FCS
Math: Functions

Massimo Caboara

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Definition 1. *If A, B are sets, then we say that*

$$| A | \leq | B |$$

if there is an injective function $F : A \longrightarrow B$

Remark 1. *Given A, B, C sets*

1. $| A | = | B |$ *if and only if there is an invertible function $F : A \longrightarrow B$.*
2. $| A | \neq | B |$ *if and only if there no invertible function $F : A \longrightarrow B$.*
3. $| A | \leq | B |$ *if and only if there is an injective function $F : A \longrightarrow B$.*
4. $| A | < | B |$ *if and only if there is an injective function $F : A \longrightarrow B$ but there no invertible function $F : A \longrightarrow B$.*
5. $| A | = | B |$ *and* $| B | = | C |$ *then* $| A | = | C |$. *Since the composition of invertible functions is an invertible function.*
6. *If A, B sets $A \subseteq B$ then $| A | \leq | B |$.*
7. *(Cantor-Schröder-Bernstein theorem)*

   $$| A | \leq | B | \text{ and } | B | \leq | A | \Longrightarrow | A | = | B |$$

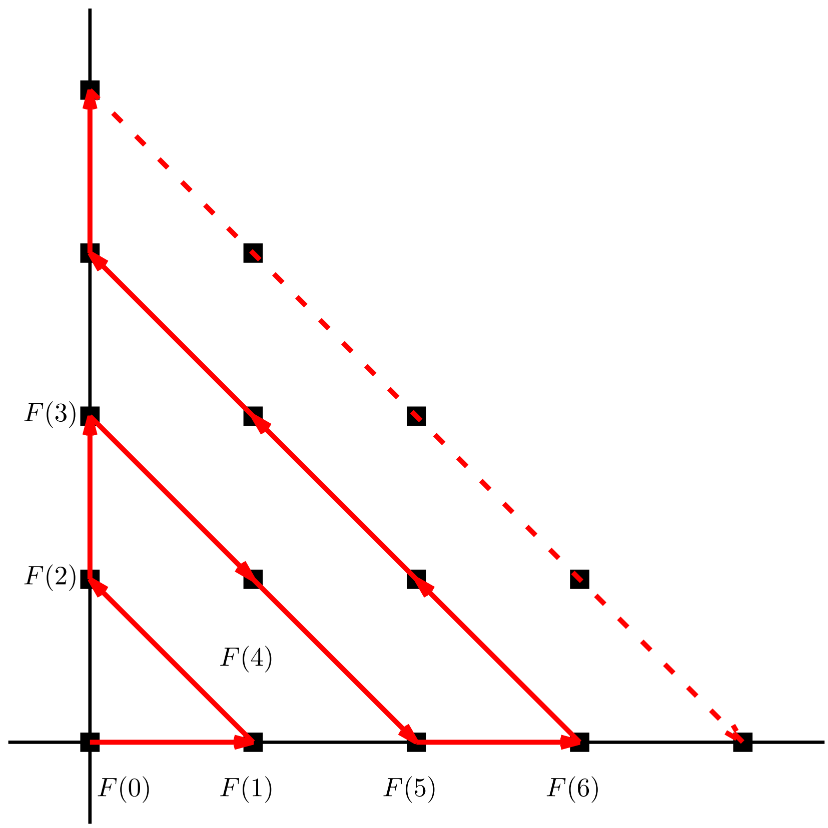
   We can define the sum for infinite cardinalities, but it is weird.
8. *If A, B infinite sets and $| A | = | B |$ then $| A | + | B | = | A | = | B |$.*
9. *If A, B infinite sets and $| A | > | B |$ then $| A | + | B | = | A |$.*

   A consequence of Cantor-Schröder-Bernstein theorem:
10. *If A, B sets $| A \cup B | = | A | + | B |$ and $| A \cap B | \leq | A |, | B |$. [This will be shown in class]*

Exercise 1. *We prove that* $\mid \mathbb{N} \mid = \mid \mathbb{N} \times \mathbb{N} \mid$ *by building a one-to-one correspondence* $F : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$.

We say that $F(0) = (0,0)$, $F(1) = (1,0)$, $F(2) = (0,1)$, $F(3) = (0,2)$, $F(4) = (1,1)$, $F(5) = (2,0)$ *and so on as in the imagine below.*

GNU1 - The one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$



The function F *is a one-to-one correspondence, and so*

$$\mid \mathbb{N} \mid = \mid \mathbb{N} \times \mathbb{N} \mid$$

Exercise 2. *We prove that*

$$\mid \mathbb{N} \mid = \mid \mathbb{Q} \mid$$

by building a one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{Q}$. *The idea is to mimick the one-to-one correspondence between* $\mathbb{N}$ *and* $\mathbb{N} \times \mathbb{N}$.

There is a natural function between $\mathbb{Q}$ *and* $\mathbb{Z} \times \mathbb{N}$,

$$\begin{array}{rccc} T: & \mathbb{Q} & \longrightarrow & \mathbb{N} \times \mathbb{N} \\ & p/q & \mapsto & (p,q) \end{array}$$

with the usual provisos that $q > 0$ and p, q are coprime (no common factor, we don't consider the rationals like $4/2$) to ensure that the definition of p/q is unique.

If we restrict the codomain to the couples (p, q) such that $q > 0$ and p, q are coprime we have a one-to one correspondence

$$T' : \begin{array}{ccc} \mathbb{Q} & \longrightarrow & A \\ p/q & \mapsto & (p,q) \end{array} \quad \textit{where } A = \{(p,q) \in \mathbb{N}\times\mathbb{N} \mid q > 0 \textit{ and } p,q \textit{ are coprime}\}$$

So $\mid \mathbb{Q} \mid = \mid A \mid$.

We give a description of A, detailing the elements of $\mathbb{Z}\times\mathbb{N}$ that we are taking out. We proceed diagonally

$$(0,0) \qquad \textit{we take it out because denominator is } 0$$

Listed elements of $\mathbb{Q}$: none.

$$\begin{array}{rl} (-1,0) & \textit{we take it out because denominator is } 0 \\ (0,1) & = 0/1 = 0 \in \mathbb{Q} \textit{ OK} \\ (1,0) & \textit{we take it out because denominator is } 0 \end{array}$$

Listed elements of $\mathbb{Q}$: 0.

$$\begin{array}{rl} (-2,0) & \textit{NO, denominator is } 0 \\ (-1,1) & = -1/1 = -1 \in \mathbb{Q} \textit{ OK} \\ (0,2) & = 0/1 = 0 \in \mathbb{Q} \textit{ we take it out, repeat} \\ (1,1) & = 1/1 = 1 \in \mathbb{Q} \textit{ OK} \\ (2,0) & \textit{NO, denominator is } 0 \end{array}$$

Listed elements of $\mathbb{Q}$: $0, \pm 1, \pm 2, \pm 1/2$.

$$\begin{array}{rl} (-3,0) & \textit{NO, denominator is } 0 \\ (-2,1) & = -2/1 = -2 \in \mathbb{Q} \textit{ OK} \\ (-1,2) & = -1/2 \in \mathbb{Q} \textit{ OK} \\ (0,3) & = 0/1 = 0 \textit{ we take it out, repeat} \\ (1,2) & = 1/2 \in \mathbb{Q} \textit{ OK} \\ (2,1) & = 2/1 = 2 \in \mathbb{Q} \textit{ OK} \\ (3,0) & \textit{NO, denominator is } 0 \end{array}$$

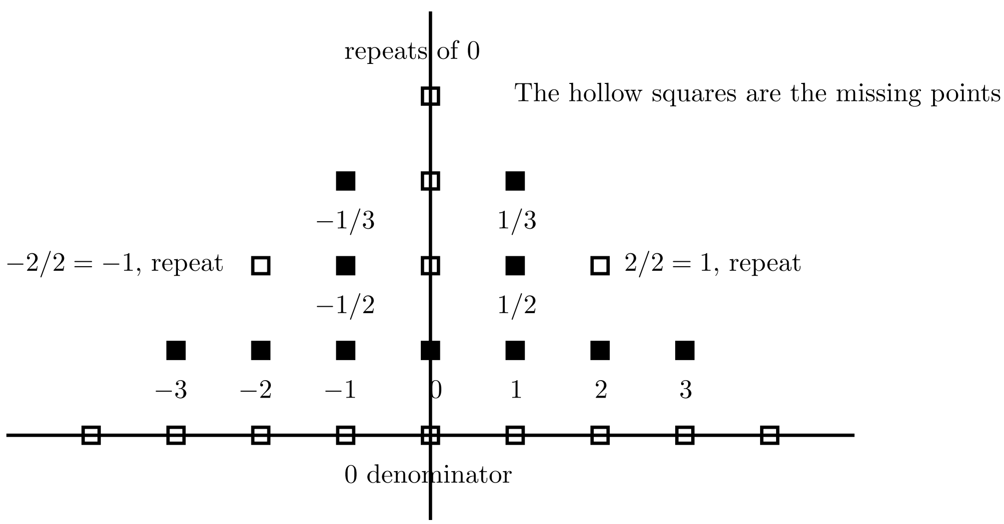
Listed elements of $\mathbb{Q}$*:* $0, \pm 1$.

$$
\begin{array}{rl}
(-4,0) & \textit{NO, denominator is } 0 \\
(-3,1) & = -3/1 = -3 \in \mathbb{Q} \ \textit{OK} \\
(-2,2) & = -2/2 = -1 \ \textit{NO, repeat} \\
(-1,3) & = -1/3 \in \mathbb{Q} \ \textit{OK} \\
(0,4) & = 0/1 = 0 \ \textit{NO, repeat} \\
(1,3) & = 1/3 \in \mathbb{Q} \ \textit{OK} \\
(2,2) & = 2/2 = 1 \ \textit{NO, repeat} \\
(3,1) & = 3/1 = 3 \in \mathbb{Q} \ \textit{OK} \\
(4,0) & \textit{NO, denominator is } 0
\end{array}
$$

Listed elements of $\mathbb{Q}$*:* $0, \pm 1, \pm 2, \pm 1/2, \pm 3, \pm 1/3$.

graphically

GNU2 - The set $A \subseteq \mathbb{N} \times \mathbb{N}$

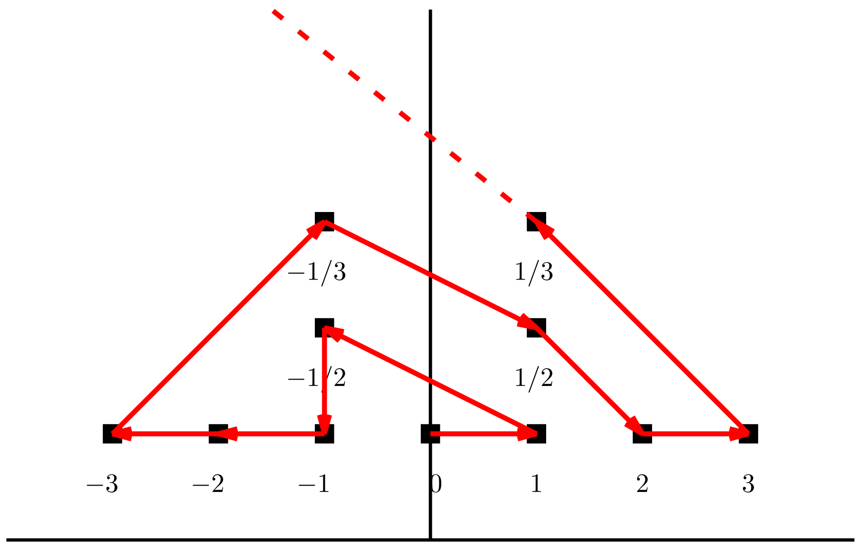


From the description above and the drawing, it is clear that T' is a one-to-one correspondence.

If we prove that there is a one-to-one correspondence $G : \mathbb{N} \longrightarrow A$, and we are done, because then $\mid \mathbb{N} \mid = \mid A \mid$ and $\mid \mathbb{Q} \mid = \mid A \mid$ give $\mid \mathbb{N} \mid = \mid \mathbb{Q} \mid$.

We define G as in the imagine below:

GNU3 - The one-to-one-correspondence $G : \mathbb{N} \longrightarrow A$



The function G is a one-to-one correspondence, and so

$$\mid \mathbb{N} \mid = \mid A \mid$$

Exercise 3. *Prove $\mid \mathbb{N} \mid \neq \mid \mathbb{R} \mid$.*

We have to prove that there is no one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{R}$. Since $\mid (0,1) \mid = \mid \mathbb{R} \mid$, we can prove that there is no one-to-one correspondence $F : \mathbb{N} \longrightarrow (0,1)$.

By contradiction, we say that there is a one-to-one correspondence $F : \mathbb{N} \longrightarrow (0,1)$. That means that we can enumerate ALL the real numbers *in* $(0,1)$. *That means that we can put them ALL in a list, where the a's, b's etc are digits (numbers in* $0,\ldots,9$*)*

$$\begin{aligned} F(0) &= 0, a_0 a_1 a_2 a_3 \cdots \\ F(1) &= 0, b_0 b_1 b_2 b_3 \cdots \\ F(2) &= 0, c_0 c_1 c_2 c_3 \cdots \\ F(3) &= 0, d_0 d_1 d_2 d_3 \cdots \\ \vdots &= \vdots \end{aligned}$$

or, if you prefer, changing a_0 *in* a_0^0*, etc. etc.*

$$\begin{aligned} F(0) &= 0, a_0^0 a_1^0 a_2^0 a_3^0 \cdots \\ F(1) &= 0, a_0^1 a_1^1 a_2^1 a_3^1 \cdots \\ F(2) &= 0, a_0^1 a_1^2 a_2^2 a_3^2 \cdots \\ F(3) &= 0, a_0^3 a_1^3 a_2^3 a_3^3 \cdots \\ \vdots &= \vdots \\ F(n) &= 0, a_0^n a_1^n a_2^n a_3^n \cdots \\ \vdots &= \vdots \end{aligned}$$

to find a contradiction, we build a real y *number belonging to* $(0,1)$ *that is NOT in the above list.*

$$y = 0, b_0 b_1 b_2 b_3 \cdots b_n \cdots$$

where the b_i*'s are any digit but we have that*

$$b_0 \neq a_0^0,\ b_1 \neq a_1^1,\ b_2 \neq a_2^2,\ b_3 \neq a_3^3, \ldots b_n \neq a_n^n, \ldots$$

And y *is different from* $F(0)$ *since their first digit is different,* y *is different from* $F(1)$ *since their second digit is different,* y *is different from* $F(2)$ *since their third digit is different and so on.*

It is clear that $y \in (0,1)$ *but* y *can't be in the list above, that supposedly contains all the elements of* $(0,1)$*, and here it is our contradiction. The hypotesis that the list above contains all the elements of* $(0,1)$ *is hence false, and it is so impossible to find a one-to-one correspondence between* $\mathbb{N}$ *and* $(0,1)$*, as so it is impossible to find a one-to-one correspondence between* $\mathbb{N}$*and* $\mathbb{R}$*. Hence*

$$| \mathbb{N} | \neq | \mathbb{R} |$$