FCS
Math: Functions

Massimo Caboara

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We remind the definition of increasing (growing) function

Definition 1. *If $A, B \subset \mathbb{R}^2$, a function $f : A \longrightarrow B$ is* increasing *if*

$$\forall\ a, b \in A \quad a < b \Longrightarrow f(a) < f(b)$$

and is weakly increasing *if*

$$\forall\ a, b \in A \quad a < b \Longrightarrow f(a) \leq f(b)$$

We use the specular definitions for *decreasing/weakly decreasing* functions.

Definition 2. *If $A, B \subset \mathbb{R}^2$ a function $f : A \longrightarrow B$ is* decreasing *if*

$$\forall\ a, b \in A \quad a < b \Longrightarrow f(a) > f(b)$$

is weakly decreasing *and if*

$$\forall\ a, b \in A \quad a < b \Longrightarrow f(a) \geq f(b)$$

We say that the function f flips the values.

Definition 3. *If $A, B \subset \mathbb{R}^2$ a function $f : A \longrightarrow B$ is* increasing/weakly increasing/decreasing/weakly decreasing in $D \subset A$ *if the function $f_{|D}$ is increasing/weakly increasing/decreasing/weakly decreasing*

Definition-Proposition 1. *increasing/weakly increasing/decreasing/weakly decreasing functions are called* monotone/weakly monotone *functions.*

Remark 1. *Some text calls* increasing *a weakily increasing function and* strictly increasing *an increasing function. Some for decreasing.*

Proposition 1. *If $F : A \longrightarrow B$ is an increasing function, then the inequality*

$$H(x) > G(x) \textit{is equivalent to the inequality } F(H(x)) > F(G(x))$$

IF $\forall\ x\ H(x), G(x) \in A$.

Proposition 2. *If $F : A \longrightarrow B$ is an decreasing function, then the inequality*

$$H(x) > G(x) \textit{is equivalent to the inequality } F(H(x)) < F(G(x))$$

IF $\forall\ x\ H(x), G(x) \in A$. (The inequality flips*).*

A full example

Example 1. *We want to solve, with all the details, the inequality*

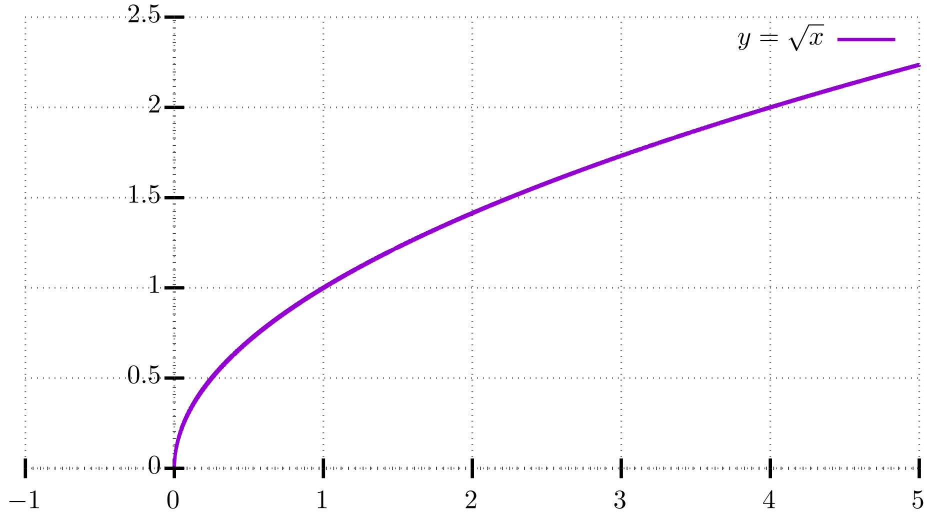
$$x + 2 < \sqrt{-5x - 2}$$

with $x \in \mathbb{R}$

Please remark that this exercise and the one done in class differ by the verse of the inequality.

First of all, since it involves the function $\sqrt{\cdot}$, *we recall the graph of this elementary function*

GNU8: Graph of $\sqrt{x}$



we note that the domain is $\mathbb{R}^+$ *and the codomain* $\mathbb{R}$. *To be more precise, the function is*

$$\begin{array}{rccc} \sqrt{\cdot}: & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x} \end{array}$$

Fot the inequality to have a meaning, we need to check when $\sqrt{-5x-2}$ *exists. Equivalently, for which* $x \in \mathbb{R}$ *we have that* $-5x - 2$ *belongs to the domain of* $\sqrt{\cdot}$. *Equivalently, for which* $x \in \mathbb{R}$ *we have that* $-5x - 2 \geq 0$. *That is an easy inequality*

$$\begin{aligned} -5x - 2 &\geq 0 \\ 5x + 2 &\leq 0 \\ 5x &\leq -2 \\ x &\leq -2/5 \end{aligned}$$

Note that to reach he second step we used the rule if we multiply noth parts of an inequality by -1 we flip the inequality. *That is an occurence of the function applying method. Multiplying both parts of the inequality by -1 is exactly the same as applying to the inequality the function*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & -x \end{array}$$

whose domain is $\mathbb{R}$ and decreasing, and that hence flips the inequality.

So we have the condition

$$x \leq -2/5 \quad \textit{or if you prefer } x \in (-\infty, -2/5]$$

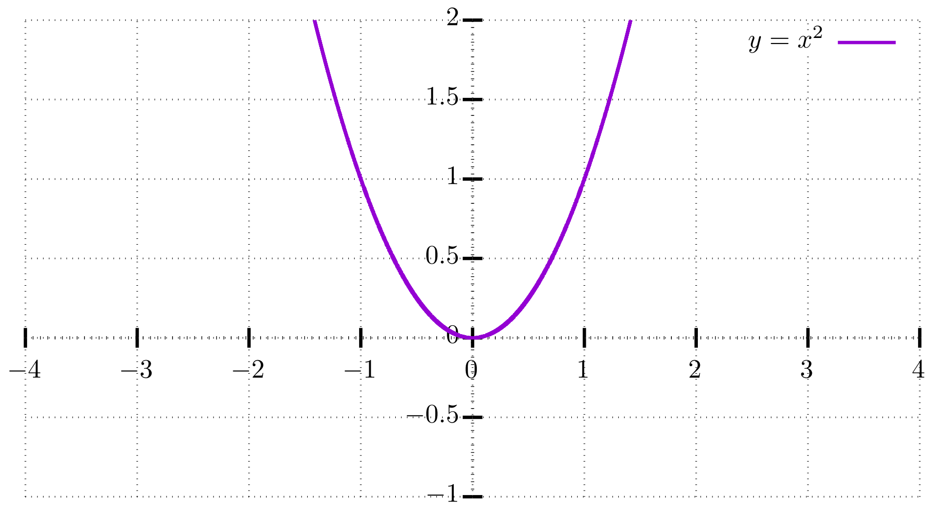
for the inequality $x + 2 < \sqrt{-5x-2}$ to have a meaning.

We want to solve the inequality using the function method. We want to apply some function to the inequality in such a way to simplify it. Since a square root is involved, we are thinking af the square function

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}_0^+ \\ x & \mapsto & x^2 \end{array}$$

whose graph is

GNU9: Graph of x^2

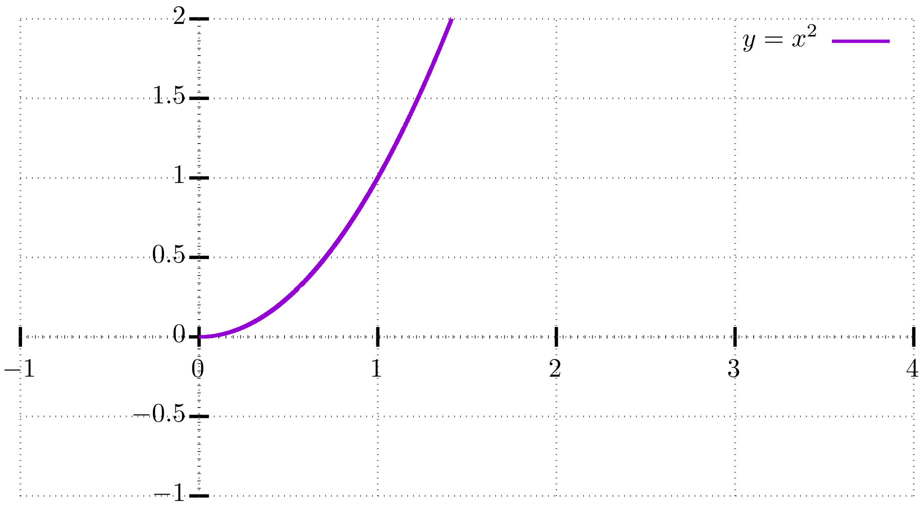


but this function is not always increasing or decreasing. We have to restrict its domain to have, for example, an increasing function. Let's use the function

$$f_{\mathbb{R}_0^+}: \begin{array}{ccc} \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}^+ \\ x & \mapsto & x^2 \end{array}$$

(we remember that when we write $\mathbb{R}_0^+$ *we mean* $[0, +\infty)$ *o* $\{x \in \mathbb{R} \mid x \geq 0\}$*).*

GNU10:Graph of positive or null x^2



that is clearly increasing and has domain $\mathbb{R}^+$.

Remembering that $x \leq -2/5$ *we want to apply this function to the inequality*

$$x + 2 < \sqrt{-5x - 2}$$

To do that we need

- *For* $f_{\mathbb{R}^+}$ *to be increasing - and this is OK*
- *For both parts of the inequality,* $x + 2$, $-5x - 2$, *to belong to the domain of* $f_{\mathbb{R}^+}$, *e.g.* $\mathbb{R}^+$. *That means that we need to have*

  $$\sqrt{-5x - 2} > 0 \text{ and } x + 2 > 0$$

  The first inequality always holds (since $\sqrt{-5x - 2}$, *when in exists (and it do exist given the constraint), is always positive, as we can see from the graph). For the second inequality, we cannot always guarantee that - for example* $x = -3$ *is acceptable (it satisfy the condition* $x \leq -2/5$*) but for* $x = -3$ *the expression* $x + 2$ *has value* -1, *not positive. Not OK.*

What can we do then? We can use a divide and conquer technique: when $x + 2$ *is positive, we apply to the inequality the function* $f_{\mathbb{R}^+}$, *and when* $x + 2$ *is negative or null, we do something else.*

So we split our solution search in two beanches. First we suppose that $x + 2$ *is positive, then we suppose that* $x + 2$ *is negative or null. In both cases, we have to remember the constraint* $x \in (-\infty, -2/5]$ *fot the inequality to have a meaning.*

- *We examine the case $x+2 \geq 0$ with the constraint $x \in (-\infty, -2/5]$ The inequality is*
$$x + 2 < \sqrt{-5x-2}$$
We want to apply the function
$$\begin{array}{rccc} f_{\mathbb{R}_0^+}: & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & x^2 \end{array}$$
to the inequalty. It is increasing and both $x+2$ and $\sqrt{-5x-2}$ are in int domain e.g. they are bot positive or null. The first by construction, the second because the square root is always positve or null (check the graph).

We can proceed. Since $f_{\mathbb{R}_0^+}$ is increasing, there is no flip when we apply it to the inequality.
$$\begin{array}{rcl} x+2 & < & \sqrt{-5x-2} \\ (x+2)^2 & < & (\sqrt{-5x-2})^2 \end{array}$$

Now, we want to check that $f_{\mathbb{R}_0^+}$ is the inverse of $\sqrt{\cdot}$.

We check that the functions
$$\begin{array}{rccc} \sqrt{\cdot}: & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & \sqrt{x} \end{array} \text{ and } \begin{array}{rccc} f_{\mathbb{R}_0^+}: & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & x^2 \end{array}$$
are reciprocally inverses. We have to check that applying the first and then the second or vice versa we get the identity
$$\forall\ x \in \mathbb{R}_0^+\ (\sqrt{x})^2 = x \text{ and } \forall\ x \in \mathbb{R}_0^+\ (\sqrt{x^2}) = x$$
and that is true since we know that
$$(\sqrt{x})^2 = \left(x^{1/2}\right)^2 = x^{2 \cdot \frac{1}{2}} = x^1 = x \quad \text{if } x \geq 0 \quad \text{and } \sqrt{x} \text{ exists}$$
and
$$(\sqrt{x^2}) = \mid x \mid = x \quad \text{if } x \geq 0$$
remembering that
$$\mid x \mid = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$
Since we have found explicitly that they are the inverse of one another, it is not necessary to check using the horizontal lines rule the invertibility of
$$\begin{array}{rccc} \sqrt{\cdot}: & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & \sqrt{x} \end{array} \text{ and } \begin{array}{rccc} f_{\mathbb{R}_0^+}: & \mathbb{R}_0^+ & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & x^2 \end{array}$$

The functions are thus reciprocally inverses and

$$\forall\ x \in \mathbb{R}_0^+\ (\sqrt{x})^2 = x \Longrightarrow \forall\ x \in (-\infty, -2/5]\ (\sqrt{-5x-2})^2 = -5x-2$$

since $5x-2 \in \mathbb{R}_0^+$ *for our constraint* $x \in (-\infty, -2/5]$.

$$\begin{array}{rcl} (x+2)^2 & < & (\sqrt{-5x-2})^2 \\ x^2+4x+4 & < & -5x-2 \\ x^2+9x+6 & < & 0 \end{array}$$

We draw the parabola $y = x^2+9x+6$ *in red and check when it is below (since the inequality is* $x^2+9x+6<0$*) the horizontal line* $y = 0$*, in green. We can easily draw the parabola. The zeroes* $x_{1,2}$ *we by the formula*

$$x_{1,2} = \frac{-9 \pm \sqrt{81-24}}{2} = \frac{-9 \pm \sqrt{57}}{2} \simeq \frac{-9 \pm 7.54}{2} = -8.27, -0.73$$

For the vertex $V = (V_x, V_y)$

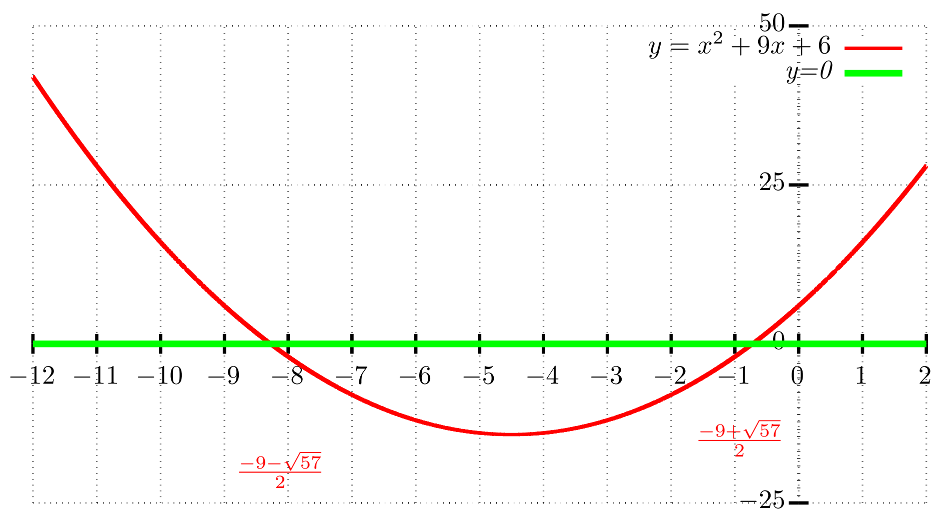
$$V_x = \left(-\frac{b}{2a}\right) = -\frac{9}{2} = -9/2 = -4.5$$

and, if $p(x) = x^2+9x+6$

$$V_y = p(V_x) = (V_x)^2+9(V_x)+6 = (-9/2)^2+9(-9/2)+6 = \frac{36}{4}-\frac{36}{2}+6 = -\frac{57}{4} = -14.25$$

and the intersection with the x-axis is $(0, 6)$

GNU11: $y = x^2+9x+6$



It is easy to see from the graph that the interval where the red parabola is *below the green line is*

$$\left[\frac{-9-\sqrt{57}}{2}, \frac{-9+\sqrt{57}}{2}\right] \simeq [-8.27, -0.73]$$

and the solutions for the last inequality are

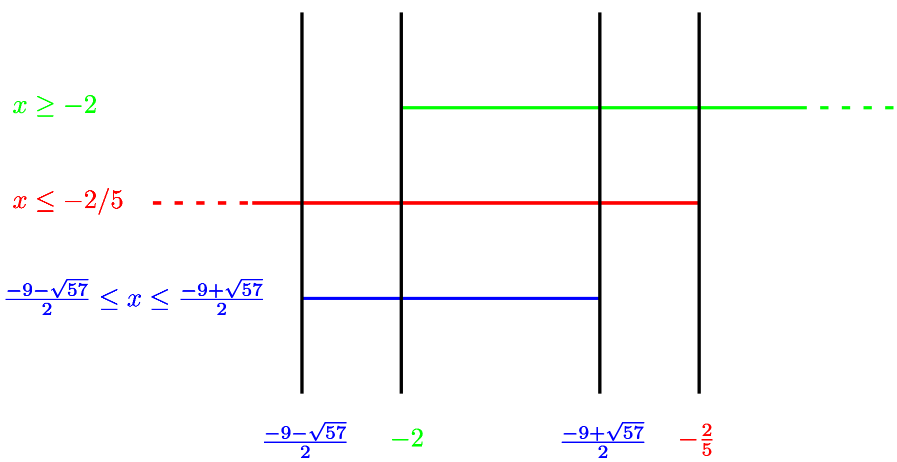
$$\frac{-9-\sqrt{57}}{2} \leq x \leq \frac{-9+\sqrt{57}}{2} \text{ or if you prefer } x \in \left[\frac{-9-\sqrt{57}}{2}, \frac{-9+\sqrt{57}}{2}\right]$$

We have to consider them, the constraint $x \in (-\infty, -2/5]$ *and the fact that we suppose* $x + 2 \geq 0 \iff x \in [-2, +\infty]$. *All three have to hold at the same time, so the system is*

$$\begin{cases} x \in [-2, +\infty] \iff \ x \geq -2 \\ x \in (-\infty, -2/5] \iff \ x \leq -2/5 \simeq -0.4 \\ x \in \left[\frac{-9-\sqrt{57}}{2}, \frac{-9+\sqrt{57}}{2}\right] \simeq [-8.27, -0.73] \end{cases}$$

or, graphically,

GNU12: Table for the system above



Note that the distance between lines is arbitrary to ease the reading of the table. It is immediate to see that the only x*'s for which all three the conditions hold (the only* x*'s for which the horiziontal lines are colored at the same time) are*

$$-2 \leq x \leq \frac{-9+\sqrt{57}}{2} \iff x \in \left[-2, \frac{-9+\sqrt{57}}{2}\right] \simeq [-2, -0.73]$$

- *We examine the case $x+2$ negative with the constraint $x \in (-\infty, -2/5]$*
  The inequality is
  $$x + 2 < \sqrt{-5x-2}$$
  and since $x+2$ is negative, it is always smaller than $\sqrt{-5x-2}$, that is positive when it exists (again, we can just check the graph of $\sqrt{\cdot}$), and it does exists exists because $x \in (-\infty, -2/5]$.

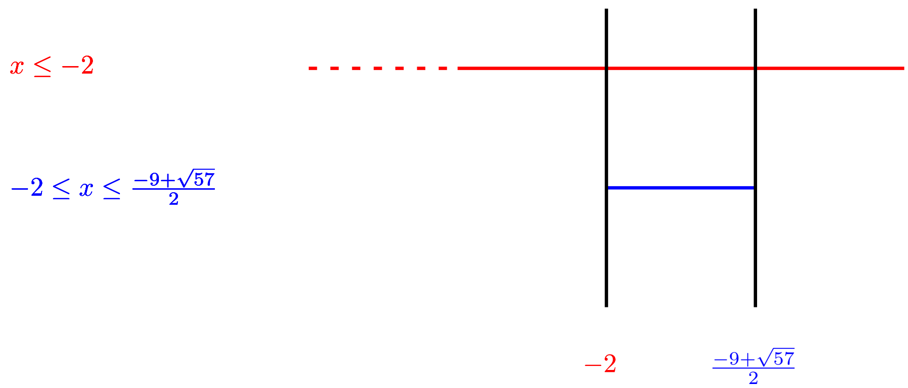
  In this case so the inequality holds if $x+2 < 0$ and $x < -2/5$. These conditions have to hold at the same time, so we have to solve the system
  $$\begin{cases} x+2 < 0 \\ x < -2/5 \end{cases} \iff \begin{cases} x < -2 \\ x < -2/5 \end{cases} \iff x < -2$$
  So, in the case $x+2 < 0$ all the x are solutions.

We have to put together the results:

- *Case $x+2 \geq 0$: Solutions are*
  $$x \in \left[-2, \frac{-9+\sqrt{57}}{2}\right]$$
- *Case $x+2 < 0$: every x is a solution, so solutions are*
  $$x \in (-\infty, -2)$$

The union of these two intervals, $(-\infty, -2)$ and $\left[-2, \frac{-9+\sqrt{57}}{2}\right]$ is the full solution of the inequality. It is immediate to see that this union is
$$\left(-\infty, \frac{-9+\sqrt{57}}{2}\right]$$
but here is the table in this simple case

GNU13: Table for the union above



ad it is easy to see that the x's for which at least one on the condition holds are indeed $x \in \left(-\infty, \frac{-9+\sqrt{57}}{2}\right]$