FCS
Math: Functions
Some solved exercises

Massimo Caboara

April 15^{th}, 2021

Some solutions will be discussed in class

Exercise 1. *Is the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 2^x + x - 1 \end{array}$$

injective, surjective, invertible? Can we make it invertibile by restriciting domain and/or codomain? In the latter case, what is the inverse function formula?

We use the algebraic method. We want to solve the equation

$$2^x + x - 1 = b$$

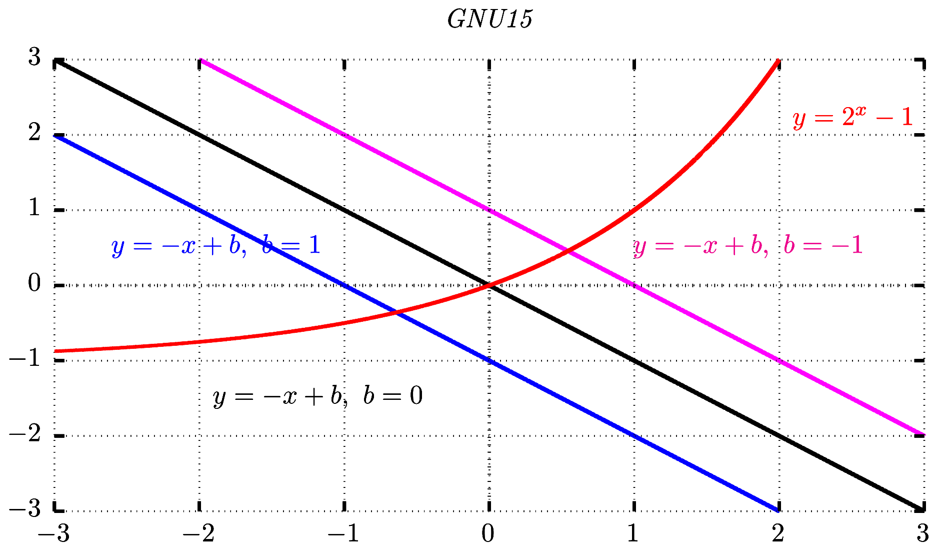
with respect to the unknown $x \in \mathbb{R}$ and the parameter $b \in \mathbb{R}$. This is not an easy equation to solve, let's try the graphical resolution. We split the equation as

$$2^x - 1 = -x + b$$

We want to know if and when the graphs of the two functions

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & 2^x - 1 \end{array}, \qquad \begin{array}{rccc} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & -x + b \end{array}$$

intersects depending on the parameter $b \in \mathbb{R}$.



We can see that there always one and only one intersection, and thus F is injective, surjective and invertible. An inverse

$$F^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$$

does exists, but we are not able to find the formula for F^{-1}

Exercise 2. *Is the function*

$$\begin{array}{rccc} h: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \arctan\left((x-1)^3\right) \end{array}$$

injective, surjective, invertible? Can we make it invertibile by restriciting domain and/or codomain? In the latter case, what is the inverse function formula?

We use the algebraic method. We solve the equation

$$\arctan\left((x-1)^3\right) = b$$

for the unknown $x \in \mathbb{R}$ and the parameter $b \in \mathbb{R}$.

We note that

$$-\pi/2 \leq \arctan(a) \leq \pi/2$$

for any a. Hence if $b \geq \pi/2$ or $b \leq -\pi/2$ there are no solution. For example, we take $b = 10 > \pi/2$, and the equation

$$\arctan\left((x-1)^3\right) = 10$$

has no solution. The function is not surjective.

Let's try to see if the function is injective. If

$$b \geq \pi/2 \text{ or } b \leq -\pi/2$$

there is no solution, OK. We have to check the number of solutions when

$$-\pi/2 < b < \pi/2$$

We can apply to the equation the invertible function

$$\begin{array}{rccc} \tan(\cdot): & (-\pi/2, \pi/2) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \tan(x) \end{array}$$

since both b *and* $\arctan\left((x-1)^3\right)$ *are always in* $(-\pi/2, \pi/2)$ *we get*

$$\begin{array}{rcl} \tan\left(\arctan\left((x-1)^3\right)\right) & = & \tan(b) \\ (x-1)^3 & = & \tan(b) \end{array}$$

We can apply to the equation the invertible function

$$\begin{array}{rccc} \sqrt[3]{\cdot}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{x} \end{array}$$

$$\begin{array}{rcl} \sqrt[3]{(x-1)^3} & = & \sqrt[3]{\tan(b)} \\ x & = & \sqrt[3]{\tan(b)} \end{array}$$

and this the one and only solution for any $b \in (-\pi/2, \pi/2)$. *The function is thus injective.*

Since the function is injective but not surjective, it is not invertible. But the function

$$\begin{array}{rccc} h': & \mathbb{R} & \longrightarrow & (-\pi/2, \pi/2) \\ & x & \mapsto & \arctan\left((x-1)^3\right) \end{array}$$

obtained from h *by restricting the codomain, is surjective. Since it is a restriction, it is still injective. It is hence invertible and its inverse is*

$$\begin{array}{rccc} {h'}^{-1}: & (-\pi/2, \pi/2) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{\tan(x)} \end{array}$$

As an exercise, check that

$$\forall\ x \in \mathbb{R} \quad h' \circ {h'}^{-1}(x) = x \text{ and } \forall\ x \in \mathbb{R} \quad {h'}^{-1} \circ h'(x) = x$$