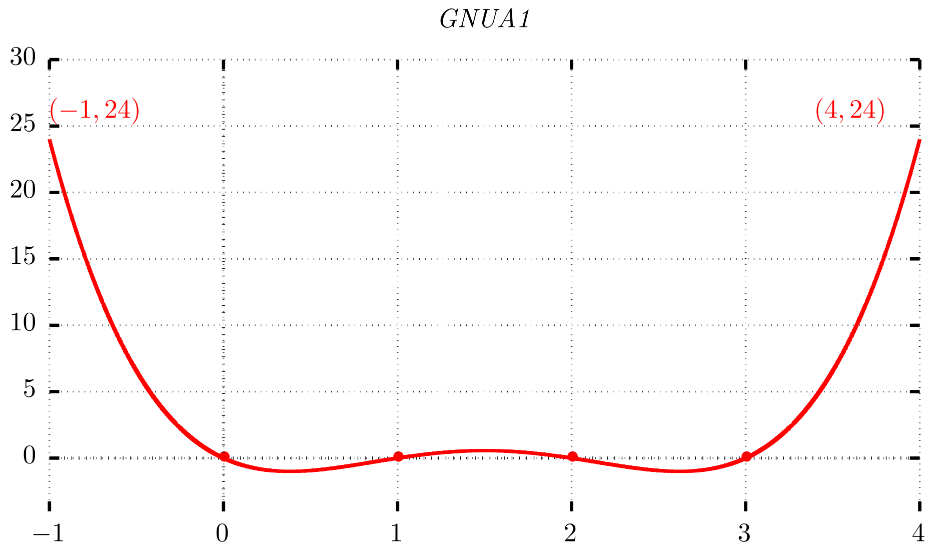
4. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sin(x) \end{array}$ *[Difficult - think about that]*

5. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \cos(x) \end{array}$ *[Difficult - think about that]*

Exercise 4. *Is it possibile to find a one-to-one correspondence between the sets* $A = \{n \in \mathbb{N} \mid n \text{ is multiple of } 3\}$ *and* $B = \{n \in \mathbb{N} \mid n \text{ is multiple of } 4\}$*? [YES]*

Exercise 5. *Is it possibile to find a one-to-one correspondence between the sets* $A = \{2n+2 \mid n \in \mathbb{N}\}$ *and* $B = \{n^2 \mid n \in \mathbb{N}\}$*. [YES]*

Exercise 6. *Is the function* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *whose graph in shown below invertible? If not, find some restrictions of the domanin/codomain that produces and invertible function*



Some restrictions that give invertible functions are

$$F' : \begin{array}{ccc} [3,4] & \longrightarrow & [0,24] \\ x & \mapsto & F(x)) \end{array}, \quad F'' : \begin{array}{ccc} [-1,0] & \longrightarrow & [0,24] \\ x & \mapsto & F(x)) \end{array}$$

Exercise 7. *The sets* $\mathbb{N}, \mathbb{Z}$ *have the same cardinality? [YES]*

Exercise 8. *The sets* $\mathbb{N}, \mathbb{Q}$ *have the same cardinality? [Very difficult, we will speak about that]*

Exercise 9. *The sets* $\mathbb{N}, \mathbb{R}$ *have the same cardinality? [Very difficult, we will speak about that]*

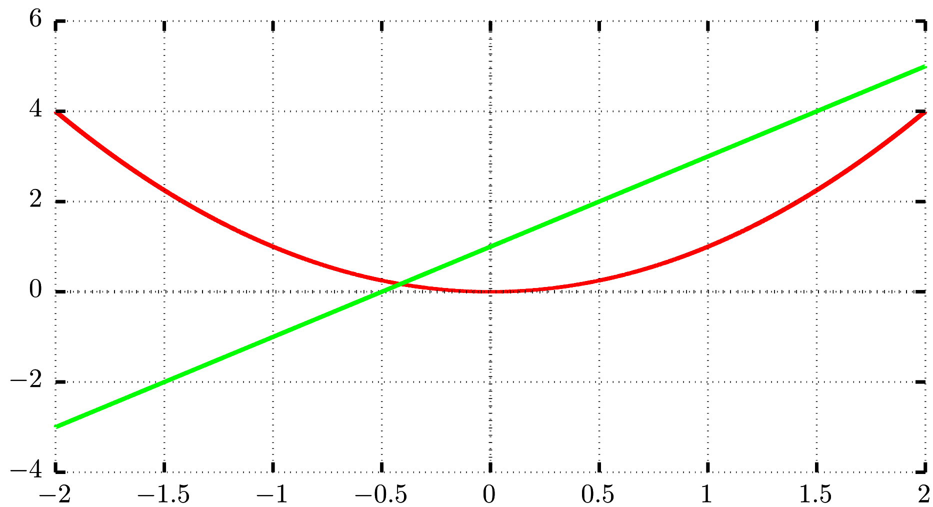
Exercise 10. *The sets* $\mathbb{N}, \mathbb{N}^2$ *have the same cardinality? [Very difficult, we will speak about that]*

Exercise 11. *The sets* $\mathbb{N}, \mathbb{N}^3$ *have the same cardinality? [Very difficult, we will speak about that]*

Exercise 12 (Hilbert hotel)**.** *We have an hotel with infinite rooms, all occupied. If a new customer comes, can we find a free room for him? [YES]*

Exercise 13. *Do the subsets* A*,* B *in* $\mathbb{R}^2$ *have the same cardinality?*

GNUA1



[YES - project one graph onto the other]