FCS
Math: Functions
Exercises

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Exercise 1. *Given the sets*

$$A = \{1, 3, 5, 7, 14\} \text{ and } B = \{-2, 3, 4, 8\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution: The first solutions are trivial,

$$\begin{aligned} A \times B \;\; = \;\; & \{(\ (\ 1, -2\ ), (\ 1, 3\ ), (\ 1, 4\ ), (\ 1, 8\ ), (\ 3, -2\ ), (\ 3, 3\ ), (\ 3, 4\ ), (\ 3, 8\ ), (\ 5, -2\ ), (\ 5, 3\ ), (\ 5, 4\ ), \\ & (\ 5, 8\ ), (\ 7, -2\ ), (\ 7, 3\ ), (\ 7, 4\ ), (\ 7, 8\ ), (\ 14, -2\ ), (\ 14, 3\ ), (\ 14, 4\ ), (\ 14, 8\ )\ )\} \end{aligned}$$

Exercise 2. *Given the sets*

$$A = \{a \in \mathbb{N} \ \mid \ a \text{ is a multiple of } 12\} \text{ and } B = \{k \in \mathbb{N} \ \mid \ k \text{ is a multiple of } 15\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution:

$$\begin{aligned} A \cup B &= \{12, 15, 24, 30, 36, 45, 48, 60, 72, 65, \ldots\} \\ A \cap B &= \{k \in \mathbb{N} \ \mid \ k \text{ is a multiple of } LCM(12, 15) = 60\} \\ A - B &= \{12, 24, 36, 48, 72, \ldots\} \\ A \times B &= \{(a, b) \in \mathbb{N}^2 \ \mid \ a \text{ is a multiple of } 12 \text{ and } b \text{ is a multiple of } 15\} \end{aligned}$$

Exercise 3. *Given the sets*

$$A = \{(a, a^2) \ \mid \ a \in \mathbb{R}\} \text{ and } A = \{(b, b) \ \mid \ b \in \mathbb{R}\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution:

The set A *is the parabola* $y = x^2$, *the set* B *the line* $y = x$. *The set* $A \cup B$ *is the union of the two drawing, the set* $A \cap B$ *is just the two points* $\{(0, 0), (1, 1)\}$, *the set* $A - B$ *is the parabola minus the points* $\{(0, 0), (1, 1)\}$ *and*

$$A \times B = \{(a, a^2, b, b) \in \mathbb{R}^4 \ \mid \ a, b \in \mathbb{R}\}$$

Exercise 4. *Draw on the* $\mathbb{R}^2$ *plane the sets*

$$[1,2]\times[1,1],\ [-1,2]\times[2,+\infty],\ \{(1,1),(2,3),(3,7)\}$$

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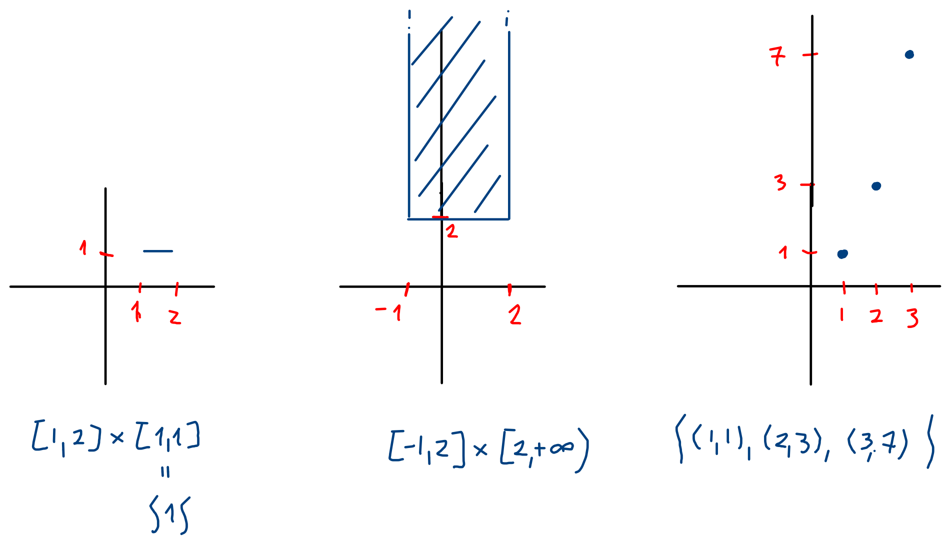

Figure 1

Exercise 5. *Given the sets*

$$A = \{a \in \mathbb{R} \ \mid \ x^3 - 4x^2 + x + 6 = 0 \in \mathbb{R}\}$$
$$B = \{a \in \mathbb{R} \ \mid \ x^4 - 5x^2 + 4 = 0 \in \mathbb{R}\}$$
$$C = \{a \in \mathbb{R} \ \mid \ x^4 - 2x^3 + 4x^2 - 6x + 3 = 0 \in \mathbb{R}\}$$

Detail the equalities and inclusions between A, B, C

Solution:
We have that

$$\begin{aligned} x^3 - 4x^2 + x + 6 &= (x+1)(x-2)(x-3) \\ x^4 - 5x^2 + 4 &= (x+2)(x-2)(x+1)(x-1) \\ x^4 - 2x^3 + 4x^2 - 6x + 3 &= (x^2+3)(x-1)^2 \end{aligned}$$

Hence

$$\begin{array}{rcl} A & = & \{1,2,3\} \\ B & = & \{\pm 1, \pm 2\} \\ C & = & \{1\} \end{array}$$

Hence there are no equalities and the only inclusions are

$$C \subset A, B$$

Exercise 6. *We have the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 3x^2 - x + 1 \end{array}$$

Compute

1. $F(1) = F(-1)$*? [* $F(-1) = 5 \neq 3 = F(1)$ *]*
2. $F(1), F(0), F(5)$. *[*$F(-1) = 5,\ F(0) = 1,\ F(5) = 71$*]*
3. *Given* $a \in \mathbb{R}$*, compute* $F(a-1), F(3a^2-2), F(\sqrt{a^2+1})$.

$$\begin{array}{rcl} F(a-1) & = & 3(a-1)^2 - (a-1) + 1 = 3a^2 - 7a + 5 \\ F(3a^2-2) & = & 3(3a^2-2)^2 - (3a^2-2) + 1 = 27a^4 - 39a^2 + 15 \\ F(\sqrt{a^2+1}) & = & 3(\sqrt{a^2+1})^2 - (\sqrt{a^2+1}) + 1 = 3a^2 - \sqrt{a^2+1} + 4 \end{array}$$

4. *Given* $\clubsuit \in \mathbb{R}$*, with* $\clubsuit > 0$*, compute* $F(\clubsuit - 2)$*,* $F(2\clubsuit)$*,* $F(\sqrt{\clubsuit})$.

   $F(\clubsuit - 2) = 3\clubsuit^2 - 13\clubsuit + 15$. $F(2\clubsuit) = 12\clubsuit^2 - 2\clubsuit + 1$, $F(\sqrt{\clubsuit}) = 3\clubsuit - \sqrt{\clubsuit} + 1$

Exercise 7. *We have the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^2 + 1 \end{array}$$

Compute

1. $F(1) = F(-1)$*? [YES]*
2. *For which* $a \in \mathbb{R}$ *we have* $F(a) = F(-a)$. *[Solutions of* $a^2 + 1 = a^2 + 1$*, so for all* $a \in \mathbb{R}$*]*
3. *For which* $y \in \mathbb{R}$ *we have* $F(y) = F(y+1)$. *[Solutions of* $y^2 + 1 = (y+1)^2$*, so* $y = 0$*]*

4. *For which $b \in \mathbb{R}$ we have $F(b+2) = F(2b+3)$. [Solutions of $(b+2)^2+1 = (2b+3)^2+1$, so $b = -5/3, -1$]*

Exercise 8. *We have the functions*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array}$$

1. *Is it true that $F \equiv G$? [NO, there is an $a \in \mathbb{R}$ such that $F(a) \neq G(a)$, for exmple $a = 2$, for which $F(2) = 4 \neq 2 = G(2)$]*
2. *Is there an $a \in \mathbb{R}$ such that $F(a) = G(a)$? [YES, for example $a = 1$, since $F(1) = 1 = G(1)$]*

Exercise 9. *We try to describe a function by*

$$F: \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ p/q & \mapsto & p^2/q^2 \end{array}$$

Is F a well defined function?

Solution: YES.

Exercise 10. *We try to describe a function by*

$$F: \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ p/q & \mapsto & p+q \end{array}$$

Is F a well defined function? If not, how can we modify the formula of F to have a well defined function?

Solution: NO, because $\frac{6}{3} = \frac{4}{2}$ but $F\left(\frac{6}{3}\right) = 9 \neq 6 = F\left(\frac{4}{2}\right)$

Exercise 11. *In the plane $\mathbb{R}^2$, draw the graphs of the functions*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}, \quad F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ b & \mapsto & b^2-1 \end{array}, \quad F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ c & \mapsto & 3c^2 \end{array}, \quad F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \heartsuit & \mapsto & \heartsuit^2+3 \end{array}$$

GNU1: *Graphs*