FCS
Math: Functions
Exercises

Massimo Caboara

April 29^{th}, 2021

Solved exercise

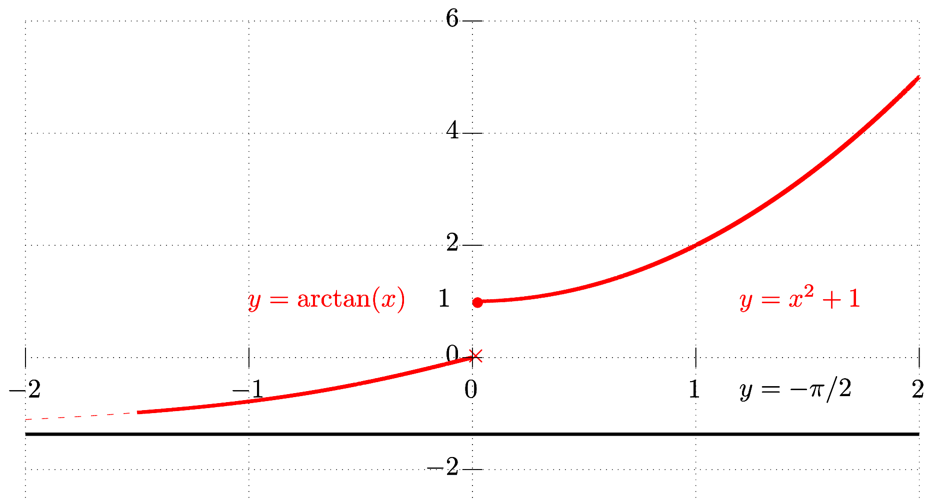
Example 1. *We have the function*

$$F: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} \arctan(x) & x < 0 \\ x^2 + 1 & x \geq 0 \end{cases}$$

1. *Draw the graph of F. Mark the intersections with the axis and the interesting points.*
2. *Is F injective, surjective, invertible? What is the image of F?*
3. *Determine $F(0)$, $F(\pi)$, $F([-\pi/2, \pi/4])$.*
4. *Determine $F^{-1}(0)$, $F^{-1}(1)$, $F^{-1}((-\sqrt{2}/2, \sqrt{2}/2])$, $F^{-1}((-\infty, 0])$.*
5. *Build an invertible function from F by restricting its domain and/or codomain.*
6. *Determine the formula for this inverse.*

1 The graph is

GNU1: Graph of $F(x)$



2 *It is easy to see that every horiziontal line $y = a$, $a \in \mathbb{R}$ intersects the graph of F once or not at all, so F is injective.*

The horizontal line $y = 1/2$ does not intersects the graph of F, so F is not surjective. We could have chosen another horizontal line with no intersection, for example $y = -2$. The graphical rule says that a function is surjective if EVERY horizontal line $y = a$ with a in the domain of F intersects the graph of the function at least once.

Being injective but not surjective, F is not invertible.

The image of F is $(-pi/2, 0] \cup [1, +\infty)$

3 $F(0) = 1$, $F([1, 2]) = [F(1), F(2)] = [2, 5]$ *because F is increasing. To determine the image of $(-\pi/2, \pi/4]$ it is useful to split this interval*

$$(-\pi/2, \pi/4] = (-\pi/2, 0) \cup [0, \pi/4]$$

and we have

$$\begin{aligned} F((-\pi/2, \pi/4]) &= F((-\pi/2, 0) \cup [0, \pi/4]) \\ &= F((-\pi/2, 0)) \cup F([0, \pi/4]) \\ &= (F(-\pi/2), F(0)) \cup [F(0), F(\pi/4)] \\ &= (-1, 0) \cup [1, \pi^2/16 + 1] \end{aligned}$$

4 $F^{-1}(1/2) = \emptyset$ *because the graph of F and the line $y = 1/2$ have no intersection.* $F^{-1}(-2) = \emptyset$ *because the graph of F and the line $y = -2$ have no intersection. For the counterimage of the interval $[-1, 2]$ it is useful to split this interval*

$$[-1, 2] = [-1, 0) \cup [0, 1) \cup [1, 2]$$

and we have

$$\begin{aligned}
F^{-1}([-1,2]) &= F^{-1}([-1,0)\cup[0,1)\cup[1,2]) \\
&= F^{-1}([-1,0))\cup F^{-1}([0,1))\cup F^{-1}([1,2]) \\
&= [F^{-1}(-1),F^{-1}(0))\cup\emptyset\cup[F^{-1}(1),F^{-1}(2)] \\
&= [-\pi/4,0)\cup\emptyset\cup[0,1] \\
&= [-\pi/4,0)\cup[0,1] \\
&= [-\pi/4,1]
\end{aligned}$$

5 *The function is injective, so we keep its domain. To make the function surjective, we restrict the codomain to the image. The new, invertible, function is*

$$\begin{array}{rccl} F': & \mathbb{R} & \longrightarrow & (-\pi/2,0]\cup[1,+\infty) \\ & x & \mapsto & \begin{cases}\arctan(x) & x<0\\ x^2+1 & x\geq 0\end{cases}\end{array}$$

We compute its inverse piece by piece

First piece

$$\arctan(x)=y \iff \tan(\arctan(x))=\tan(y)\iff x=\tan(y)$$

and we can apply the tangent to the equation because $\arctan(x)\in(\pi/2,\pi/2)$ *for every* x *and* $y\in(-pi/2,0]$ *for this piece. In both cases the values are in the domain of THIS tangent (restriction of the tangent to* $(-\pi/2,\pi/2)$*), that for the sake of simplicity we still call tangent*

$$\begin{array}{rccc} \tan: & (-\pi/2,\pi/2) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \tan(x)\end{array}$$

and so we can apply it to the equation. Since THIS tangent is invertible, the equation does not change and we have $\tan(\arctan(x))=x$

Second piece

$$x^2+1=y\iff x^2=y-1\iff\sqrt{x^2}=\sqrt{y-1}\iff x=\sqrt{y-1}$$

and we can apply the square root to the equation because $x^2\geq 0$ *and* $y-1\geq 0$ *in this piece. In both cases the values are in the domain of the square root, and so we can apply it to the equation. Since the square root*

$$\begin{array}{rccc} \sqrt{\cdot}: & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & \sqrt{x}\end{array}$$

is invertible with inverse

$$\begin{array}{rccc} (\cdot)^2: & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & x^2\end{array}$$

the equation does not change and we have $\sqrt{x^2} = x$ for the x's in $\mathbb{R}_0^+$.

The inverse of F' *is then*

$$F'^{-1}: \quad (-\pi/2, 0] \cup [1, +\infty) \longrightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} \tan(x) & x \in (-\pi/2, 0] \\ x^2 + 1 & x \in [1, +\infty) \end{cases}$$

Proposed exercises

Exercise 1. *We have the function*

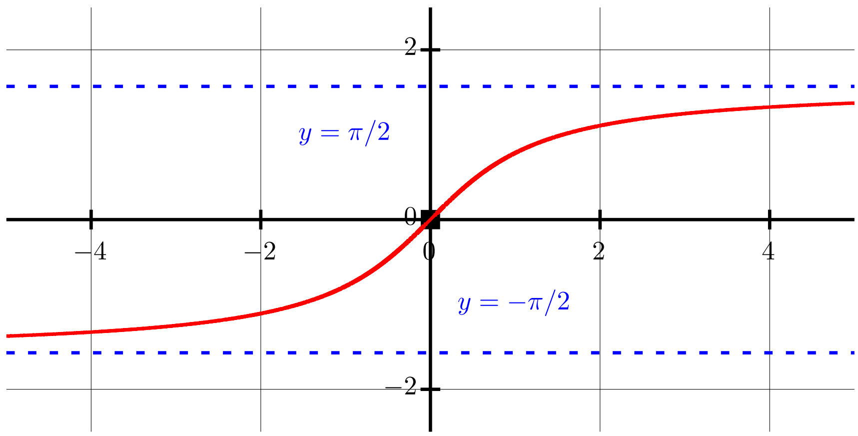
$$F: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto \arctan x$$

1. *Draw the graph of* F*. Mark the intersections with the axis and the interesting points.*
2. *Is* F *injective, surjective, invertible?*
3. *Determine* $F(0)$*,* $F(\pi)$*,* $F([-\pi/2, \pi/4])$*.*
4. *Determine* $F^{-1}(0)$*,* $F^{-1}(1)$*,* $F^{-1}((-\sqrt{2}/2, \sqrt{2}/2])$*,* $F^{-1}((-\infty, 0])$*.*
5. *Build an invertible function from* F *by restricting its domain and/or codomain.*
6. *Determine the formula for this inverse and draw its graph.*

Solution: The graph of F is

Draw the graph of F. Mark the intersections with the axis and the interesting points.

GNU2: The Graph of F

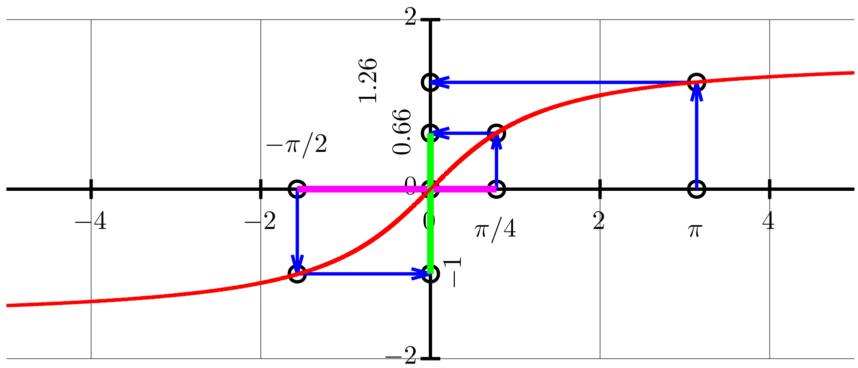


Clearly, F in injective, not surjective and hence not invertible.

We have that $F(0) = 0$, $F(\pi) \sim 1.26$, $F(-\pi/2) \sim -1$,$F(\pi/4) \sim 0.66$, using the arctan function on a pocket computer. Since the function F is increasing and continuous,

$$F([-\pi/2, \pi/4]) = [F(-\pi/2), F(\pi/4)] \sim [-1, 0.66]$$

GNU2: The Graph of F



The image of $[-\pi/2, \pi/4]$ is $[-1, 0.66]$

The function F is not invertible, but if we restrict its codomain we get the invertible function

$$\begin{array}{rccc} F': & \mathbb{R} & \longrightarrow & (-\pi/2, \pi/2) \\ & x & \mapsto & \arctan(x) \end{array}$$

whose inverse is

$$\begin{array}{rccc} F': & (-\pi/2, \pi/2) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \tan(x) \end{array}$$

and we have $F(x) = F'(x) = \tan(x)$ for all the $x \in \mathbb{R}$ we have, using the tan function on a pocket computer, that

$$F^{-1}(0) = \tan(0) = 0$$
$$F^{-1}(1) = \tan(1) \sim 1.55$$
$$F^{-1}(-\sqrt{2}/2) = \tan(\pm\sqrt{2}/2) \sim \pm 0.85$$

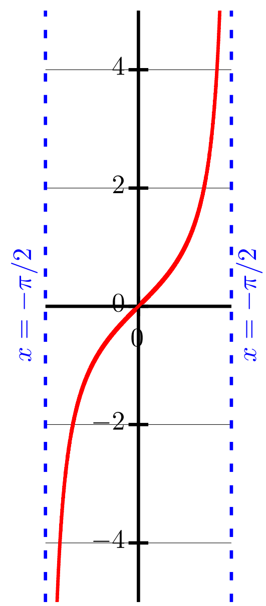
Since the function F is increasing and continuous, so is the function F^{-1} and

$$F'^{-1}((-\sqrt{2}/2, \sqrt{2}/2]) = (F'^{-1}(-\sqrt{2}/2), F'^{-1}(\sqrt{2}/2)] \sim (-0.85, 0.85]$$

From the graph it is easy to see that

$$F'^{-1}((-\infty, 0]) = (-\pi/2, 0]$$

The graph of F'^{-1} is



⋈

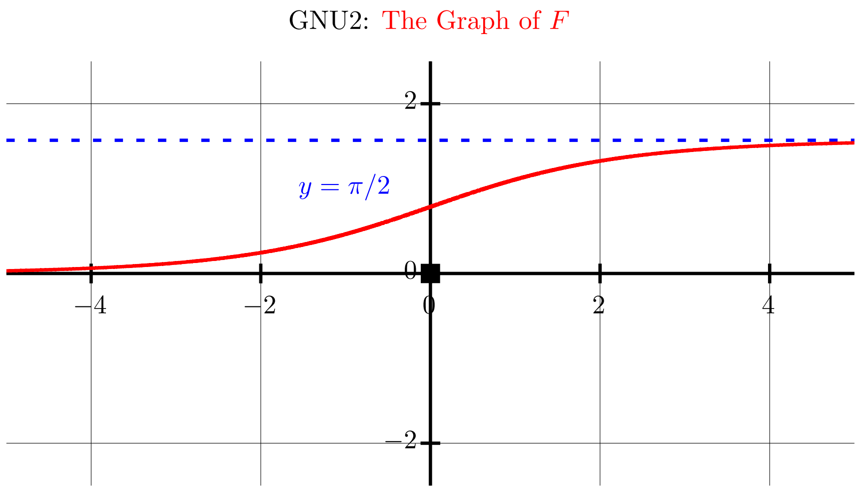
For the following functions follow the same procedure as the functions above. Only the graph is provided.

Exercise 2. *We have the function*

$$F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan 2^x \end{array}$$

1. *Draw the graph of F. Mark the intersections with the axis and the interesting points.*
2. *Is F injective, surjective, invertible?*
3. *Determine $F(0)$, $F(-1)$, $F([-\pi/2, \pi/4])$.*
4. *Determine $F^{-1}(0)$, $F^{-1}(1)$, $F^{-1}((0, 1])$, $F^{-1}((-\infty, 0])$.*
5. *Build an invertible function from F by restricting its domain and/or codomain.*
6. *Determine the formula for this inverse and draw its graph.*

Solution: The graph of F is



⋈

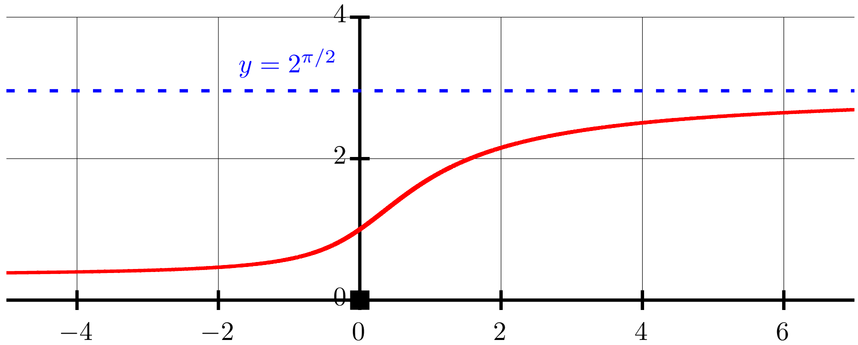
Exercise 3. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 2^{\arctan x} \end{array}$$

1. *Draw the graph of F. Mark the intersections with the axis and the interesting points.*
2. *Is F injective, surjective, invertible?*
3. *Determine $F(0)$, $F(\pi)$, $F([-\pi/2, \pi/4])$.*
4. *Determine $F^{-1}(0)$, $F^{-1}(1)$, $F^{-1}((-\sqrt{2}/2, \sqrt{2}/2])$, $F^{-1}((-\infty, 0])$.*
5. *Build an invertible function from F by restricting its domain and/or codomain.*
6. *Determine the formula for this inverse and draw its graph.*

Solution: The graph of F is

GNU2: The Graph of F



$\bowtie$

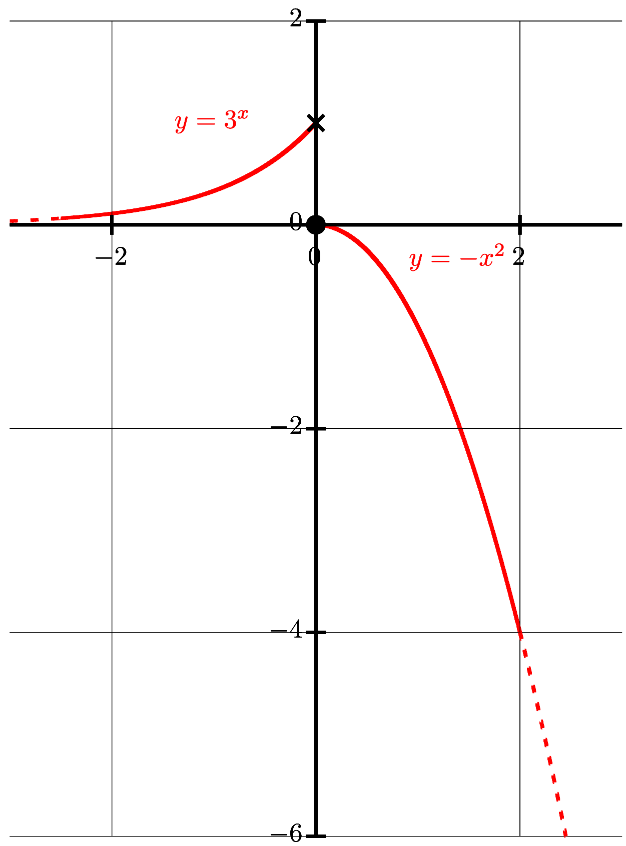
Exercise 4. *We have the function*

$$F: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} 3^x & x < 0 \\ -x^2 & x \geq 0 \end{cases}$$

1. *Draw the graph of F. Mark the intersections with the axis and the interesting points.*
2. *Is F injective, surjective, invertible?*
3. *Determine $F(0)$, $F(-1)$, $F([-1, 1])$.*
4. *Determine $F^{-1}(0)$, $F^{-1}(-1)$, $F^{-1}([-2, 2])$*
5. *Build an invertible function from F by restricting its domain and/or codomain.*
6. *Determine the formula for this inverse and draw its graph.*

Solution: *The graph of F is*

GNU2: The Graph of F



$\bowtie$

Exercise 5. *We have the function*

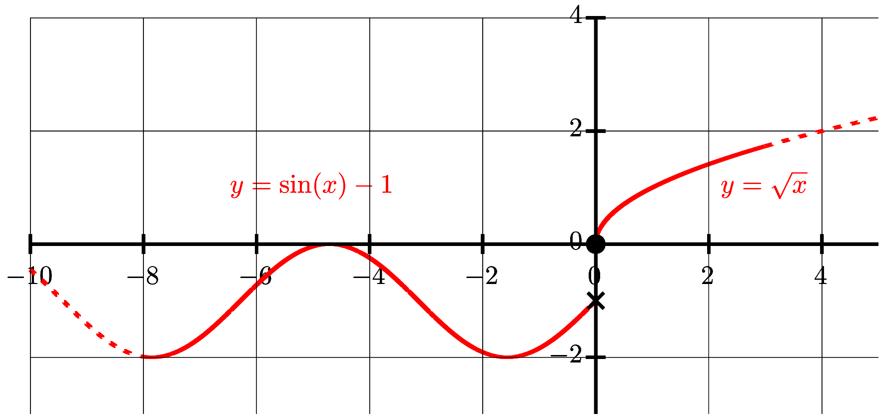
$$F: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} \sin(x) - 1 & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$$

1. *Draw the graph of F. Mark the intersections with the axis and the interesting points.*
2. *Is F injective, surjective, invertible?*
3. *Determine $F(0)$, $F(-\pi)$, $F([-\pi/2, 2])$.*
4. *Determine $F^{-1}(0)$, $F^{-1}(1)$, $F^{-1}([-1, 1])$.*
5. *Build an invertible function from F by restricting its domain and/or codomain.*

6. Determine the formula for this inverse and draw its graph.

Solution: The graph of F is

GNU2: The Graph of F



$\bowtie$