FCS
Math: Functions
Definitions

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Definition 1. *Given the sets A, B, the function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

is injective *if and only if*

$$\forall b \in B \quad \text{the equation } F(a) = b \text{ has at most } 1 \text{ solution in } A$$

Corollary 1. *A function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) \end{array}$$

is injective *if and only if every horizontal line $y = a$ intersects the graph of F once or not at all.*

Remark 1. *Consider the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) \end{array}$$

which graph is known, the elements $A, B, C, D \in \mathbb{R}$, of which only one is different from 0 and the function

$$\begin{array}{rccc} F': & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & AF(Bx + C) + D \end{array}$$

1. *The graph of F' is the graph of F shifted up or down by D. Up if $D > 0$, down otherwise.*

2. *The graph of F' is the graph of F shifted left or right by C. Left if $D > 0$, right otherwise.*

3. *The graph of F' is the graph of F amplified by C.*

4. If F is periodic of period p, F' is periodic of period $B \cdot p$.

Definition 2. *Given a function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

the set

$$Im(F) = \{F(a) \ | \ a \in A\} = \{b \in B \ | \ \exists \ a \in A \text{ such that } b = F(a)\} \subseteq B$$

is the image *of F.*

Definition 3. *Given two functions*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}, \qquad G: \begin{array}{ccc} D & \longrightarrow & C \\ d & \mapsto & G(d) \end{array}$$

such that $D \subseteq Im(F)$ the function

$$G \circ F: \begin{array}{ccc} A & \longrightarrow & C \\ a & \mapsto & G(F(a)) \end{array}$$

is the composition *of G, F.*

Proposition 1. *If we have the two functions*

$$F: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & F(a) \end{array}, \qquad G: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & G(a) \end{array}$$

it may happen that $F \circ G \not\equiv G \circ F$. Consider for example

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) = 5x - 2 \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & G(x) = x^2 - 1 \end{array}$$

It is easy to verify that for most $x \in \mathbb{R}$

$$F \circ G(x) = F(G(x)) \neq G(F(x)) = G \circ F(x)$$

and hence $F \circ G(x) \not\equiv G \circ F(x)$

Definition 4. *The function*

$$id_A: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & id_A(a) = a \end{array}$$

is called the identity *of A.*