FCS
Math: Functions

Massimo Caboara

April 15^{th}, 2021

Definition 1. *If $F : A \longrightarrow B$ is a function, we say it is a* surjective *function if, equivalently*

1. *In the case $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, every horizontal line $y = b$ intersects the graph of F at least once.*
2. *The equation $F(x) = b$ has al least one solution for each $b \in B$.*
3. $\forall\ b \in B\ \exists\ a \in A$ *s.t.* $F(a) = b$.

Definition 2. *Let X be a set. Then the set of all the subsets of X is called* the parts of X *and indicated as $\mathcal{P}(\mathcal{X})$.*

Remark 1. *If X has cardinality $n \in \mathbb{N}$, then $\mid \mathcal{P}(X) \mid = 2^n$.*

Example 1. *If $X = \{a, b\}$ then*

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

$\mid X \mid = 2$ *and* $\mid \mathcal{P}(X) \mid = 2^2 = 4$.

Example 2. *If $X = \{a, b, c\}$ then*

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$\mid X \mid = 3$ *and* $\mid \mathcal{P}(X) \mid = 2^3 = 8$.

Teorem 1. *Let X be a set, then $\mid X \mid < \mid \mathcal{P}(X) \mid$*

Proof. If X, finite of cardinality n, the thesis follows from the previous remark, $\mid X \mid = n \Longrightarrow \mid \mathcal{P}(X) \mid = 2^n$.

If X is an infinite set, the function

$$\begin{array}{cccc} F: & X & \longrightarrow & \mathcal{P}(X) \\ & x & \mapsto & \{x\} \end{array}$$

is clearly injective, since $F(a) = F(b) \Longleftrightarrow \{a\} = \{b\} \Longleftrightarrow a = b$, and so

$$\mid X \mid \leq \mid \mathcal{P}(X) \mid$$

To prove that $| X | \neq | \mathcal{P}(X) |$ we have to prove that there is no one-to-one correspondence between X and $\mathcal{P}(X)$. We prove that by contradiction. We suppose one such one-to-one correspondence $T : X \longrightarrow \mathcal{P}(X)$ does exists and we find a contradiction.

Let $A = \{x \in X \mid x \notin T(x)\}$ be a subset of X, and hence an element of $\mathcal{P}(X)$. Since T is a one-to-one correspondence, there exists $a \in A$ such that $T(a) = A$. Since $a \in X$ and $A \subseteq X$ there are only two possibile cases: $a \in A$ or $a \notin A$.

- If $a \in A$, the a has to fulfill the condition of A, hence $a \notin A$, absurd.
- If $a \notin A$, the a is an element of X that does not fulfill the condition of A, hence $a \notin A$ is false, hence $a \in A$, absurd.

So a one-to-one correspondence like T cannot exists. Thus

$$| X | \leq | \mathcal{P}(X) | \text{ and } | X | \neq | \mathcal{P}(X) | \Longrightarrow | X | < | \mathcal{P}(X) |$$

□

Remark 2. *There are a infinite number of infinite sets with different (increasing) cardinality.*

$$| \mathbb{N} | < | \mathcal{P}(\mathbb{N}) | < | \mathcal{P}(\mathcal{P}(\mathbb{N})) | < \cdots$$

Definition 3. *If $F : A \longrightarrow B$ is a function and $C \subseteq A$ then the set*

$$Im(A) = \{F(a) \mid a \in A\} \subseteq B$$

is called the image *of A.*

Definition 4. *If $F : A \longrightarrow B$ is a function and $D \subseteq B$ then the set*

$$\{a \in A \mid F(a) \in D\} \subseteq A$$

is called the counterimage *of A.*