# FCS
# Math: Functions
# Exercises

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## Exercises

**Exercise 1.** *We have the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 2x^2 - 3x - 2 \end{array}$$

1. *Draw the graph of $F$. Mark the intersections with the axis and the vertex.*
2. *Determine $F([1,3])$.*
3. *Determine $F((-\infty,3])$.*
4. *Find $F^{-1}(-4)$, $F^{-1}(0)$, $F^{-1}(1)$, $F^{-1}(5)$.*
5. *Determine $F^{-1}([1,5])$.*
6. *Build an invertible function from $F$ by restricting its domain and codomain.*
7. *Determine the formula for this inverse.*

*Solution.*

*First of all we draw the graph of the parabola. We find the parabola roots*

$$x_{1,2} = \frac{3 \pm \sqrt{9 + 4 \cdot 2 \cdot 2}}{4} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4} = \{-1/2, 2\}$$

*and its vertex. The $x$-coordinate of the vertex is*

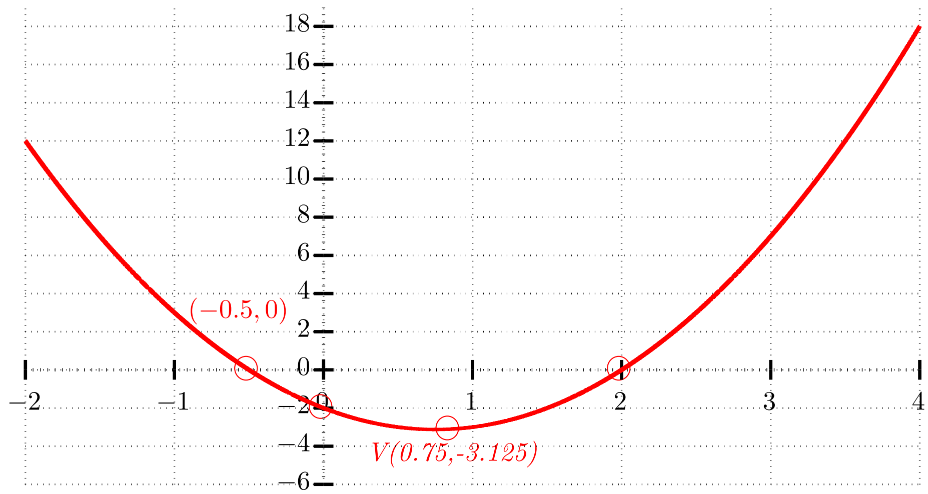
$$V_x = -\frac{b}{2a} = -\frac{-3}{4} = \frac{3}{4}$$

*and thus the $y$-coordinate of the vertex is*

$$V_y = F(3/4) = 2 \cdot \frac{9}{16} - 3\frac{3}{4} - 2 = -\frac{25}{8} \simeq -3.125$$

*The vertex is* $V = (3/4, 25/8) \simeq -3.125$. *The intersection with the* $x = 0$ *axis is* $(0, -2)$

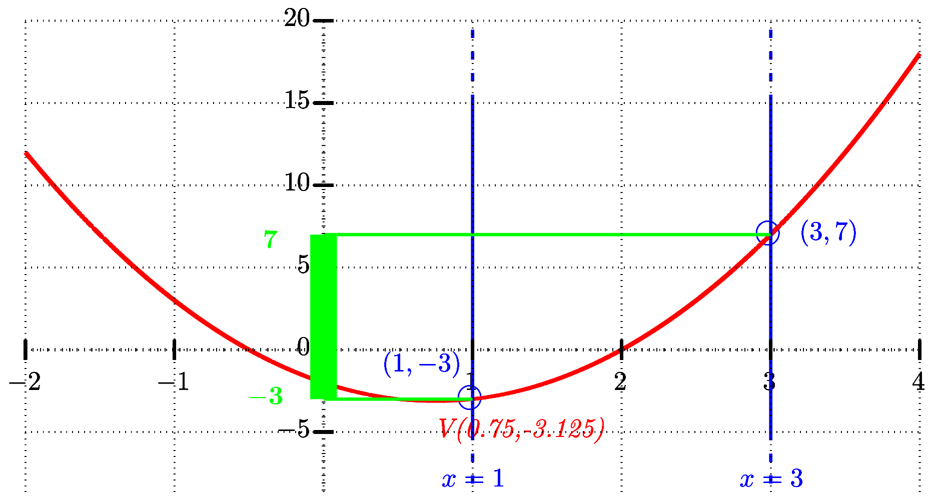
*GNU1:* $y = 2x^2 - 3x - 2$




*To compute* $F([1, 3])$ *and* $F((-\infty, 1])$ *we draw the parabola and the two vertical lines* $x = 1$ *and* $x = 3$. *The two intersections are*

$$(1, -3) \text{ and } (3, 7)$$

*GNU2:* $y = 2x^2 - 3x - 2$



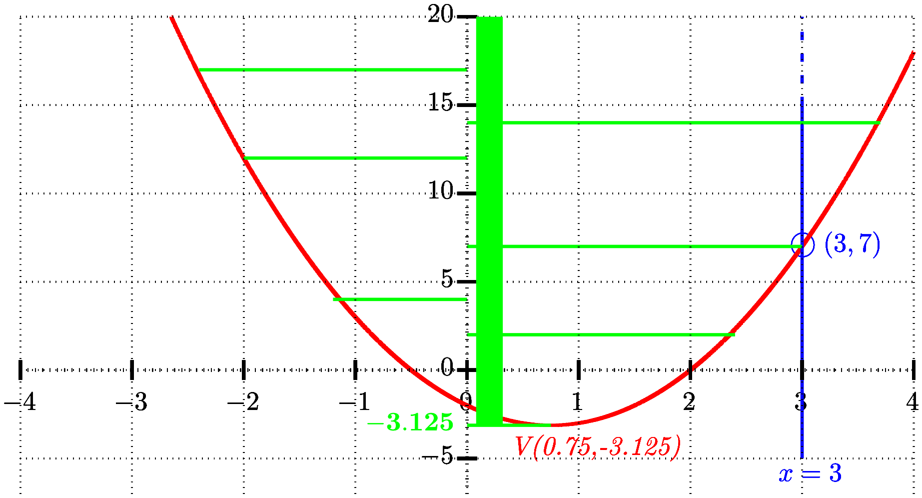
*We see that*

$$F([1,3]) = [-3,7] \text{ the thick green line}$$

*and since the second intersection,* $(1,-3)$*, is higher than the vertex, and to the right of the vertex,*

$$F((-\infty,3]) = [-25/8,+\infty) =\simeq [-3.125,+\infty) \text{ the thick green line}$$

*GNU3:* $y = 2x^2 - 3x - 2$




*To compute the intersections of the parabola* $y = 2x^2 - 3x - 2$ *and the horizontal lines* $y=-4$*,* $y=0$*,* $y=1$*,* $y=5$ *we solve the equations*

$$2x^2 - 3x - 2 = -4, \quad 2x^2 - 3x - 2 = 0, \quad 2x^2 - 3x - 2 = 1, \quad 2x^2 - 3x - 2 = 5$$

*whose solutiona are, respectively*

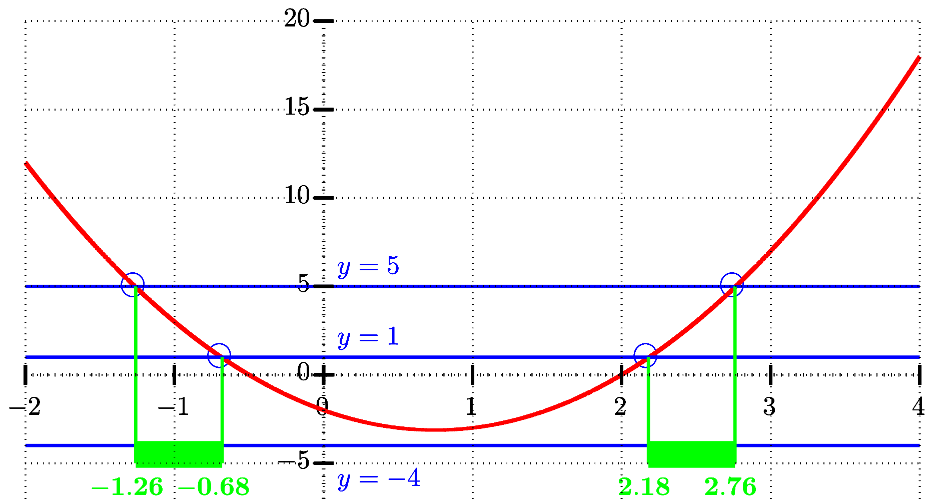
- *for the first,* $\emptyset$*, even without using the formula, obvious from the graph or from the fact that the lowest point in the parabola, the vertex, has* $y$*-coordinate* $-3.125$*, higher than* $-4$*.*
- *For the second,* $x_{1,2} = -1/2, 2$*, we have already computed the roots of the parabola, the points of intersection with the line* $y = 0$*.*
- *For the third,*

$$x_{1,2} = \frac{-9 \pm \sqrt{9+4\cdot 2\cdot 3}}{4} = \frac{-9 \pm \sqrt{33}}{4} = \frac{3 \pm 5.74}{4} =\simeq -0.68, 2.18$$

- *For the fourth,*

$$x_{1,2} = \frac{3 \pm \sqrt{9+4\cdot 2\cdot 7}}{4} = \frac{3 \pm \sqrt{65}}{4} = \frac{3 \pm 8.06}{4} =\simeq -1.26, 2.76$$

*GNU4:* $y = 2x^2 - 3x - 2$




*So*

$$F^{-1}(-4) = \emptyset \quad F^{-1}(0) = \{-1/2, 2\}$$

$$F^{-1}(1) = \{\frac{-9-\sqrt{33}}{4}, \frac{-9+\sqrt{33}}{4}\} \simeq -0.68, 2.18\} \quad F^{-1}(1) = \{\frac{3-\sqrt{65}}{4}, \frac{3+\sqrt{65}}{4}\} \simeq \{-1.26, 2.76\}$$

*and*

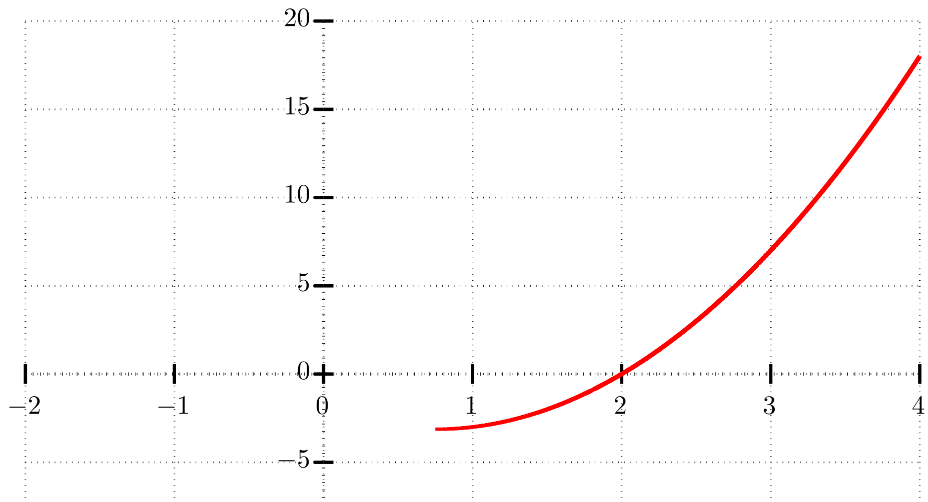
$$F^{-1}([1,5]) = [\frac{3-\sqrt{65}}{4}, \frac{-9-\sqrt{33}}{4}] \cup [\frac{-9+\sqrt{33}}{4}, \frac{3+\sqrt{65}}{4}] \simeq [-1.26, -0.68] \ \cup \ [2.18, 2.76]$$

*Looking at the graph of $F$ we see that the function is invertible, for example, to the right of the vertex if we restrict the codomain to $[-25/8, +\infty)$, so the function we are considering is*

$$F_{|[\ \frac{3}{4}, +\infty)} : \quad \begin{array}{ccc} [\ \frac{3}{4}, +\infty) & \longrightarrow & [-25/8, +\infty) \\ x & \mapsto & 2x^2 - 3x - 2 \end{array}$$

*whose graph is*

*GNU5:* $y = 2x^2 - 3x - 2$ *restricted to* $[\frac{3}{4}, +\infty)$




*We want to invert this function, getting*

$$F^{-1}_{|[\frac{3}{4},+\infty)} : \left[-\frac{25}{8}, +\infty\right) \longrightarrow \left[\frac{3}{4}, +\infty\right)$$

*To compute the formula for the inverse, with* $x \in [3/4, +\infty)$ *and* $y \in [-25/8, +\infty)$, *we solve*

$$\begin{aligned} 2x^2 - 3x - 2 &= y \\ 2x^2 - 3x - 2 - y &= 0 \\ x_{1,2} &= \frac{3 \pm \sqrt{9 + 4 \cdot 2 \cdot (2 + y)}}{4} \\ x_{1,2} &= \frac{3 \pm \sqrt{25 + 8y}}{4} \end{aligned}$$

*And since* $y \in [-25/8, +\infty)$ *the square root exists. Since* $x \in [3/4, +\infty)$, *we take only the positive in the* $\pm$ *and so*

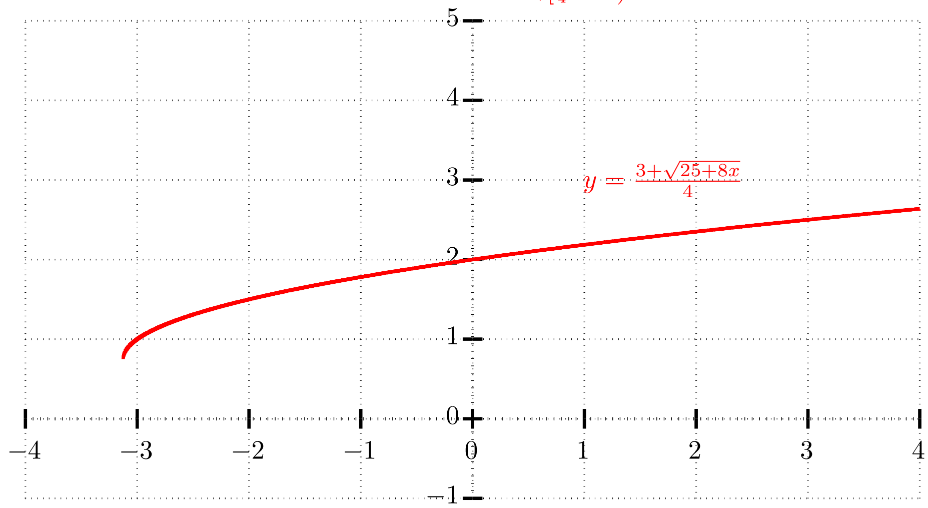
$$x = \frac{3 + \sqrt{25 + 8y}}{4}$$

*and the inverse formula is so* $\frac{3+\sqrt{25+8x}}{4}$. *The whole inverse is*

$$\begin{aligned} F^{-1}_{|[\frac{3}{4},+\infty)} : \quad [-\tfrac{25}{8}, +\infty) &\longrightarrow [\tfrac{3}{4}, +\infty) \\ x &\mapsto \frac{3 + \sqrt{25 + 8x}}{4} \end{aligned}$$

*and we draw it*

*GNU6: Graph of* $F^{-1}_{|[\frac{3}{4},+\infty)}$




**Exercise 2.** *We want to solve, with all the details, the inequality*

$$\log_3(x-2) < \log_3(3x+2)$$

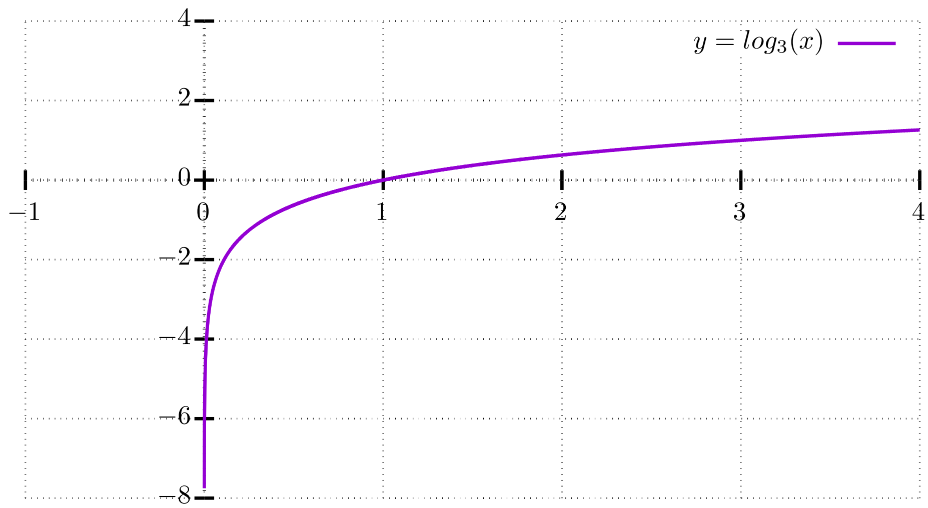
*with* $x \in \mathbb{R}$.

*Solution: the first solution is with full, excruciating details. The second is a quick and dirty exposition that would suffice as an answer in a test.*

*First solution:*

*First of all, since it involves the function* $\log_3$, *we recall the graph of this elementary function*

*GNU7: Graph of $log_3$*




*We see that the domain (also called range) is $\mathbb{R}^+$, while the codomain is $\mathbb{R}$. The function, we recall, is more precisely $\log_3 : \mathbb{R}^+ \longrightarrow \mathbb{R},\ \log_3(x) = \log_3(x)$.*

*Since we are applying the function $\log_3$ to $x-2$ and $3x+2$, we have to check for which $x \in \mathbb{R}$ we have that $x-2, 3x+2$ are in the domain of $\log_3$, which is $\mathbb{R}^+$. Since both quantities have to be oin the domani, we have the system*

$$\begin{cases} x-2>0 \\ 3x+2>0 \end{cases} \iff \begin{cases} x>2 \\ x>-2/3 \end{cases}$$

*and the solution for the system (both inequalities have to held ) is $x > 2$.*

*We now know that the constraint for the meaningful existence of the inequality is $x > 2$.*

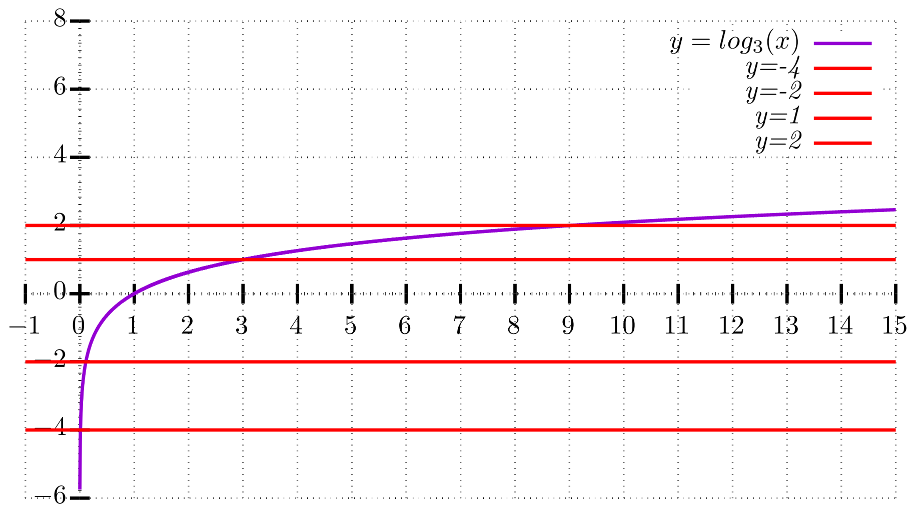
*Now we want to solve the inequality. We want to do that using the inverse function method, e.g. we want to apply a function to both parts of the inequality to simplify it.*

*We check if the function $\log_3$ is invertible, to use its inverse.*

*We can check if $\log_3$ is invertible.*

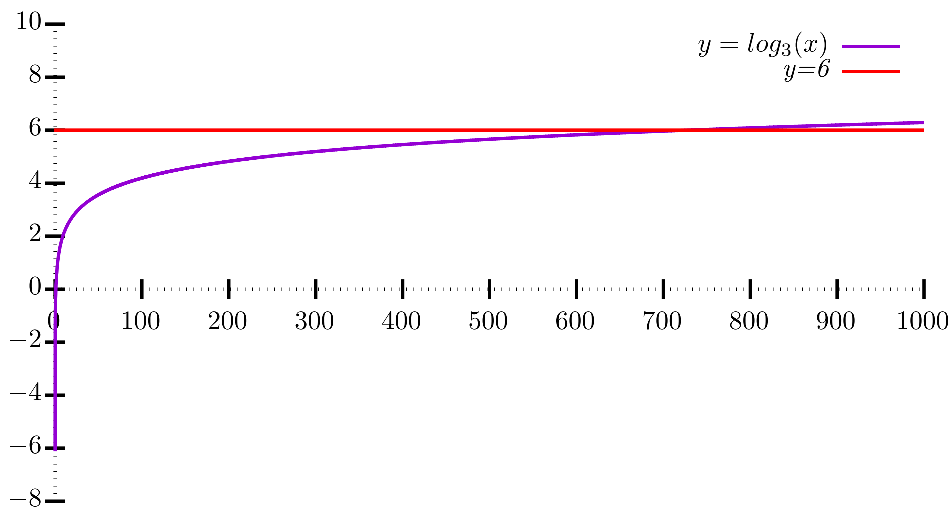
*To check the invertibility of $\log_3$, since we have the graph of the function, we check that every horizontal line $y = a$, $a \in \mathbb{R}$, the codomain of $\log_3$, intersect the graph once and only once.*

*GNU8: Invertibility of $log_3$*

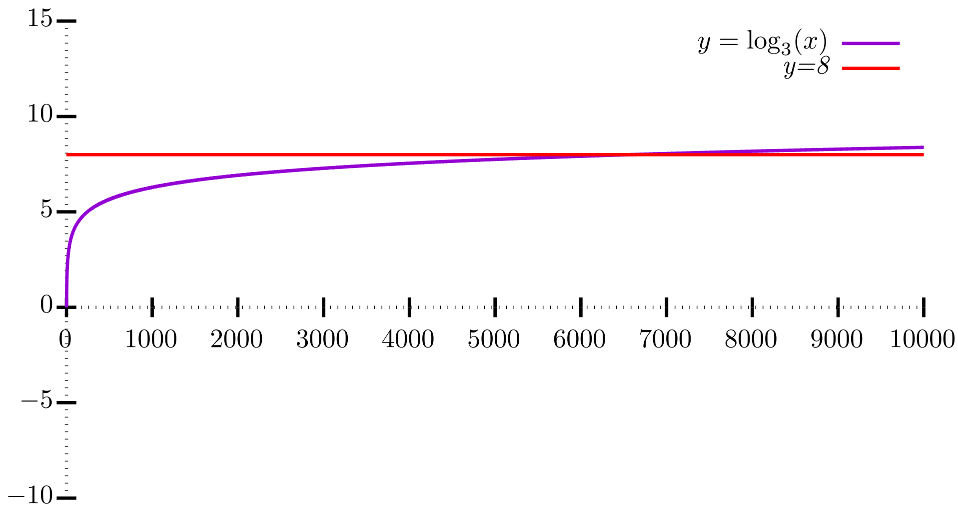



*The graph is physically limited, but we know that the* $\log_3$ *grows very slowly but without limit*

*GNU9: Invertibility of $log_3$ - big $x$*

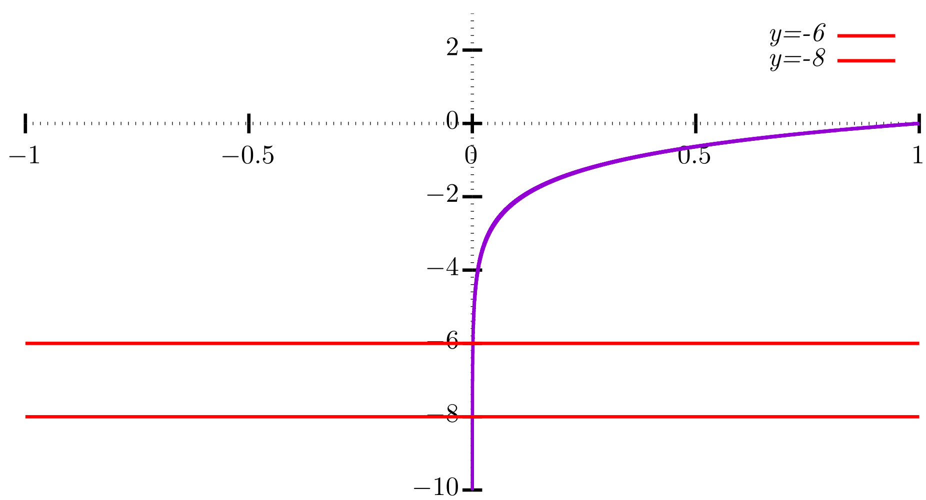


*GNU10: Invertibility of* $\log_3$ *- very big* $x$




*and that near* 0 *the function* $\log_3$ *decreases without limit*

*GNU11: Invertibility of log3 - very small x*




*So, by looking at the graph, we know that every horiziontal line* $y = a$*,* $a \in \mathbb{R}$*, intersect the graph once and only once. The function is thus invertible.*

*Indeed, we now remember that the inverse of the function*

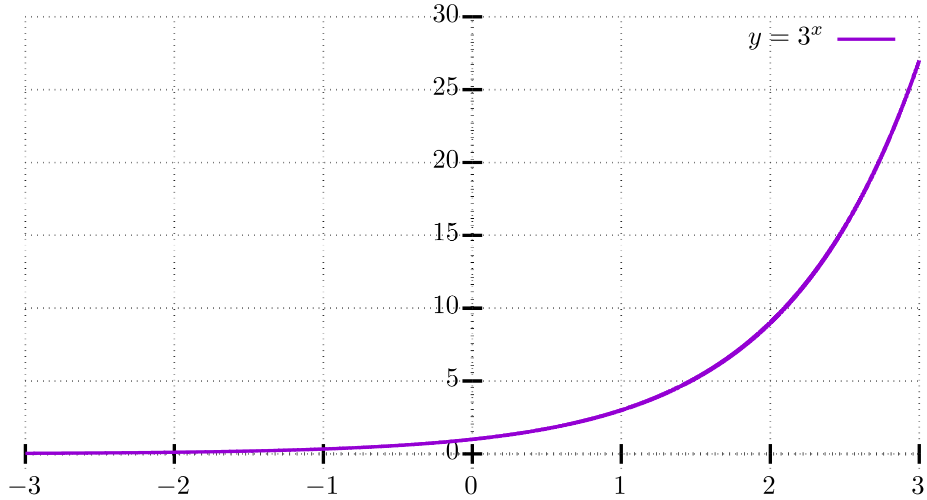
$$\begin{array}{rccc} \log_3 : & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_3(x) \end{array}$$

*is the function*

$$\begin{array}{cccc} 3^{\cdot} & \mathbb{R} & \longrightarrow & \mathbb{R}^{+} \\ & x & \mapsto & 3^{x} \end{array}$$

*whose graph is, we recall*

*GNU12: Graph of* $3^x$




*We note that the domain and codomain of the functions* $\log_3(\cdot), 3^{\cdot}$ *are switched, as necessary for one to be the inverse of the other.*

*We want to apply the function* $3^{\cdot}$ *to both parts of the inequality*

$$\log_3(x-2) < \log_3(3x+2)$$

*rembering the constraint* $x > 2$ *that we have found previously.*

*To do that, we need*

- *to check that both parts,* $\log_3(x-2), \log_3(3x+2)$ *are for every* $x > 2$, *in the domain of* $3^x$. *Since that, just by looking at the graph, this domain is* $\mathbb{R}$ *(we can also look at the graph if we don't remember), we are sure that* $\log_3(x-2), \log_3(3x+2)$ *are in the domain of* $3^x$ *for every* $x > 2$. *To apl*
- *to check that the function* $3^{\cdot}$ *is increasing or decreasing. It is obvious by looking at the graph that* $3^{\cdot}$ *is increasing.*

*We can then apply the function* $3^{\cdot}$ *to the inequality and we keep the same verse, because* $3^{\cdot}$ *is increasing.*

$$3^{\log_3(x-2)} < 3^{\log_3(3x+2)}$$

*Now, since* $\log_3(\cdot)$ *and* $3^{\cdot}$ *are the inverse of one other, we have that*

$$\forall \heartsuit \in \mathbb{R}^{+} \; 3^{\log_3(\heartsuit)} = \heartsuit \; \textit{and} \; \forall \clubsuit \in \mathbb{R} \;\; \log_3(3^{\clubsuit}) = \clubsuit$$

*because we know that*

$$f \circ f^{-1} = \text{id} \textit{ and } f^{-1} \circ f = \text{id}$$

*So we have*

$$\begin{aligned}
3^{\log_3(x-2)} &< 3^{\log_3(3x+2)} \\
x-2 &< 3x+2 \\
-2x &< 4 \\
-x &< 2 \\
x &> -2
\end{aligned}$$

*Note that to reach he last step we used the rule* if we multiply noth parts of an inequality by $-1$ we flip the inequality. *That is an occurence of the function applying method. Multiplying both parts of the inequality by* $-1$ *is exactly the same as applying to the inequality the function*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & -x \end{array}$$

*whose domain is* $\mathbb{R}$ *and decreasing, and that hence flips the inequality.*

*We have that the solutions of the inequality is*

$$x > -2$$

*We remember that we have the condition*

$$x > 2$$

*We have to consider both inequalities at the same time, e.g. we have to solve the system*

$$\begin{cases} x > -2 \\ x > 2 \end{cases}$$

*whose solution is clearly (both inequalities have to hold at the same time)*

$$x > 2$$

*The solution of the inequality*

$$\log_3(x-2) < \log_3(3x+2)$$

*is thus*

$$x > 2 \textit{ or, if you prefer, } x \in (2, +\infty)$$

*Second solution:*

*The existence field of the two* $\log_3$ *are* $x-2>0$ *and* $3x+2>0$*. We get the condition*

$$\begin{cases} x-2>0 \\ 3x+2>0 \end{cases} \Longrightarrow \begin{cases} x>2 \\ x>-2/3 \end{cases} \Longrightarrow x>2$$

*Since* $\log_3$ *is an increasing function, we have*

$$\log_3(x-2) < \log_3(3x+2) \Longrightarrow x-2 < 3x+2 \Longrightarrow x > -2$$

*If we put this together with the condition* $x>2$ *we get*

$$\begin{cases} x>2 \\ x>-2 \end{cases} \Longrightarrow x>2$$

*The solution to the inequality is thus* $(2,+\infty)$ *or the* $x \in (2,+\infty)$ *or* $x>2$*.*

**Exercise 3.** *Given the sets*

$$A = \{n^2 \mid n \in \mathbb{N}\} \text{ and } B = \{k^3 - 1 \mid k \in \mathbb{N}\}$$

*say if* $\mid A \mid = \mid B \mid$ *and find an explicit one-to-one correspondence.*

*Solution.*

*We have the functions*

$$\begin{array}{cccc} F: & \mathbb{N} & \longrightarrow & A \\ & n & \mapsto & n^2 \end{array} \quad \text{and} \quad \begin{array}{cccc} G: & \mathbb{N} & \longrightarrow & B \\ & k & \mapsto & k^3-1 \end{array}$$

*those are clearly one-to-one correspondences with inverses*

$$\begin{array}{cccc} F^{-1}: & A & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \sqrt{n} \end{array} \quad \text{and} \quad \begin{array}{cccc} G^{-1}: & B & \longrightarrow & \mathbb{N} \\ & k & \mapsto & \sqrt[3]{k+1} \end{array}$$

*So* $\mid A \mid = \mid B \mid$ *and an explicit one-to-one correspondence between* $A$ *and* $B$ *is*

$$\begin{array}{cccc} G \circ F^{-1}: & A & \longrightarrow & B \\ & n & \mapsto & G \circ F^{-1}(n) = (\sqrt{n})^3 - 1 \end{array}$$

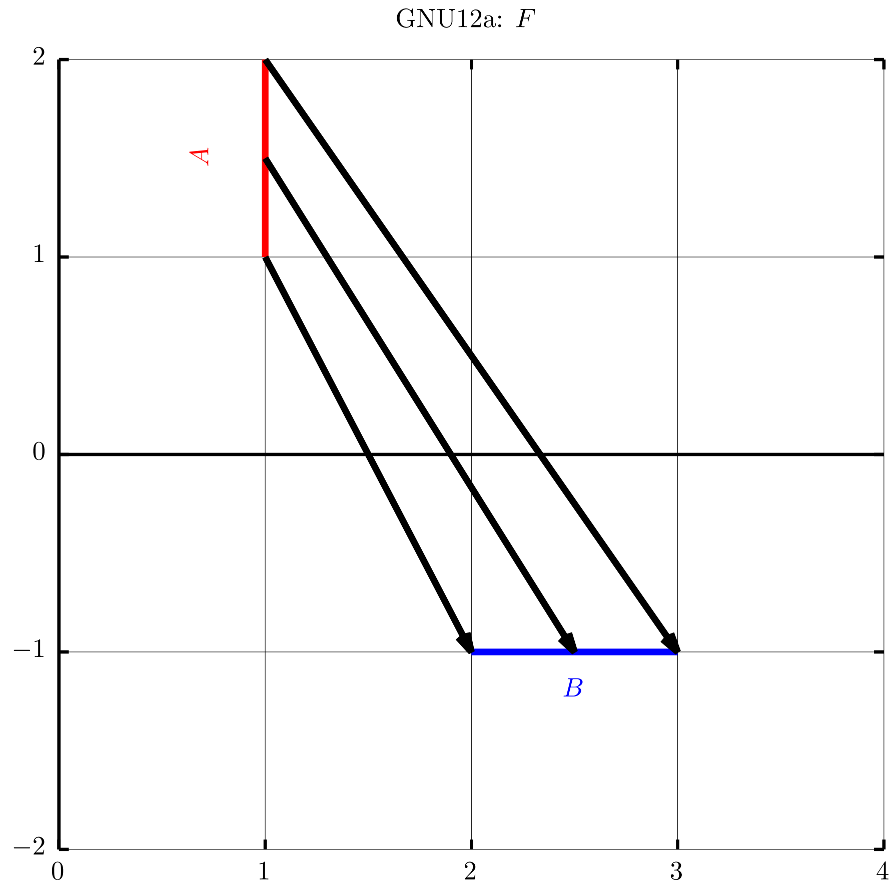
**Exercise 4.** *Given the sets*

$$A = \{(x,y) \in \mathbb{R}^2 \mid x = 1, y \in [1,2]\} \subset \mathbb{R}^2 \text{ and } B = \{(x,y) \in \mathbb{R}^2 \mid x = \in [2,3], y = -1\} \subset \mathbb{R}^2$$

*say if* $\mid A \mid = \mid B \mid$ *and find an explicit one-to-one correspondence.*

*Solution*

*We build the one-to-one correspondence* $F$ *between* $A$ *and* $B$*, thus* $\mid A \mid = \mid B \mid$*.*




**Exercise 5.** *Given the sets*

$$A = \{(x, y) \in \mathbb{R}^2 \ \mid \ x = 1, y \in [1, 2]\} \subset \mathbb{R}^2 \text{ and } B = \{(k, 1) \ \mid \ k \in \mathbb{N}\} \subset \mathbb{R}^2$$

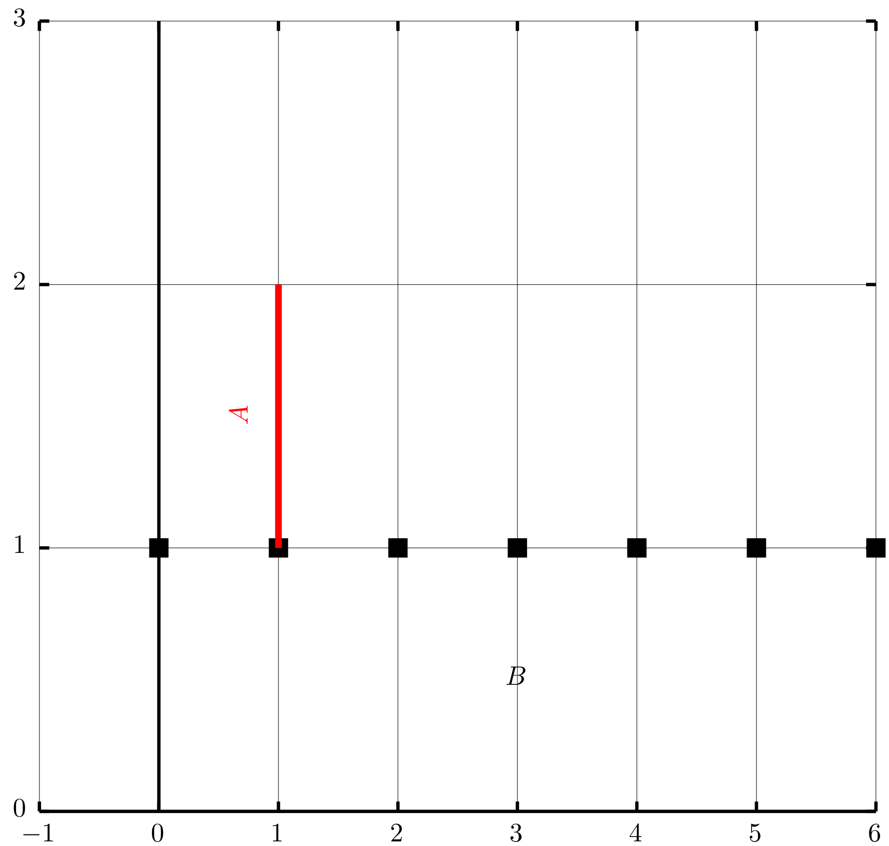
*say if* $\mid A \mid = \mid B \mid$ *and find an explicit one-to-one correspondence.*

*Solution*

*The set* $A$*, being a segment in* $\mathbb{R}^2$ *has the same carinality of* $\mathbb{R}$*, as previously shown.*

*The set* $B$ *is shown graphically below, and has clearly the same cardinality of* $\mathbb{N}$*.*

GNU12b: sets $A$ and $B$




*Since* $\mid \mathbb{R} \mid \neq \mid \mathbb{N} \mid$, *we have that* $\mid A \mid \neq \mid B \mid$

## Exercises

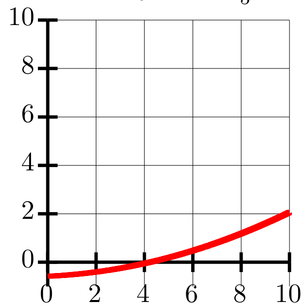
**Exercise 6.** *Are the following functions invertible? If they are, produce the inverse and draw its graph*

1.
$$\begin{array}{rccc} f: & [-3/5, +\infty) & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & 3\sqrt{5x+3}-1 \end{array}$$

*Solution: inverse is*

$$\begin{array}{rccc} f^{-1}: & \mathbb{R}_0^+ & \longrightarrow & [-3/5, +\infty) \\ & x & \mapsto & \frac{\frac{x^2+2x+1}{9}-3}{5} \end{array}$$

GNU13: $y = \frac{\frac{x^2+2x+1}{9}-3}{5}$



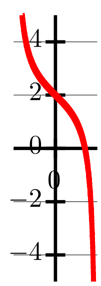

*2.*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & (-\pi/2, \pi/2) \\ & x & \mapsto & \arctan(-x+2) \end{array}$$

*Solution: inverse is*

$$\begin{array}{rccc} f^{-1}: & (-\pi/2, \pi/2) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 2-\tan(x) \end{array}$$

GNU14: $y = 2 - tan(x)$



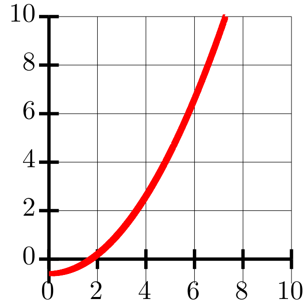

*3.*

$$\begin{array}{rccc} f: & [-3/5, +\infty) & \longrightarrow & \mathbb{R}_0^+ \\ & x & \mapsto & \sqrt{5x+3} \end{array}$$

*Solution: inverse is*

$$\begin{array}{rccc} f^{-1}: & \mathbb{R}_0^+ & \longrightarrow & [-3/5, +\infty) \\ & x & \mapsto & \frac{x^2-3}{5} \end{array}$$

GNU15: $y = \frac{x^2-3}{5}$




*4.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & (-\pi/2, \pi/2) \\ x & \mapsto & \frac{\arctan(-x+2)}{2} - 3 \end{array}$$

*Solution: function is not defined, some values are outside the codomain.*

**Exercise 7.** *Determine which of the following functions are monotone by drawing the graphs.*

*1.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sin(x) \end{array}$$

*[NO]*

*2.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \tan(x) \end{array}$$

*[NO]*

*3.*

$$f: \begin{array}{ccc} (-\pi/2, \pi/2) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \tan(x) \end{array}$$

*[YES]*

*4.*

$$f: \begin{array}{ccc} (-\pi, \pi) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sin(x) \end{array}$$

*[NO]*

*5.*

$$f: \begin{array}{ccc} (0, \pi) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \cos(x) \end{array}$$

*[YES]*

*6.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 5^x \end{array}$$

*[YES]*

*7.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 1/5^x \end{array}$$

*[YES]*

*8.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \arctan(x) \end{array}$$

*[YES]*

*9.*

$$f: \begin{array}{ccc} \mathbb{R}-\{0\} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 1/x^2 \end{array}$$

*[NO]*

*10.*

$$f: \begin{array}{ccc} \mathbb{R}-\{0\} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \log_{1/2}(x) \end{array}$$

*[Not defined]*

**Exercise 8.** *Determine where the following functions are monotone. The solutions show in any case ONE possibile monotone restriction of the functions*

*1.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2+2x+1 \end{array}$$

$[0, +\infty)$

*2.*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & -x^2+4x-3 \end{array}$$

$[2, +\infty)$

*3.*

$$f: \begin{array}{ccc} [0,4\pi] & \longrightarrow & \mathbb{R} \\ x & \mapsto & -\sin(x) \end{array}$$

$[\pi/2, 3/2\pi)$

*4.*

$$f: \begin{array}{ccc} [-\pi,\pi]-\{-\pi/2,\pi/2\} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \tan(x) \end{array}$$

$[-\pi/2, \pi/2)$ *or* $(-\pi, -\pi/2)$ *or* $(\pi/2, \pi)$.

**Exercise 9.** *Solve the following inequalities. If you use the function application method, remember to check that the function can be applied to the inequality, e.g. both parts of the inequalies belong to the function domain.*

1. $2^x < 2^{x^2+1}$. *Hint: solve with* $y = 2^x$
2. $\sqrt{x-3} > x$. *We will solve this in class.*
3. $\frac{1}{x} < \frac{1}{x+2}$. *Hint: solve* $\frac{1}{x} - \frac{1}{x+2} < 0$.
4. $\log_3(x^2) < \log_3(x)$. *Hint: solve* $\begin{cases} x > 0 \\ x^2 < x \end{cases}$.
5. $\left(\frac{1}{5}\right)^x > 5$. *Hint: solve* $\left(\frac{1}{5}\right)^x > \left(\frac{1}{5}\right)^{-1}$.
6. $\left(\frac{1}{5}\right)^{3x-1} > \left(\frac{1}{5}\right)^{2x+3}$. *Hint: Solve* $3x - 1 < 2x + 2$
7. $\log_{1/2}(x-1) < \log_{1/2}(3x+1)$. *Hint: solve* $\begin{cases} x - 1 > 0 \\ 3x + 1 > 0 \\ x - 1 > 3x + 1 \end{cases}$.