Ex $|\mathbb{N}| = |\mathbb{Z}[x]|$ ?

$\mathbb{Z}[x]$ = {POLYNOMIALS WITH $\mathbb{Z}$-COEFFICIENTS}

$x^2-3x+2$, $x^{12}-x+1$

WE KNOW $\boxed{|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Z}\times\mathbb{Z}| = |\mathbb{Z}^3| = \dots |\mathbb{Z}^n| = \dots}$

$\mathbb{Z}\times\mathbb{Z} = \mathbb{Z}^2$

$f \in \mathbb{Z}[x]$ $\quad \boxed{\begin{array}{l} \bar{f}: \mathbb{Z} \to \mathbb{Z} \\ x \rightsquigarrow f(x) \qquad x \in \mathbb{Z} \end{array}}$

---

$\mathbb{Z}[x]_{\leq 3} = \{ f \in \mathbb{Z}[x] \mid \text{degree of } f \text{ is } \leq 3 \}$

$x^3+2$, $x-1$ $\quad$ $\cancel{x^4+2}$

NEW QUESTION

$|\mathbb{N}| = |\mathbb{Z}[x]_{\leq 3}|$ ?

$f: \mathbb{N} \to \mathbb{Z}[x]_{\leq 3}$ 1-TO-1

POLY OF DEG 0
1
2
3

WE KNOW

$|\mathbb{N}| = |\mathbb{Z}^3|$

$= |\mathbb{Z}^4|$

$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

$\mathbb{Z}^3 = \{(a,b,c) \mid a,b,c \in \mathbb{Z}\}$

POLY OF DEGREE 0 : $\mathbb{Z}$ $\quad$ 0, 3, 1, 101, −32 OK

POLY OF DEGREE 1 : $ax+b$ $\quad$ $a, b \in \mathbb{Z}$

$f_2$ : {POLY OF DEGREE 1} $\longrightarrow \mathbb{Z}\times\mathbb{Z} = \mathbb{Z}^2 = \{(a,b) \mid a,b \in \mathbb{Z}\}$

$ax+b \rightsquigarrow (a,b)$

$f_2(3x-1) = (3,-1)$

$5x+2 \leftsquigarrow (5,2)$

$f_2$ IS 1-TO-1 (Ex)

POLYS OF DEGREE 2 : $f_3$ : {POLYS OF DEG 2} $\longrightarrow \mathbb{Z}^3$

$ax^2+bx+c \rightsquigarrow (a,b,c)$

$f_3$ IS 1-TO-1

$f_n$ : {POLYS OF DEGREE $n-1$} $\longrightarrow \mathbb{Z}^n$ IS 1-TO-1

$a_{n-1}x^{n-1}+\dots+a_1x+a_0 \rightsquigarrow (a_{n-1}, a_{n-2}, \dots, a_1, a_0)$

WHAT WE HAVE PROVED : $|$ {POLYS OF DEG. $n$} $| = |\mathbb{N}|$

$|\mathbb{Z}[x]_{\leq 4}| = |\mathbb{N}|$

$x^3+1$, $x^2-1$, $x^4$, ......

$\mathbb{Z}[x]_{\leq 4}$ = {POLYS OF DEG 0} — $|\mathbb{N}|$
$\cup$ {POLYS OF DEG 1} — $|\mathbb{N}|$
$\cup$ {POLYS OF DEG 2} — $|\mathbb{N}|$
$\cup$ {POLYS OF DEG 3} — $|\mathbb{N}|$
$\cup$ {POLYS OF DEG 4} — $|\mathbb{N}|$

$\Rightarrow |\mathbb{Z}[x]_{\leq 4}| = |\mathbb{N}|$ $\quad$ (Ex FIND THE EXACT FUNCTION)

---

$|\mathbb{Z}[x]_{\leq n}| = |\mathbb{N}|$ AS BEFORE

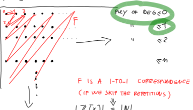
WE STILL HAVE $|\mathbb{Z}[x]| = |\mathbb{N}|$ ?

{1, 2, 3} FINITE

{1, 2, ..., n} FINITE

{1, 2, 3, .... } = $\mathbb{N}$ NOT FINITE

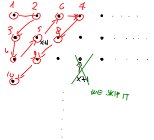
$\mathbb{Z}[x]$




F IS A 1-TO-1 CORRESPONDENCE (IF WE SKIP THE REPETITIONS)

$\Downarrow$

$|\mathbb{Z}[x]| = |\mathbb{N}|$




DEG ≤ 0 $|\mathbb{N}|$

DEG ≤ 1

DEG ≤ 2

IF WE SKIP THE REPET. IS 1-TO-1

$2/2 = 1$ = $(2,2)$

$(3,3)$ = $3/3 = 1$

$|\mathbb{Z}[x]| = |\mathbb{N}|$ IS EASY TO SHOW

$|\mathbb{Q}| = |\mathbb{N}|$ $\Rightarrow$ $|\mathbb{Q}[x]| = |\mathbb{N}|$

---

WE KNOW $|\mathbb{Q}| \leq |\mathbb{R}|$

$\mathbb{Q}$

$3/2, 1, -9$

$\mathbb{R}$

$3/2, 1, -9$

$\sqrt{3}, \sqrt{9+\sqrt{2}}$

$\pi, e, (\sqrt{2})^{\sqrt{2}}$

QUESTION: IF WE ADD ALL THE ROOTS $\sqrt{p/q}$ TO $\mathbb{Q}$, WE CHANGE CARDINALITY

NO EVERY ROOT IS A SOLUTION OF A POLY. EQ. $\sqrt{2} \Rightarrow x^2 - 2 = 0$

THERE ARE AS MANY POLY EQ AS $|\mathbb{N}|$

$\Downarrow$ $|\mathbb{Q} + \text{ROOTS}| = |\mathbb{N}|$

$|\mathbb{Q} + \text{ROOTS} + \pi + e| \overset{?}{=} |\mathbb{N}|$ YES

$\sqrt{3} + \pi + e^2$ /

$\sqrt{5}/\pi + \pi^2/\sqrt{2}$

WE ADD $\pi$ $\mathbb{Q} + \pi = \{a + b\pi \mid a, b \in \mathbb{Q}\}$

$|\mathbb{Q} + \pi| = |\mathbb{N}|$

$= |\mathbb{Q}^2|$

A "REAL" REAL NUMBER IS CALLED TRANSCENDENT

A REAL NUMBER "LINKED TO $\mathbb{Q}$" IS CALLED ALGEBRAIC

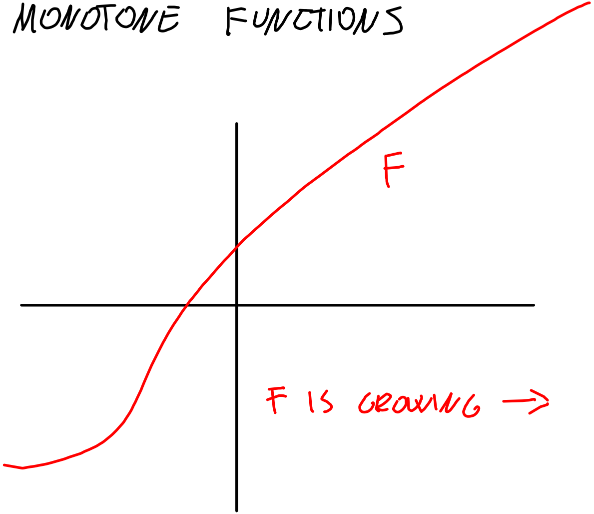
SOLUTION OF ALG. EQUATION $x^2 - 2 = 0$ FOR $\sqrt{2}$

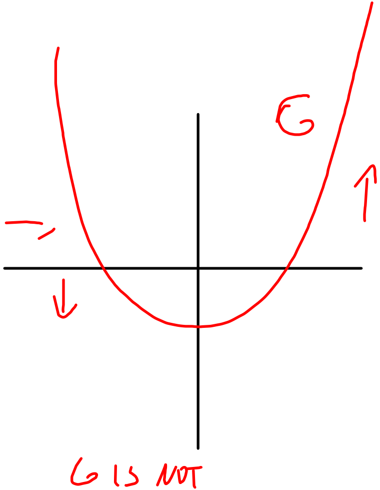
WHAT WE HAVE SHOWN IS THAT MOST REAL NUMBERS ARE TRANSENDENT.

: HOME WORK : FIND AS MANY TRANSCENDENT NUMBERS YOU CAN (USE INTERNET)

$\pi$, $e$ ARE TRANSCENDENT

# MONOTONE FUNCTIONS



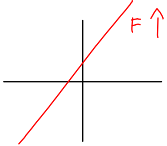

DEFINITION: $F: \mathbb{R} \to \mathbb{R}$

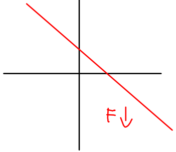
F IS INCREASING ($\uparrow$) IF $a > b \Rightarrow F(a) > F(b)$

F IS DECREASING ($\downarrow$) IF $a > b \Rightarrow F(a) < F(b)$

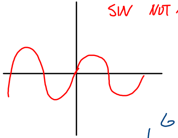
IF F IS $\uparrow$ OR $\downarrow$ F IS CALLED MONOTONE

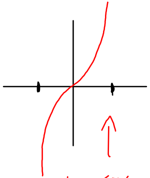
F




SIN NOT MONOTONE



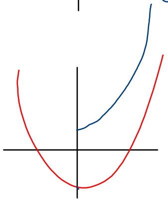


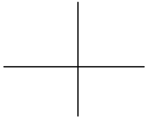
tg : $(-\pi/2, \pi/2) \to \mathbb{R}$

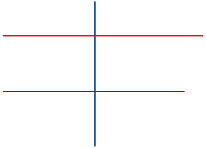
G IS MONOTONE

G: $\mathbb{R}_0^+ \to \mathbb{R}$

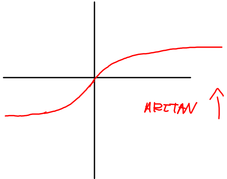
F NOT MONOTONE

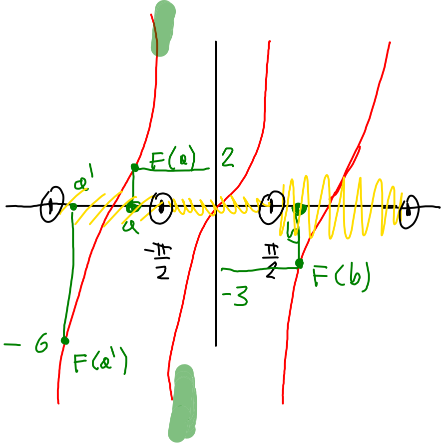






NOT MONOTONE






Eg: $\mathbb{R} - \{k\frac{\pi}{2}\} \longrightarrow \mathbb{R}$

NOT CONTINUOUS

2 -3

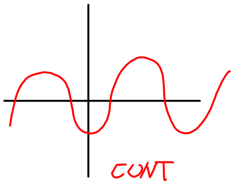
$F(a) > F(b)$ ? YES

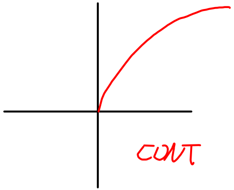
$F(a') > F(b)$

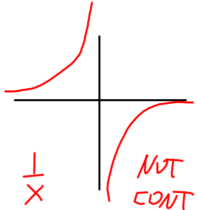
-6 -3 NO

DEF: A $F: A \to \mathbb{R}$ IS CALLED <u>CONTINUOUS</u>

IF WE CAN DRAW F WITHOUT RAISING THE PEN FROM THE PAPER



$$3^x = 4 \longrightarrow \log_3 3^x = \log_3 4$$

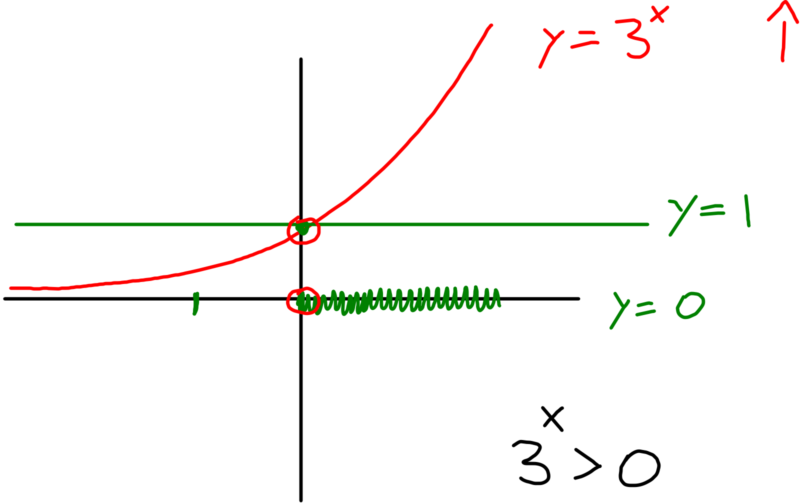
$$\Downarrow$$

$$x = \log_3 4$$

APPLY INJECTIVE FUNCTION $\log_3 x$

$$3^{x^2-2} = 3^x \longrightarrow \log_3 3^{x^2-2} = \log_3 3^x$$

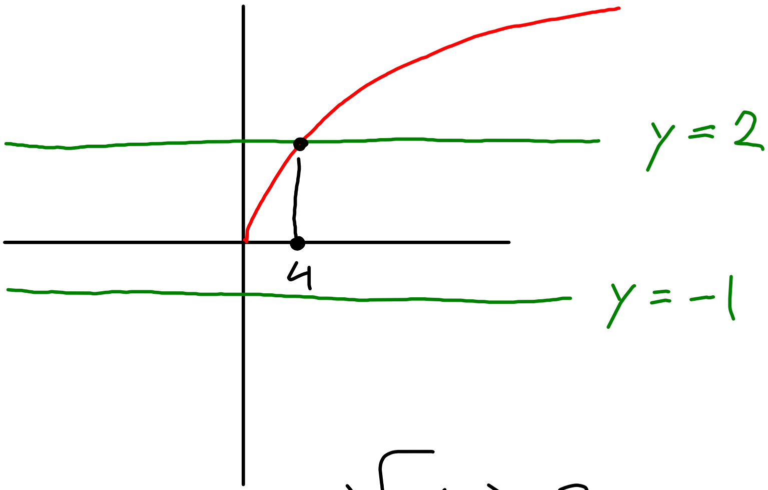
$$x^2 - 2 = x \quad \ldots\ldots\ldots$$




$3^x > 0$ $\qquad \mathbb{R}$

$3^x > 1$ $\qquad x > 0$

$y = \sqrt{x}$




$F : \mathbb{R}^+_0 \to \mathbb{R}^+_0$

$x \rightsquigarrow \sqrt{x}$

1-TO-1 (INJ)

↑

$\sqrt{x} \geq 2 \qquad x \geq 4$

$\sqrt{x} \geq -1 \qquad \mathbb{R}$

$$3^{x^2+2} > 3^{x-1}$$

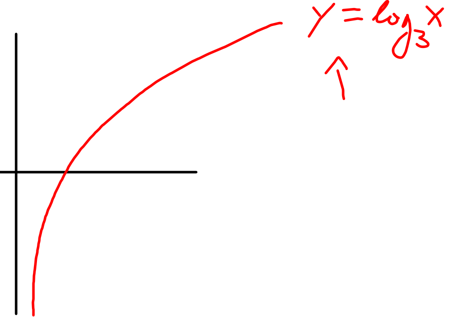
$$\Updownarrow$$

$$\log_3 3^{x^2+2} > \log_3 3^{x-1}$$

$$\Updownarrow$$

$$x^2+2 > x-1$$

$\log_3$ is $\uparrow$




WE WANT TO APPLY A FUNCTION H

$$H\left(3^{x^2+2}\right) > H\left(3^{x-1}\right) \iff 3^{x^2+2} > 3^{x-1}$$

THEOREM : $F(x) > G(H)$ INEQUALITY

H is $\uparrow$

THEN $F(x) > G(H) \iff H(F(x)) > H(G(x))$

$$\log_5(x^2+1) > \log_5(x)$$

$x > 0 = EF$

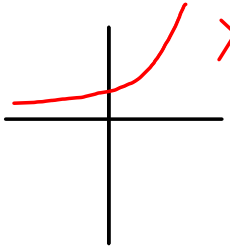
$$\Updownarrow \quad \longleftarrow \quad \log_5 \uparrow$$

$$5^{\log_5(x^2+1)} > 5^{\log_5(x)}$$

$$\Updownarrow$$

$$x^2+1 > x$$

SOLVE THIS



$y = 5^x \uparrow$

Ex

$$\log_{1/2} x > \log_{1/2}(x^2-1)$$

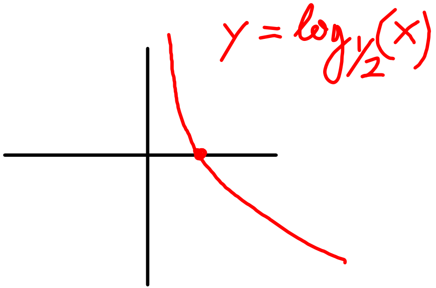
FLIPS

$$\Downarrow$$

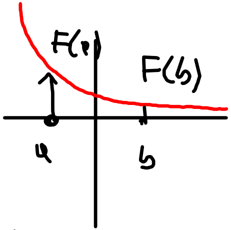
$$\left(\frac{1}{2}\right)^{\log_{1/2} x} < \left(\frac{1}{2}\right)^{\log_{1/2}(x^2-1)}$$

$$x < x^2 - 1 \quad \ldots \text{ SOLVE THIS}$$

$$\begin{cases} x > 0 \\ x^2 - 1 > 0 \end{cases} \quad EF$$




INV: $\left(1/2\right)^x$ $\Downarrow$




$$y = \left(\frac{1}{2}\right)^x = \frac{1}{2^x}$$

$b > a \qquad F(b) < F(a)$

Ex $\dfrac{1}{x+2} > \dfrac{1}{x^2-1}$

EF $\begin{cases} x+2 \neq 0 \\ x^2-1 \neq 0 \end{cases}$

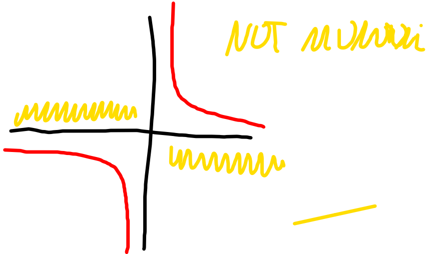
WE CAN'T APPLY $\dfrac{1}{\cancel{x}}$ TO

HOW TO SOLVE: $\boxed{\dfrac{1}{x+2} - \dfrac{1}{x^2-1}} > 0$

CAN I APPLY $1/x$

$F: \mathbb{R} - \{0\} \to \mathbb{R}$

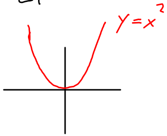
$x \mapsto 1/x$



Ex(*) $x+2 > \sqrt{-5x-2}$

$-5x-2 \geq 0$

EF




TO APPLY F WE NEED

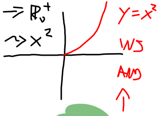
- $x+2 \in$ DOMAIN F $\iff$ $x+2 \geq 0$
- $\sqrt{-5x-2} \in$ DOMAIN F // OK

$\sqrt{\phantom{x}} \geq 0$ ALWAYS

WE RESTRICT TO $\mathbb{R}_0^+$

$F: \mathbb{R}_0^+ \to \mathbb{R}_0^+$

$x \rightsquigarrow x^2$




IF $x+2 \geq 0$

$(*) \iff (x+2)^2 > \left(\sqrt{-5x-2}\right)^2$

$x^2+4x+4 > -5x-2$

SOLVE
$$\begin{cases} x+2 \geq 0 \\ -5x-2 \geq 0 \end{cases}$$
EF

IF $x+2 < 0$ ? WHAT HAPPENS ?

WE CAN NOT APPLY F

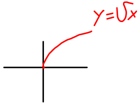
$x+2 > \sqrt{-5x-2}$ ALWAY $\geq 0$

$\Downarrow$

$x+2<0$

$\Downarrow$

NO SOLUTIONS !!




$x+2<0$ CAN'T BE BIGGER THAT A POSITIVE (OR NULL) NUMBER

WE HAD

$x+2 > \sqrt{-5x-2} \iff (x+2)^2 > -5x-2$ WITH $\begin{cases} x+2 \geq 0 \\ -5x-2 \geq 0 \end{cases}$