FCS
Math: Functions
Exercises

Massimo Caboara

April 15^{th}, 2021

Exercises, full solution will be provided

Exercise 1. *Prove that* $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$ *[Solution will be discussed in class]*

Exercise 2. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 + 2x - 15 \end{array}$$

1. *Draw the graph of* F. *Mark the intersections with the axis and the vertex.*
2. *Determine* $F([1, 2])$.
3. *Determine* $F((-\infty, 1])$.
4. *Determine the intersection of the graph of* F *with the line* $y = 0$.
5. *Determine the intersection of the graph of* F *with the line* $y = -7$.
6. *Determine the intersection of the graph of* F *with the line* $y = 1$.
7. *Determine* $F^{-1}([0, 1])$.
8. *Determine* $F([-5, +\infty))$.
9. *Build an invertible function from* F *by restricting its domain and codomain.*
10. *Determine the formula for this inverse.*
11. *Find* $F^{-1}(0)$, $F^{-1}(1)$, $F^{-1}(3)$, $F^{-1}(8)$.

Solutions:

1. *First of all we draw the graph of the parabola. We find the parabola roots*

$$x_{1,2} = \frac{-2 \pm \sqrt{4+60}}{2} = \frac{-2 \pm \sqrt{64}}{2} = \frac{-2 \pm 8}{2} = \{-5, 3\}$$

and its vertex. The x-coordinate of the vertex is

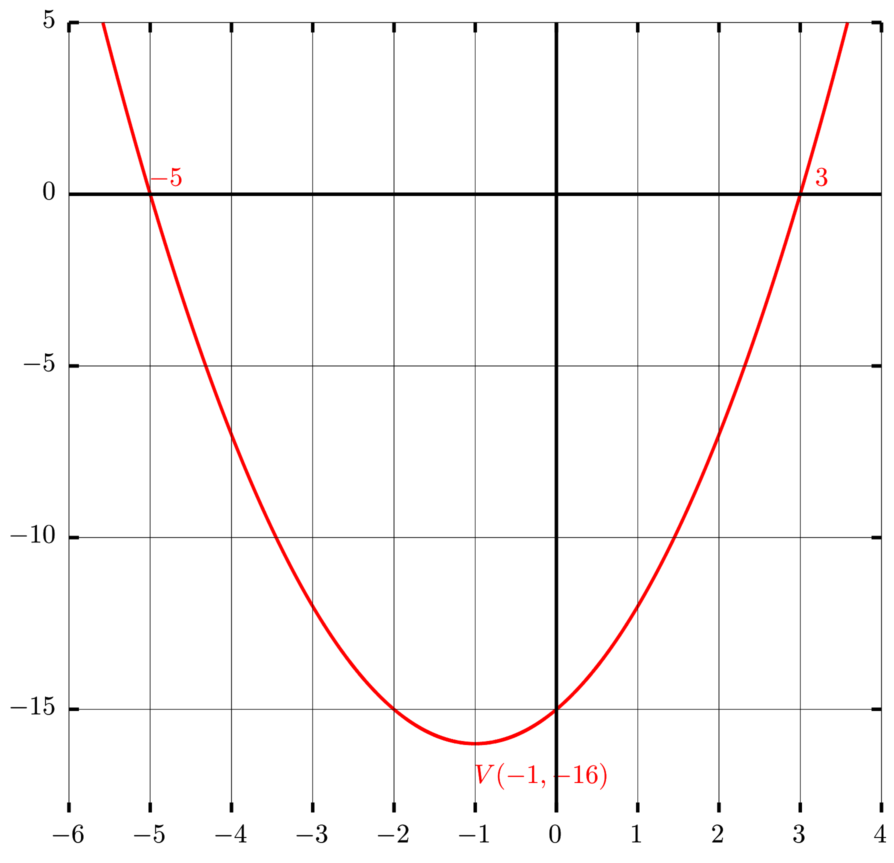
$$V_x = -\frac{b}{2a} = -\frac{2}{2} = -1$$

and thus the y-coordinate of the vertex is

$$V_y = F(-1) = (-1)^2 + 2(-1) - 15 = -6$$

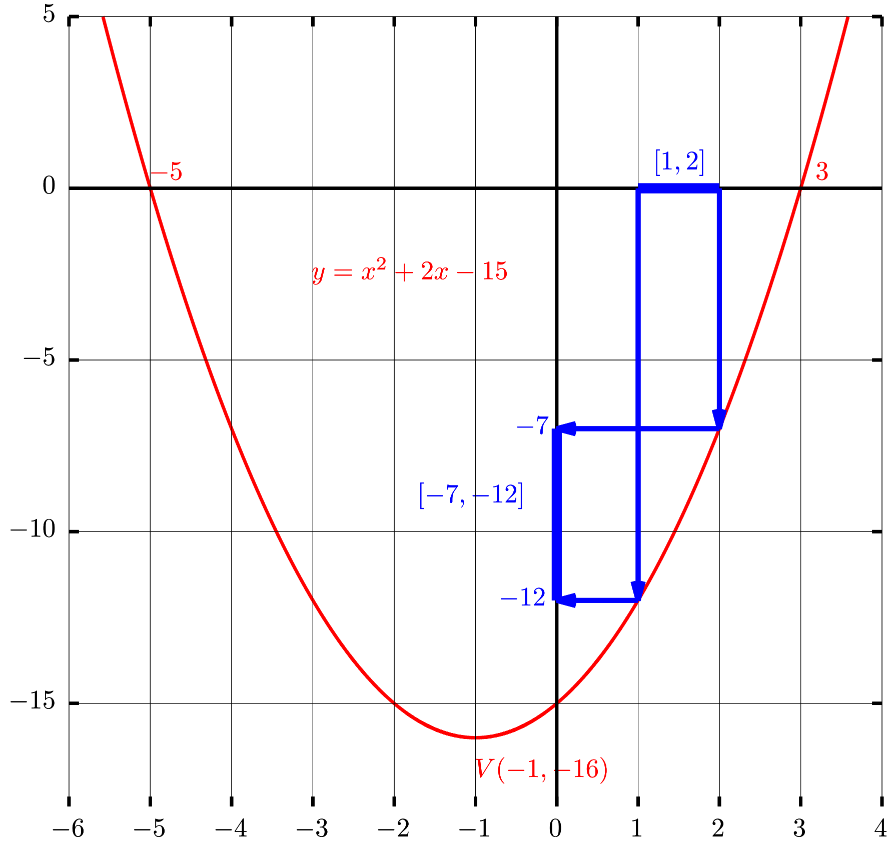
The vertex is $V = (-1, -16)$.

GNU1: Graph of F



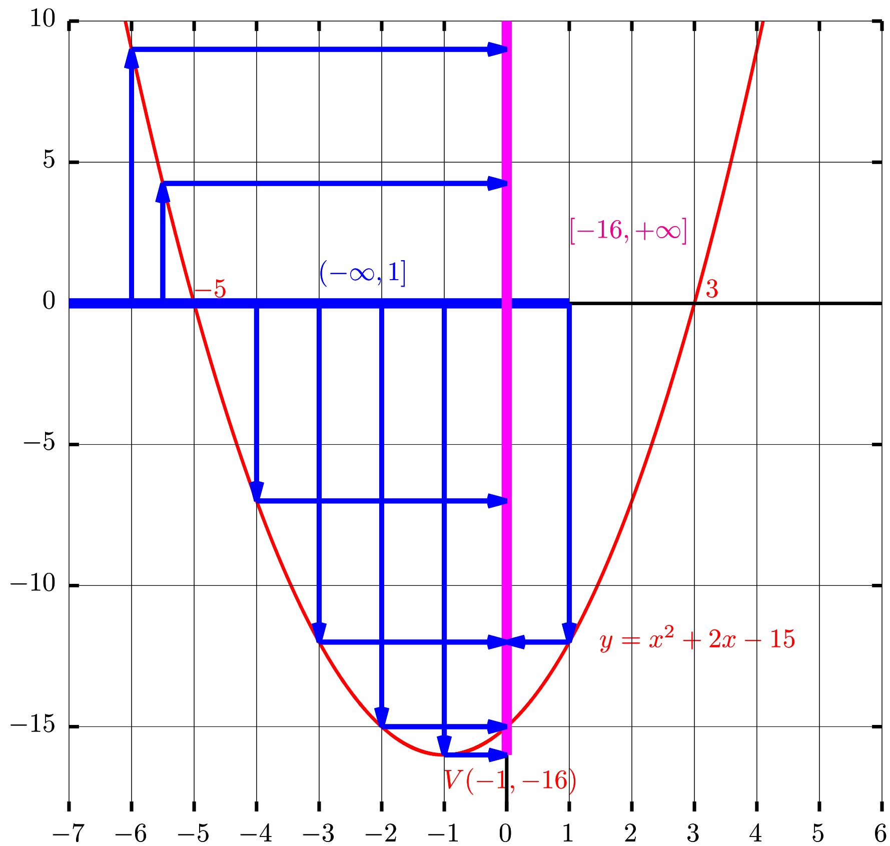
2. *Determine $F([1, 2])$.*

GNU2: $F([1,2]) = [F(2), F(1)] = [-7, -12]$



3. Determine $F((-\infty, 1])$.

GNU3: $F((-\infty,1]) = [-16,+\infty]$



4. *The intersection of* F *with the line* $y = 0$ *is* $\{(-5,0),(3,0)\}$*, the roots of the parabola.*

5. *The intersection of the graph of* F *with the line* $y = -7$ *is strictly related to solution of the equation*

$$F(x) = -7 \iff x^2+2x-15 = -7 \iff x^2+2x-8 = 0 \iff x_{1,2} = \{-4,2\}$$

The intersection is $\{(-4,-7),(2,-7)\}$.

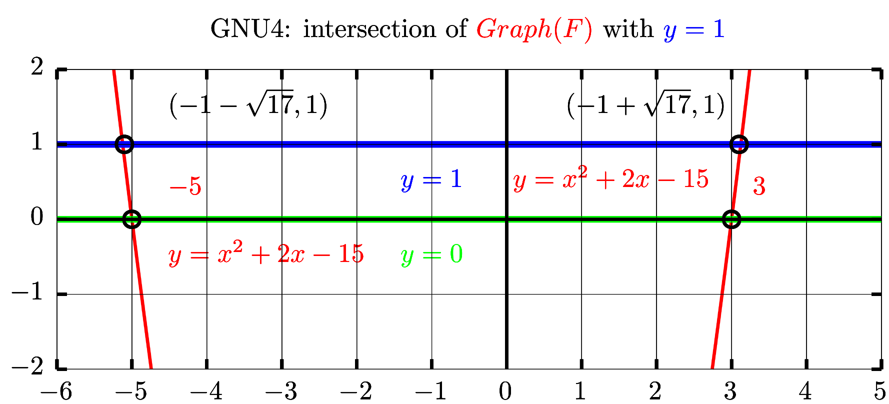
6. *The intersection of the graph of* F *with the line* $y = 1$ *is strictly related to*

solution of the equation

$$\begin{aligned} F(x) &= 1 \\ x^2 + 2x - 15 &= 1 \\ x^2 + 2x - 16 &= 0 \end{aligned}$$

$$x_{1,2} = \frac{-2 \pm \sqrt{5+64}}{2} = \frac{-2 \pm \sqrt{68}}{2} = \frac{-2 \pm \sqrt{4^2 \cdot 17}}{2} = \frac{-2 \pm 2\sqrt{17}}{2} = -1 \pm \sqrt{17}$$

The intersection is $\{(-1-\sqrt{17}, 1), (-1+\sqrt{17}, 1)\}$. *Graphically,*



Build an invertible function from F *by restricting its domain and codomain. The function*

$$\begin{array}{rccc} F' : & [-1, +\infty) & \longrightarrow & [-16, +\infty) \\ & x & \mapsto & x^2 + 2x - 15 \end{array}$$

is invertible (it follows immediatley from the graph).

The function F has inverse. To find the formula we have to solve the equation with respect to the unknown x and parameter b

$$F'(x) = b \Longleftrightarrow x^2 + 2x - 15 = b \Longleftrightarrow x^2 + 2x - (b + 15) = 0$$

We get

$$x = \frac{-2 \pm \sqrt{4 + 4(b + 15)}}{2} = \frac{-2 \pm \sqrt{64 + 4b}}{2} = -1 \pm \sqrt{16 + b}$$

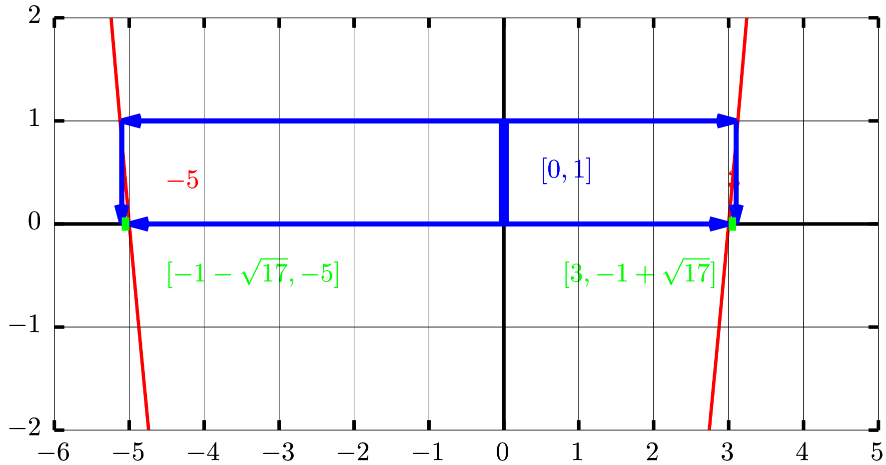
And for the inverse of F' we choose the root $x = -1 + \sqrt{16 + b}$. If we had chosen to build the inverse to the left of the vertex, and not to the right, we would have chosen the root $x = -1 - \sqrt{16 + b}$. So we have

$$\begin{array}{cccc} F'^{-1}: & [-16, +\infty) & \longrightarrow & [-1, +\infty) \\ & y & \mapsto & -1 + \sqrt{16 + y} \end{array}$$

We have

- $F^{-1}(0) = -1 + \sqrt{16 + 0} = -1 + 4 = 3.$
- $F^{-1}(1) = -1 + \sqrt{16 + 1} = -1 + \sqrt{17}.$
- $F^{-1}(3) = -1 + \sqrt{16 + 3} = -1 + \sqrt{19}.$
- $F^{-1}(8) = -1 + \sqrt{16 + 8} = -1 + \sqrt{24} = -1 + 2\sqrt{6}.$

GNU5: $F^{-1}([0,1]) = [-1-\sqrt{17}, -5] \cup [3, -1+\sqrt{17}]$



Exercise 3. *Is the function*

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & e^{2x+1}-3 \end{array}$$

injective, surjective, invertible? Can we make it invertibile by restriciting domain and/or codomain? In the latter case, what is the inverse function formula?

Solution:

We try the algebraic way, solving the equation

$$F(x) = b \Longleftrightarrow e^{2x+1} - 3 = b \Longleftrightarrow e^{2x+1} = b + 3$$

for the variable $x \in \mathbb{R}$ ($\mathbb{R}$ is the domain) and the parameter $b \in \mathbb{R}$ ($\mathbb{R}$ is the codomain).

Surgectivity: we note that the right side of the equation is always positive (check the graph of the function $f(x) = e^x$). Hence, for example, for $b = -4$ there is no solution to the equation. $\boxed{\textit{The function } T \textit{ is not surjective}}$ *. This happens for every $b \leq -3$.*

Injectivity: If $b \leq -3$ there is no solution, and this is OK. What happens if $\mathbf{b > -3}$*?*

$$e^{2x+1} = b + 3$$

We apply the invertible function (we can easily check its invertibility for example, by checking the graph)

$$\begin{array}{rccc} \log_e(\cdot): & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \log_e(x) \end{array}$$

to the equation. This is possibile since both the left and the right side are always in the domain of $\log_e(\cdot)$, $\mathbb{R}^+$.
(Remember that $b > -3$ and $\forall\ x \in \mathbb{R}\ e^{2x+1} \geq 0$). We get

$$\begin{aligned} \log_e\left(e^{2x+1}\right) &= \log_e(b+3) \\ 2x+1 &= \log_e(b+3) \\ 2x &= \log_e(b+3) - 1 \\ x &= \frac{\log_e(b+3) - 1}{2} \end{aligned}$$

we have solved the equation and there is only one solution for every b. For example

$$x = \frac{\log_e(-2+3)-1}{2} = \frac{\log_e(1)-1}{2} = -\frac{1}{2} \textit{ if } b = -2$$
$$x = \frac{\log_e(0+3)-1}{2} = \frac{\log_e(3)-1}{2} \textit{ if } b = 0$$

$\boxed{\textit{the function in injective}}$.

Since F *is surjective but not injective,* $\boxed{\textit{the function is not invertible}}$.

If we restrict the codomain form $\mathbb{R}$ *to* $[-3, +\infty)$*, we study the function*

$$\begin{array}{rccc} F' : & \mathbb{R} & \longrightarrow & [-3, +\infty) \\ & x & \mapsto & e^{2x+1} - 3 \end{array}$$

that is still injective (restricting the codomain does not change the iniectivity of a function) but is now surgjective, because

$$\textit{For all } b \in [-3, +\infty) \textit{ there is } x \in \mathbb{R} \textit{ s.t. } F'(x) = b \textit{ e.g. } x = \frac{\log_e(b+3)-1}{2}$$

So the function F'*, that is different form* F *because it has a different codomain, is invertible, and its inverse is*

$$\begin{array}{rccc} F'^{-1} : & [-3, +\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \frac{\log_e(b+3)-1}{2} \end{array}$$

Please note that

$$\forall\ x \in [-3, +\infty) \quad F' \circ F'^{-1}(x) = F'\left(F'^{-1}(x)\right) = \frac{\log_e\left(e^{2x-1} - 3 + 3\right) - 1}{2} = x$$

$$\forall\ x \in \mathbb{R} \quad F'^{-1} \circ F'(x) = F'^{-1}(F'(x)) = e^{2(\frac{\log_e(x+3)-1}{2})+1} - 3 = x$$

remember thet $\forall$ *stand for "for all". We have checked that* F'^{-1} *is indeed the inverse of* F'*. We will usually omit this check.*

Exercise 4. *Is the function*

$$\begin{array}{rccc} T: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{\sqrt[5]{x}+2} \end{array}$$

injective, surjective, invertible? Can we make it invertibile by restriciting domain and/or codomain? In the latter case, what is the inverse function formula?

We proceed algebraically, solving the equation

$$T(x) = b \quad or \quad \sqrt[3]{\sqrt[5]{x}+2} = b$$

for the unknown $x \in \mathbb{R}$ and the parameter $b \in \mathbb{R}$. We apply to the equation the invertible function

$$\begin{array}{rccc} (\cdot)^3: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^3 \end{array}$$

we obtain

$$\begin{array}{rcl} \left(\sqrt[3]{\sqrt[5]{x}+2}\right)^3 & = & b^3 \\ \sqrt[5]{x}+2 & = & b^3 \\ \sqrt[5]{x} & = & b^3-2 \end{array}$$

We apply to the equation the invertible function

$$\begin{array}{rccc} (\cdot)^5: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^5 \end{array}$$

we obtain

$$\begin{array}{rcl} \left(\sqrt[5]{x}\right)^5 & = & (b^3-2)^5 \\ x & = & \left(b^3-2\right)^5 \end{array}$$

we have always only one solution, for every $b \in \mathbb{R}$, and T is thus injective and surjective, hence invertible, and the inverse is

$$\begin{array}{rccc} T^{-1}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & (x^3-2)^5 \end{array}$$

the checking that

$$\forall\, x \in \mathbb{R} \quad F \circ F^{-1}(x) = x \text{ and } \forall\, x \in \mathbb{R} \quad F^{-1} \circ F(x) = x$$

is left as an exercise.

Exercise 5. *We have the function*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^2 - 5x + 6 \end{array}$$

1. *Determine* $f^{-1}([0, +\infty))$.
2. *Determine* $f^{-1}(\mathbb{R})$.
3. *Determine* $f^{-1}([-10, -20])$.
4. *Determine* $f^{-1}([1, 3])$.

Solution

We draw the graph of f and some lines lines, considering that the vertex V of the parabola is

$$V_x = -\frac{b}{2a} = \frac{5}{2} \text{ and } V_y = f\left(\frac{5}{2}\right) = -\frac{1}{4}$$

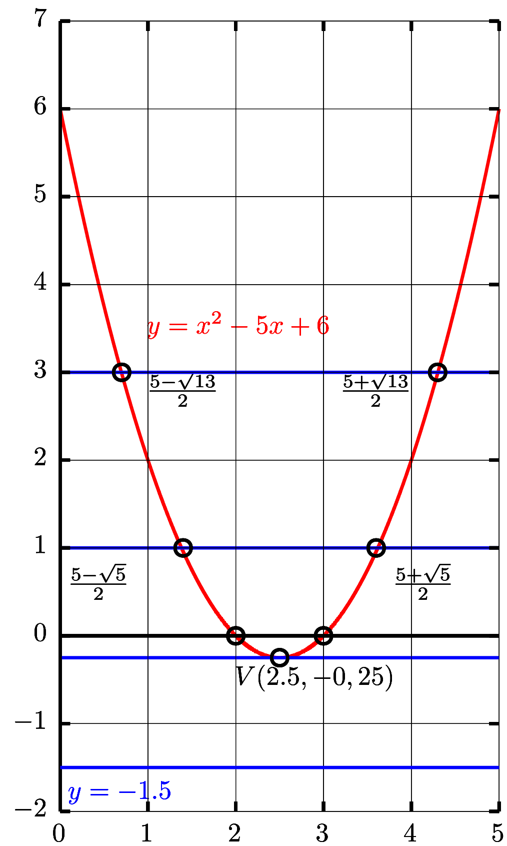
that the x-coordinates of intersections of $y = x^2 - 5x + 6$ and $y = 1$ are the solutions of

$$x^2 - 5x + 6 = 1 \text{ thus } x = \frac{5 \pm \sqrt{5}}{2} \sim 1.4, 3.6$$

and that the x-coordinates of intersections of $y = x^2 - 5x + 6$ and $y = 3$ are the solutions of

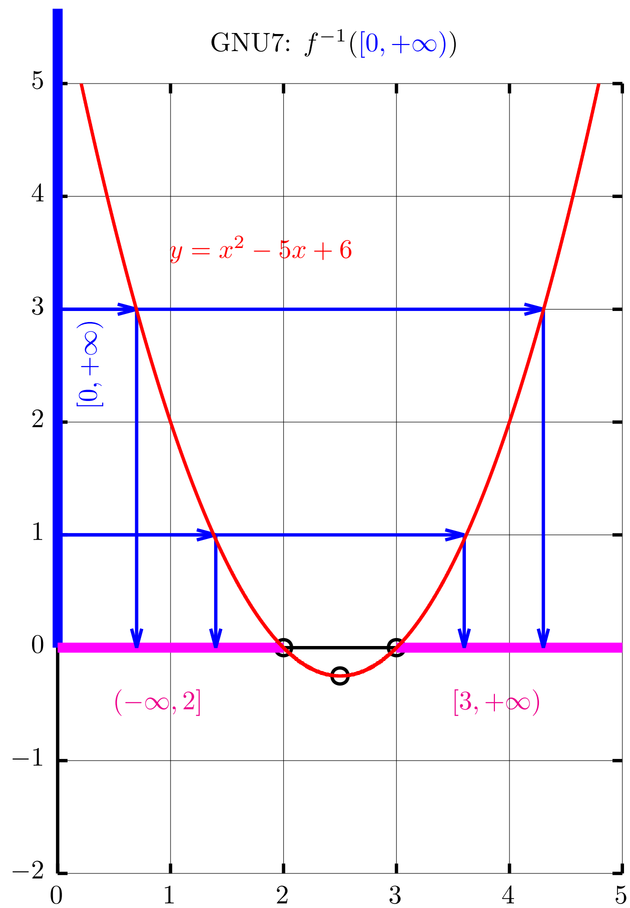
$$x^2 - 5x + 6 = 3 \text{ thus } x = \frac{5 \pm \sqrt{13}}{2} \sim 0.7, 4.3$$

GNU6: Graph of $y = x^2 - 5x + 6$

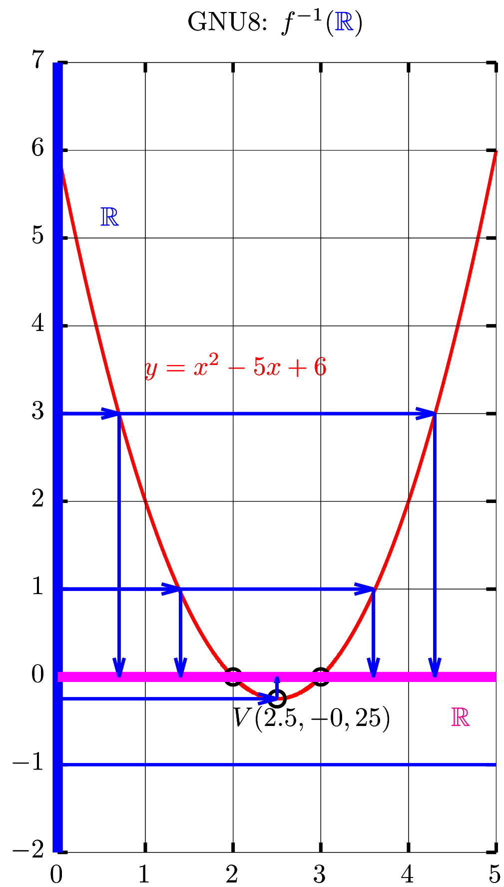


1. Looking at the graph it is easy to see that

$$f^{-1}([0, +\infty)) = (-\infty, 2] \cup [3, +\infty)$$


GNU7: $f^{-1}([0,+\infty))$
$y = x^2 - 5x + 6$
$[0,+\infty)$
$(-\infty, 2]$
$[3,+\infty)$
5
4
3
2
1
0
−1
−2
0
1
2
3
4
5

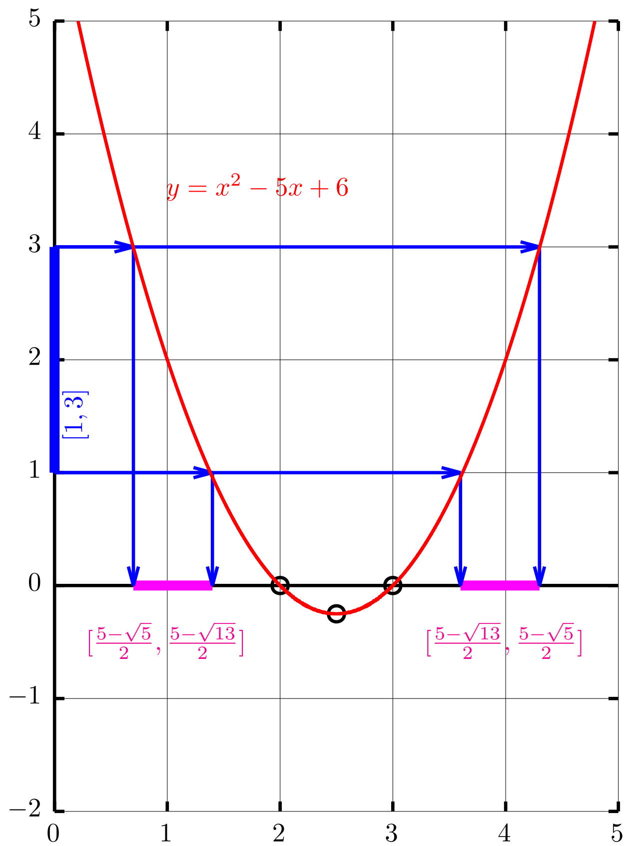
2. *Looking at the graph,* $f^{-1}(\mathbb{R}) = \mathbb{R}$.



3. *From the graph of* f *it is easy to see that* $f^{-1}([-10, -20]) = \emptyset$, *since the horizontal lines lower that* $y = -0.25$ *don't intersect the parabola, and the lines we are interested here are the ones between* $y = -10$ *and* $y = -20$.

4. *Looking at the graph* $f^{-1}([1, 3]) = [\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{13}}{2}] \cup [\frac{5-\sqrt{13}}{2}, \frac{5-\sqrt{5}}{2}]$.

GNU9: $f^{-1}([1,3])$



$y = x^2 - 5x + 6$
$[1, 3]$
$[\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{13}}{2}]$
$[\frac{5-\sqrt{13}}{2}, \frac{5-\sqrt{5}}{2}]$
5
4
3
2
1
0
−1
−2
0
1
2
3
4
5

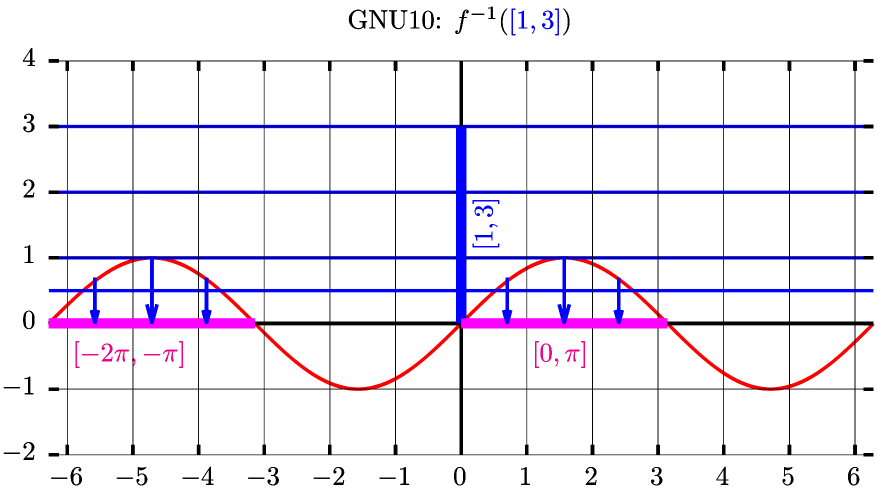
Exercise 6. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sin(x) \end{array}$$

- *Determine* $f^{-1}([0,1])$.
- *Determine* $f^{-1}([0,\sqrt{2}/2])$.

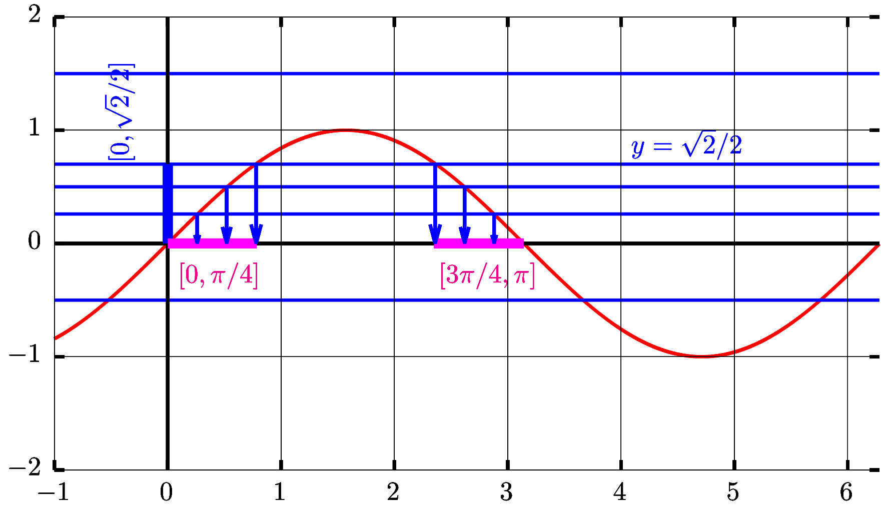
Solution

We draw the graph of the sin



and it easy to see that $f^{-1}([1,3]) = \bigcup_{k\in\mathbb{Z}}[0+2k\pi, \pi+2k\pi]$

GNU11: $f^{-1}([0,\sqrt{2}/2])$



and it easy to see that

$$f^{-1}([0,\sqrt{2}/2]) = \bigcup_{k\in\mathbb{Z}}[0+2k\pi,\pi/4+2k\pi] \cup \bigcup_{k\in\mathbb{Z}}[3/4\pi+2k\pi,2\pi+2k\pi]$$

Exercise 7. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \mapsto & 3^{x+2} \end{array}$$

1. *Say why* F *is invertible.*
2. *Find the explicit formula for* F^{-1}*.*
3. *Determine* $F^{-1}([0,1])$*.*
4. *Determine* $F^{-1}([2,4])$*.*
5. *Determine* $F^{-1}([3,+\infty))$*.*
6. *If* $a, b \in \mathbb{R}^+$*,* $a < b$*, determine* $F^{-1}([a,b])$

Solution

1. *Since* $F = f \circ g$ *is the composition of the two invertible functions*

   $$g: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x+2 \end{array} \quad \text{and} \quad f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \mapsto & 3^x \end{array}$$

   it is invertible.
2. *Solving the equation*

   $$3^{x+2} = b$$

   with respect the unknown x *and parameter* b *in the codomain* $\mathbb{R}^+$*, we find*

   $$x = \log_3(b) - 2$$

   that exists since $b > 0$*. The inverse in thus*

   $$F^{-1}: \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \mapsto & \log_3(x) - 2 \end{array}$$

3. *Determine* $F^{-1}([0,1])$*. [Later]*
4. *Determine* $F^{-1}([2,4])$*. [Later]*
5. *Determine* $F^{-1}([3,+\infty))$*. [Later]*
6. *If* $a, b \in \mathbb{R}^+$*,* $a < b$*, determine* $F^{-1}([a,b])$*. [Later]*

Exercise 8. *Consider the function*

$$\begin{array}{rccc} F: & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & n^2 - n - 6 \end{array}$$

$$\begin{array}{rcl} F^{-1}(\{0\}) & = & ? \\ F^{-1}(\{1, 2, 3\}) & = & ? \\ F^{-1}(\{-4, -6\}) & = & ? \\ F^{-1}(\{0, 1, 2, 3, 4, 5, 6\}) & = & ? \end{array}$$

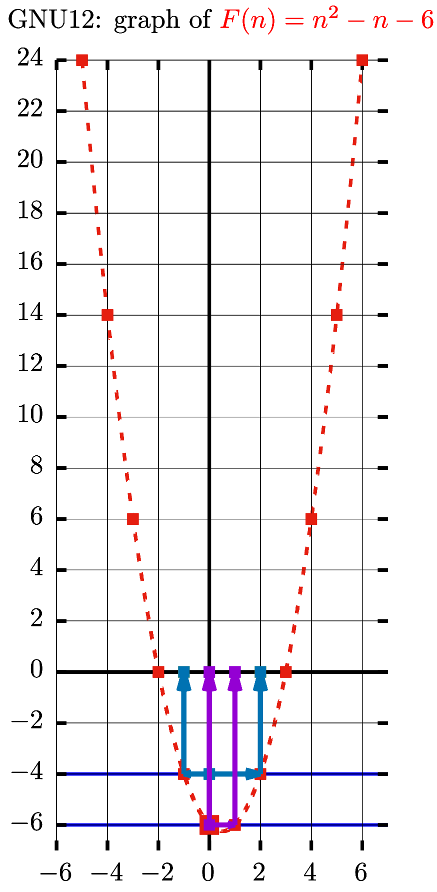
Solution

This is not a function $:\mathbb{R} \longrightarrow \mathbb{R}$, *so the graphical method is less useful. We still know that the points of the graph of* f *are on the parabola*

We compute some images

$F(-6) = 36$	$F(-5) = 24$	$F(-4) = 14$	$F(-3) = 6$
$F(-2) = 0$	$F(-1) = -4$	$F(0) = -6$	$F(0) = -6$
$F(1) = -6$	$F(2) = -4$	$F(3) = 0$	$F(4) = 6$
$F(5) = 14$	$F(6) = 24$		

We draw the "graph" of f *even if this is not a function* $:\mathbb{R} \longrightarrow \mathbb{R}$



We see that

$$F^{-1}(\{-4\}) = \{-1, 2\} \text{ and } F^{-1}(\{-6\}) = \{0, 1\} \Longrightarrow F^{-1}(\{-4, -6\}) = \{-1, 2, 0, 1\}$$

Using the same technique, we see that

$$
\begin{aligned}
F^{-1}(\{0\}) &= \{-2, 3\} \\
F^{-1}(\{1, 2, 3\}) &= \emptyset \\
F^{-1}(\{0, 1, 2, 3, 4, 5, 6\}) &= \{-3, -2, 3, 4\}
\end{aligned}
$$