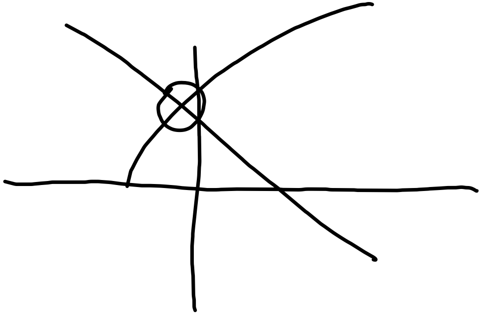
Ex 2 : $\sqrt{x+2} + x = 2$

$\sqrt{x+2} = 2 - x$

$\Updownarrow$

$\left(\sqrt{x+2}\right)^2 = (2-x)^2$ $\leftarrow$ $\left[\,2 - x \geq 0\,\right]$



$x_1 = \sim$

$x_2 = \sim$

Ex 3 $A = \{3n+5 \mid n \in \mathbb{N}\}$ $B = \{k^2+2 \mid k \in \mathbb{Z}\}$

$\mathbb{N} \longrightarrow A$

$n \rightsquigarrow 3n+5$

1-TO-1

$\mathbb{Z} \xrightarrow{G} B$

$k \rightsquigarrow k^2+2$

NOT 1-TO-1
NOT INV

$G(1) = 3$

$G(-1) = 3$

$\boxed{B = \{k^2+2 \mid k \in \mathbb{N}\}}$

$B = \{2, 3, 4, \ldots\}$

$k=0$ $k=\pm 1$ $k=\pm 2$

$B = \{ n^2 + 2 \mid n \in \mathbb{Z} \}$

$$\mathbb{Z} \xrightarrow{G'} B$$

$$n \rightsquigarrow n^2 + 2$$

BUT G' IS <u>NOT</u> INVERTIBLE

$B = \{ k^2 + 2 \mid k \in \mathbb{N} \}$

$$\mathbb{N} \xrightarrow{G} B$$

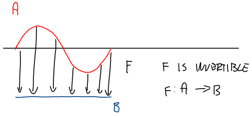
$$k \rightsquigarrow k^2 + 2$$

G INVERTIBLE

$$\mathbb{Z} \xrightarrow{H} \mathbb{N} \xrightarrow{G} B$$

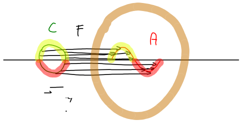
$$\boxed{G' = H \circ G}$$

Ex 5 $|A| = |B|$



F IS INVERTIBLE

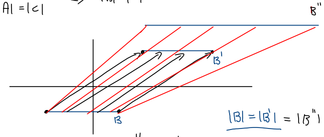
$F: A \to B$



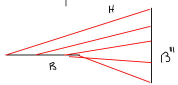
F IS $F: C \to A$ INVERTIBLE

$|A| = |B|$

$|A| = |C| \Rightarrow |B| = |C|$

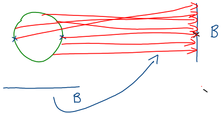


$|B| = |B'| = |B''|$



$H: B \to B'''$

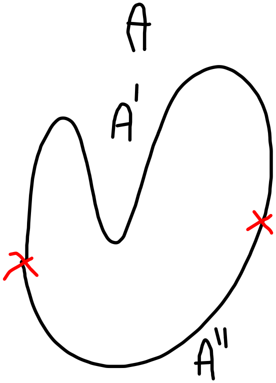
IS INV

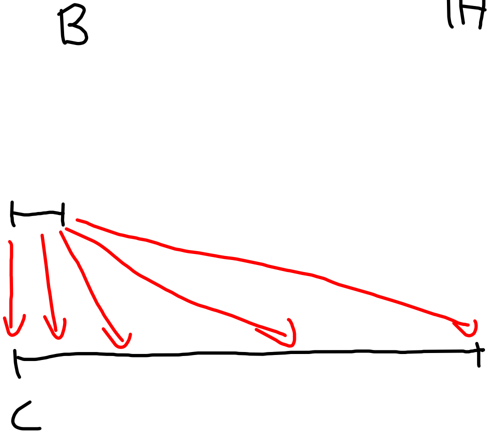


NOT 1-TO-1

BUT OTHERS ARE

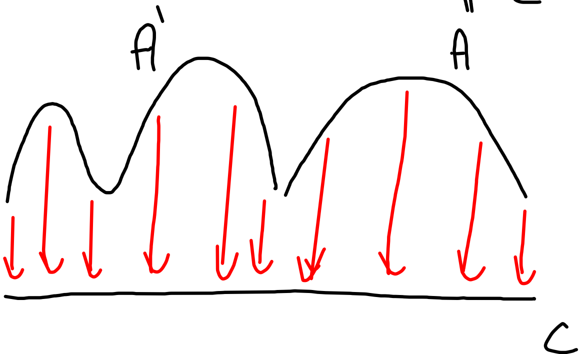
Ex

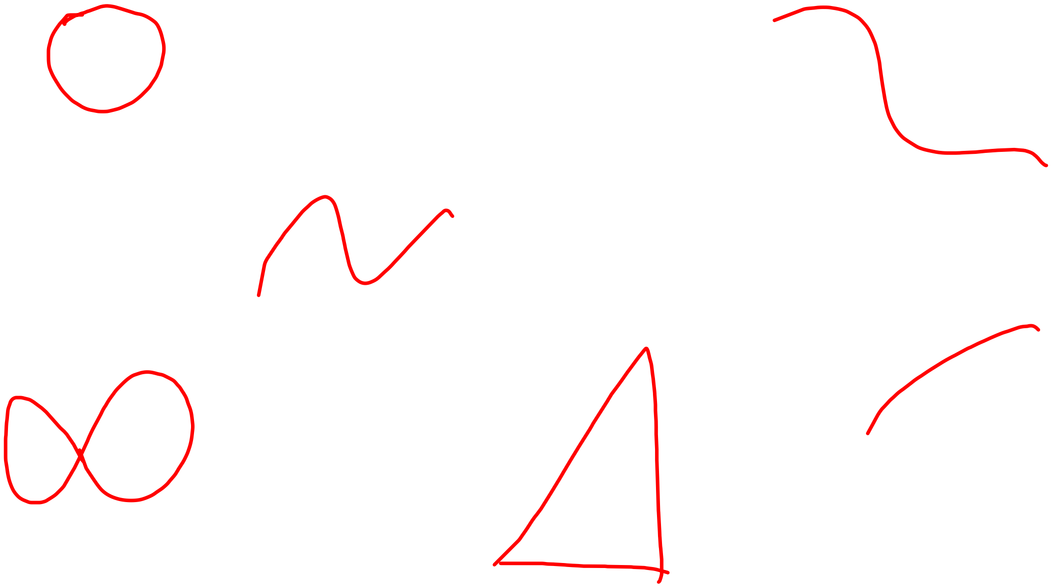




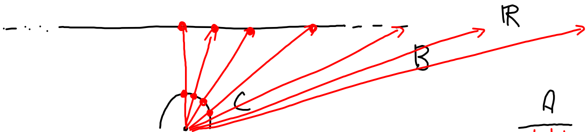
$$|A| \stackrel{?}{=} |B|$$

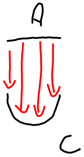
$$|B| = |C|$$ **OK**





Ex $A \longleftrightarrow (0,1)$

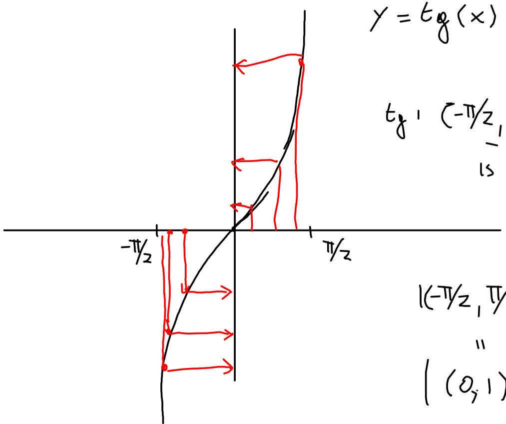




$|A| = |C|$

$|C| = |B|$

$\Downarrow \quad |A| = |B|$



$y = tg(x)$

$tg : (-\pi/2, \pi/2) \to \mathbb{R}$

is invertible

$\Downarrow$

$|(-\pi/2, \pi/2)| = |\mathbb{R}|$

$\|$

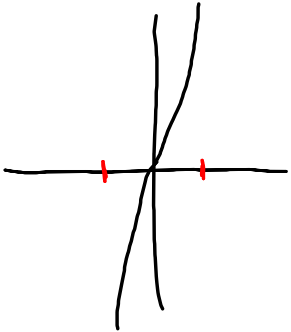
$|(0,1)|$

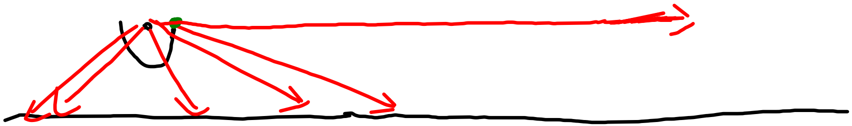
Ex $\quad |[0,1]| = |\mathbb{R}|$?

We know

$$|(0,1)| = |\mathbb{R}|$$

$$(0,1) \cup \{0,1\} = [0,1]$$





$|A|, |B|, |C|$

$$\begin{matrix} |A| = |B| \\ |B| = |C| \end{matrix} \Rightarrow |A| = |C|$$

$$|A \cup B| = |A| + |B|$$

$|A|$ IS NOT FINITE

$|B| = n \in \mathbb{N}$

$$\Rightarrow \begin{matrix} |A| + |B| = |A| \\ |A| + |A| = |A| \end{matrix}$$

$$\begin{matrix} |A| \geqslant |B| \\ |A| \leqslant |B| \end{matrix} \Rightarrow |A| = |B|$$

WE WOULD LIKE FOR IT TO BE TRUE

$|A| \neq |B|$

$\Leftrightarrow$

$\nexists\ F : A \to B$ INVERTIBLE

$|A| \leqslant |B|$? WHAT DOES IT MEAN ?

$A \subseteq B$

$|A| \leqslant |B|$

$$A \xrightarrow{F} B$$

F IS INJECTIVE

$$A \xrightarrow{F'} \{F(a) \mid a \in A\} \subseteq B$$

F' IS INVERTIBLE (RESTRICTION OF THE CODOMAIN)

$$|A| = |\{F(a) \mid a \in A\}| \leqslant |B|$$

WE CAN USE THIS THEOREM

A, B

$$\begin{matrix} |A| \leqslant |B| \\ |B| \leqslant |A| \end{matrix} \Rightarrow |A| = |B|$$

WE WANT TO PROVE

$$|[0,1]| = |\mathbb{R}|$$

 $[0,1]$

$\mathbb{R}$

$F : [0,1] \longrightarrow \mathbb{R}$ INJ ?

$\Longrightarrow$

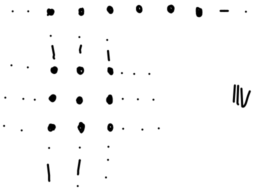
$G : \mathbb{R} \longrightarrow [0,1]$ INJ ?

LEFT AS AN EXERCISE

Ex 7

1) $|\mathbb{N}| \overset{?}{=} |\mathbb{N}\times\mathbb{N}|$

2) $|\mathbb{N}| \overset{?}{=} |\mathbb{Q}|$



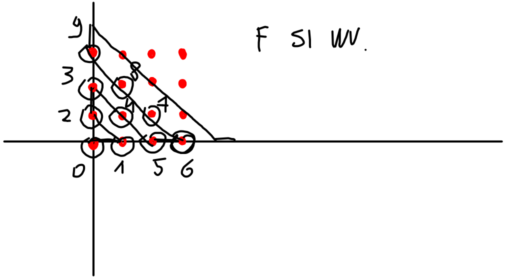
$\mathbb{N}$

$\mathbb{N}\times\mathbb{N}$

WE WANT

$F : \mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$

<u>INV</u>



F SI INV.

$(1,3) \longleftrightarrow 8$

$(3,0) \longleftrightarrow 6$

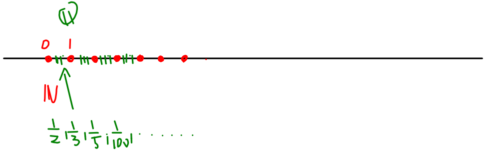
$F(93) =$ ___

WE HAVE DESCRIBED A FUNCTION (INV)

$F : \mathbb{N} \to \mathbb{N}\times\mathbb{N} \implies |\mathbb{N}| = |\mathbb{N}\times\mathbb{N}|$

$|\mathbb{N}| = |\mathbb{Q}|$?

. . . . • • • • . . . . . $\mathbb{N}$

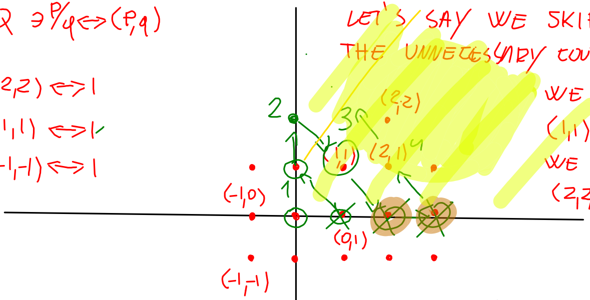


$\mathbb{Q} \ni p/q \Leftrightarrow (p,q)$

$(2,2) \Leftrightarrow 1$

$(1,1) \Leftrightarrow 1$

$(-1,-1) \Leftrightarrow 1$

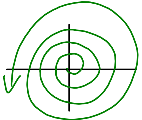


LET'S SAY WE SKIP THE UNNECESSARY COUPLES

WE KEEP (1,1)

WE DISCARD (2,2), etc

$\frac{1}{1} \sim \cancel{\frac{2}{2}} \sim \cancel{\frac{3}{3}}$



$3/10$

$\updownarrow$

$(3,10)$

$\updownarrow$

$n \in \mathbb{N}$

$|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{N} \times \mathbb{N}|$

WE HAVE SHOWN

$|A^n| = |A|$ TRUE

NOT IMMEDIATE

$|\mathbb{N}^3| = |\mathbb{N}|$

EASY

$|\mathbb{N}^n| = |\mathbb{N}|$

$|\mathbb{N}| \neq |\mathbb{R}| \iff \nexists\ F: \mathbb{N} \to \mathbb{R}$ INV

WE WANT TO PROVE THAT

$|\mathbb{R}| = |[0,1]|$ WE PROVED THAT

WE WANT TO PROVE $|\mathbb{N}| \neq |[0,1]|$ IT'S ENOUGH

$3175,312517756\ldots \in \mathbb{R}$

$0,3125111 31275\ldots \in [0,1]$

$0,\overline{9999} = 1$

WE PROVE THAT $F: \mathbb{N} \to [0,1]$ INV DOES NOT EXIST

BY R.A.A. (CONTRADICTION)

IT THER IS $F: \mathbb{N} \to [0,1]$ INV. WE CAN LIST $[0,1]$

AND ALL $x \in [0,1]$ ARE IN THE LIST

$0 \to 0,a_1 a_2 a_3 a_4 \ldots$

$1 \xrightarrow{F} 0,b_1 b_2 b_3 b_4 \ldots$

$2 \to 0,c_1 c_2 c_3 c_4 \ldots$

$3 \to 0,d_1 d_2 d_3 d_4 \ldots$

$\vdots$

BUT WE FIND A $y \in [0,1]$ NOT LISTED (CONTRAD.)

$y = 0,y_1 y_2$

$y \neq F(0)$ $y_1 \neq a_1$ $y \neq F(3)$ $y_3 \neq c_3$

$y \neq F(1)$ $y_2 \neq b_2$

$y = 0,y_1 y_2 y_3 y_4 \ldots \in [0,1]$

$y_1 \neq a_1$

$y_2 \neq b_2$

$y_3 \neq c_3$

$\vdots$

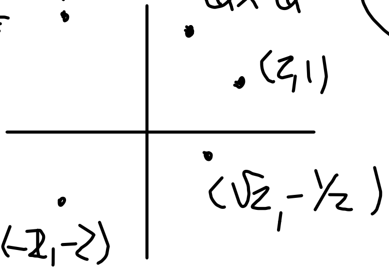
y IS NOT ON THE LIST $\implies$ CONTRADICTION !!

$|\mathbb{N}| \neq |\mathbb{R}|$

WE KNOW $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \neq |\mathbb{R}|$

I° QUESTION HOW MANY KINDS OF INFINITY ARE THERE ?

$\mathbb{Q} \times \mathbb{Q}$ YES



II° QUESTION

THERE IS SOMETHING IN BETWEEN ?

a $\mathbb{N}$ $\uparrow$ $\mathbb{R}$

$\cap$ $\mathbb{Q} \times \mathbb{Q}$

THERE IS $\mathbb{N} \subseteq A \subseteq \mathbb{R}$ $|\mathbb{N}| \neq |A| \neq |\mathbb{R}|$?

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}|$$

WE HAVE SHOWN

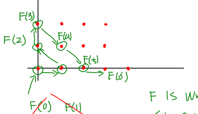
$F : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

INV

$F' : \mathbb{N} \to \mathbb{Q}$

$$|\mathbb{N}| \neq |\mathbb{R}|$$

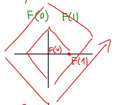
<u>THM</u> $\nexists F : \mathbb{N} \to \mathbb{R}$ INV



$|\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$

$F: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$

F IS INVERTIBLE
(1-TO-1 CORR.)

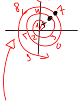


WE KEEP

$(3/5) \Longleftrightarrow (3,5)$

$3/5 \Longleftrightarrow \cancel{(9,15)}$ WE DISCARD

$a/b \Longleftrightarrow (a,b)$



$3/5 \Longleftrightarrow 9/15 \Longleftrightarrow (9,15)$

$(1,1) \to 1 = 1/1$ WE CONSIDER 1

$(2,2) \to 2/2$ ALREDY CONSIDERED
WE SKIP IT

$F: \mathbb{N} \to \mathbb{Q}$ INV

$|\mathbb{N}| \neq |\mathbb{R}| \Longleftrightarrow \nexists\ F: \mathbb{N} \to [0,1]$ INV

$|[0,1]| = |\mathbb{R}|$

$|A| > |\mathbb{R}|$ $\qquad A?$

$|\mathbb{N}| + |\mathbb{Z}| + |\mathbb{Q}| = |\mathbb{N}|$ $\qquad |A|+|A| = |A|$

EX FIND $F: \mathbb{N}\cup\mathbb{Z}\cup\mathbb{Q} \to \mathbb{N}$ INV

$\mathbb{C} = \{a+ib \mid a,b\in\mathbb{R}\}$ $\qquad |\mathbb{C}| \geq |\mathbb{R}|$

$\mathbb{C} \supseteq \mathbb{R}$ BUT $|\mathbb{C}| = |\mathbb{R}| = |\mathbb{R}\times\mathbb{R}|$ $\qquad |\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$

$|A\times A| = |A|$

- SET OF ALL SETS: NOT A SET

a_{11} $\quad b_{375}$ $\quad b_{\sqrt{2}}$ $\quad s$

- SET OF INDEXED LETTERS (ON A): SAME CARDINALITY OF A

- $\{1, 2, \dots, \} = \mathbb{N}$

PERMUTATIONS OF $\mathbb{N}$ { A PERM. OF 123 IS 132, 213, ⋮

THIS SET HAS CARDINALITY $> |\mathbb{R}|$

- SET OF $\mathcal{G} = \{F: \mathbb{R}\to\mathbb{R}\}$ $\qquad |\mathcal{G}| > |\mathbb{R}|$