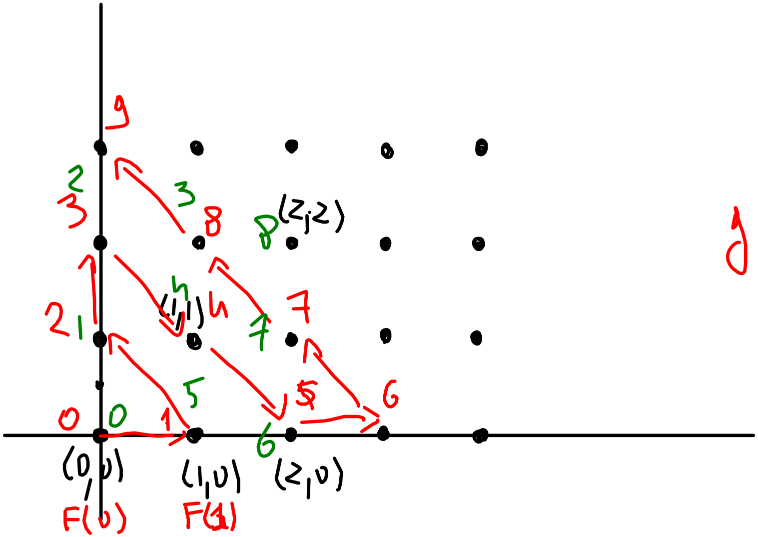
$$|\mathbb{N}| = |\mathbb{N}\times\mathbb{N}|$$



$f: \mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$

$\approx$

$g: \mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$

$\approx$

$$\mathbb{N} = \{0, 1, 2, \ldots\ldots\}$$

$$0 \in \mathbb{N}$$

$$1 \in \mathbb{N}$$

$$\vdots$$

$$\mathbb{N} \notin \mathbb{N}$$

$$\{0,1\} \notin \mathbb{N} \qquad \{0,1\} \subseteq \mathbb{N}$$

$$\mathbb{N} \subseteq \mathbb{N}$$

A = SET OF ALL SETS

$$\boxed{\begin{array}{c}\text{IF } A \text{ IS A SET} \\ A \in A\end{array}}$$

$|\mathbb{N}| = |\mathbb{N}\times\mathbb{N}|$ $\exists\, f : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ INVERTIBLE

LET' USE THE C.S.B. a) $|\mathbb{N}| \leq |\mathbb{N}\times\mathbb{N}|$

b) $|\mathbb{N}\times\mathbb{N}| \leq |\mathbb{N}|$ $\Rightarrow |\mathbb{N}| = |\mathbb{N}\times\mathbb{N}|$

WE WANT a) $f : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ INJ

$n \rightsquigarrow$

$0 \longrightarrow (0,0)$

$1 \longrightarrow (1,0)$

$2 \longrightarrow (2,0)$

$n \rightsquigarrow (n,0)$

IS f INJ? $\iff f(m) = f(n)$

$\Downarrow$

$(m,0) = (n,0)$

$\Downarrow$

$m = n$

b) $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ INJ

$(m,n) \rightsquigarrow 2^m \cdot 3^n$

$g((m,n)) = g((a,b))$

$\Downarrow$?

$(m,n) = (a,b)$

$g((m,n)) = 2^m \cdot 3^n$

$\parallel$

$g((a,b)) = 2^a \cdot 3^b$

$2^m \cdot 3^n = 2^a \cdot 3^b$

$\Downarrow$

$m = a \quad n = b$

$\Downarrow$

$(m,n) = (a,b)$

IS 6 HIT? $g((x,y)) = 6$

$g((1,1)) = 6$

$g((x,y)) = 2^x \cdot 3^y$

$g((x,y)) = 11$ NO SOLUTION

$\Downarrow$

g IS NOT 1-TO-1

$|\mathbb{N}| = |\mathbb{N}\times\mathbb{N}|$ WE KNOW

$|\mathbb{N}| = |\mathbb{N}\times\mathbb{N}\times\mathbb{N}|$? $\quad |\mathbb{N}| = |\mathbb{N}^4|$?

$|\mathbb{N}^3|$

$|\mathbb{N}| = |\mathbb{N}^n| \quad \forall n \in \mathbb{N}$ $\quad f : \mathbb{N} \to \mathbb{N}^n$ INJ

$n \rightsquigarrow (n, 0, \dots, 0)$

$\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\dots\mathbb{N}$

$g : \mathbb{N}^n \to \mathbb{N}$ INJ

$(m_1, \dots, m_n) \rightsquigarrow \sqrt[n]{m_1 \cdots m_n} \notin \mathbb{N}$

$m_1 + m_2 + \dots + m_n \in \mathbb{N}$

$(1, 0, \dots, 0) \rightsquigarrow 1$

$(0, 1, 0, \dots, 0) \rightsquigarrow 1$ $\quad$ NOT INJ

$n = 3$ $\quad f : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$

$(m_1, m_2, m_3) \rightsquigarrow m_1 + m_2 + m_3$

$(1, 0, 0) \rightsquigarrow 1 + 0 + 0 = 1$

$(0, 1, 0) \rightsquigarrow 0 + 1 + 0 = 1$

$(m_1, m_2) \rightsquigarrow 2^{m_1} \cdot 3^{m_2}$

THIS WAS OK

$(m_1, m_2, m_3, m_4) \rightsquigarrow 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdot 7^{m_4}$

$(m_1, \dots, m_5) \rightsquigarrow 2^{m_1} \cdot \dots \cdot 11^{m_5}$

$f : \mathbb{N} \to \mathbb{N}^m$ INJ

EASY

$g : \mathbb{N}^m \longrightarrow \mathbb{N}$

$(m_1, \dots, m_n) \rightsquigarrow p_1^{m_1} \cdots p_n^{m_n}$ IS INJ

p_i's ARE DIFFERENT PRIMES

BUT $g((x_1, x_2, \dots, x_n)) = 0$ NO SOLUTION

$\Downarrow$

g NO 1-TO-1

$\mathbb{Z}[x] = \{$ polynomials WITH INTEGER COEFFICIENTS $\}$

$\ni 1 + x^2, \quad x^4 + x^3 + x^{100}$

$|\mathbb{Z}[x]| = |\mathbb{N}|$?

USE $|\mathbb{N}^n| = |\mathbb{N}| \quad \forall n \in \mathbb{N}$

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \lneq |\mathbb{R}| \lneq |U_1| \lneq |U_2| \lneq |U_4| \lneq \ldots\ldots$$

$X = \{0,1\}$ $\qquad$ $\mathcal{P}(X) =$ SET OF SUBSETS OF X

$|X| = 2$ $\qquad$ $= \emptyset, \{1\}, \{0\}, \{0,1\}$ $\qquad$ $|\mathcal{P}(X)| = 4 = 2^2$

$X = \{a,b,c\}$ $\qquad$ $\mathcal{P}(X) = \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$

$|X| = 3$ $\qquad$ $|\mathcal{P}(X)| = 8 = 2^3$

$X = \{a,b,c,d\}$ $\qquad$ $|\mathcal{P}(X)| = 2^4 = 16$

$$|X| = n \overset{\text{TRUE}}{\underset{\text{BUT NOT PROVED (BY US)}}{\Longrightarrow}} |\mathcal{P}(X)| = 2^n$$

$$|\mathcal{P}(\mathbb{N})| \gneq |\mathbb{N}| \quad ?$$

A SET (INFINITE) $\quad |A| \lneq |\mathcal{P}(A)|$

$\Updownarrow$

$\nexists\ f : A \to \mathcal{P}(A)$ 1-TO-1

SUPPOSE THERE IS $f : A \to \mathcal{P}(A)$ 1-TO-1 (BY CONTRADICTION)

THIS IS AN ELEM. THIS IS A SET

WE BUILD $B = \{ x \in A \mid x \notin f(x) \}$

$\exists\ x \in B$ s.t. $f(x) = B$ YES f IS 1-TO-1

$x \in B \Rightarrow x \in B \Rightarrow x \notin B$

OR

$x \notin B \Rightarrow x \notin B \Rightarrow x \in B$

CONTRADICTION

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \dots$$

?

a) $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$ YES

b) THERE IS SOMETHING IN BETWEEN ?

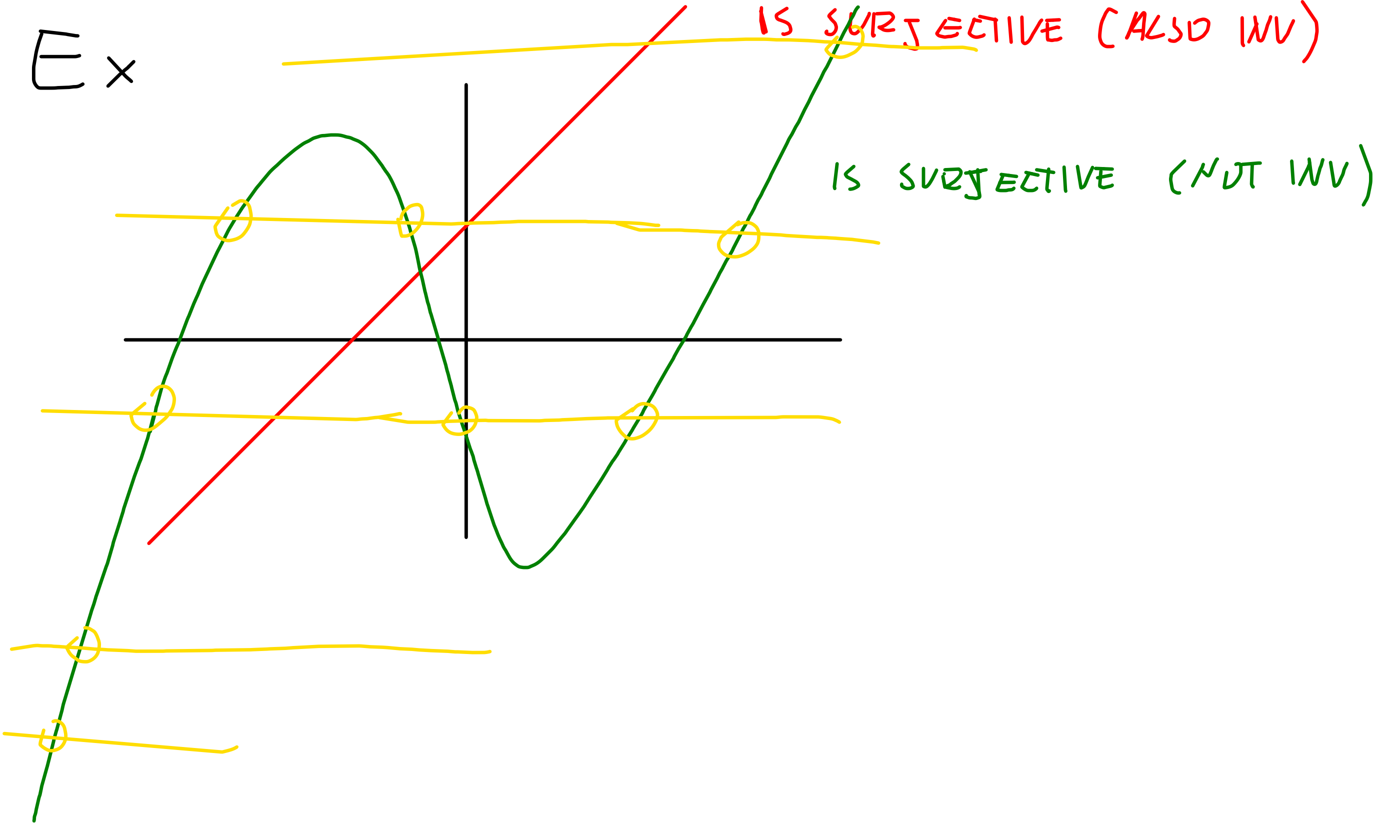
DEF $f : A \to B$ s.t. • $\forall b \in B \; \exists a \in A$ s.t. $f(a) = b$

IS CALLED A SURJECTIVE FUNCTION

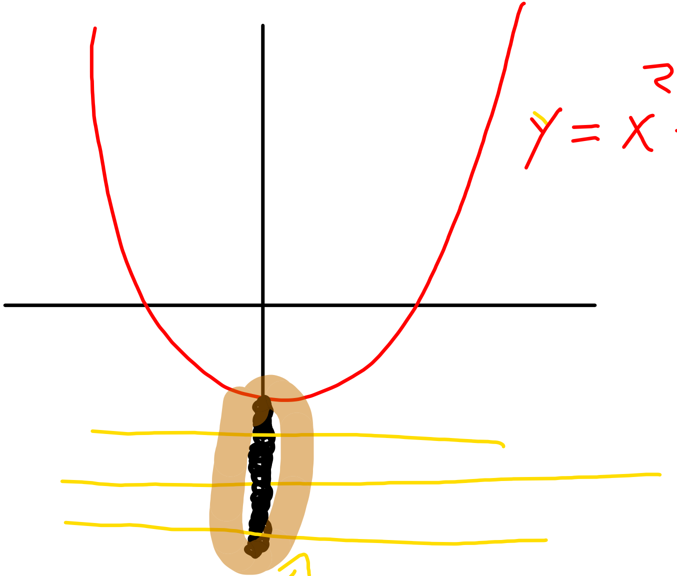
• $\forall b \in B \quad f(x) = b$ HAS SOME SOLUTIONS

• EVERY HORIZONTAL LINE $y = b$ INTERSECTS THE GRAPH OF f

Ex



Ex



$y = x^2 - 1$

NOT SURJECTIVE

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2 - 1$$

DOES NOT INTERSECT

THESE LINES ARE NOT ALLOWED

IF WE SHRINK THE CODOMAIN TO $[-1, +\infty]$

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2 - 1$$

IS SURJECTIVE

WE KNOW

FACT: $f: A \to B$ IS 1-TO-1 $\iff$ f IS INV

$\iff$ f IS INJ AND SURJECTIVE (ONTO)

VERY EASY TO SHOW

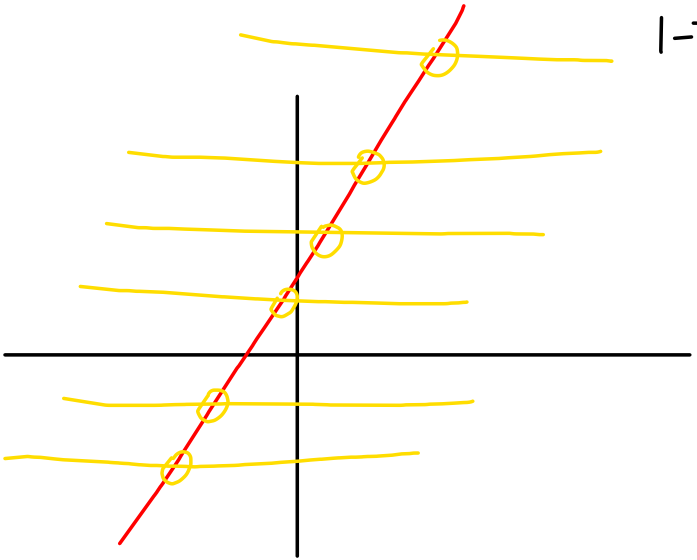
Ex $\quad f: \mathbb{R} \to \mathbb{R}$

$\quad x \rightsquigarrow 3x-2$

IS SURJ? YES

INJ? YES

1-TO-1? (INV?) YES



Ex $\quad f : \mathbb{Z} \to \mathbb{Z}$ $\quad$ is SURJ ?

$\quad\quad n \rightsquigarrow 3n-2$ $\quad$ INJ ?

INV ?

f is SURJ $\iff \forall b \in \mathbb{Z}$ $\quad f(n) = b$ HAS A SOL

$3n-2 = b$ HAS AN INTEGER SOLUTION

$$n = \frac{b+2}{3}$$ SO IF $b=2$ $\quad n \notin \mathbb{Z}$

$$n = \frac{2+2}{3} = \frac{4}{3}$$

f IS INJ $\iff \forall b \in \mathbb{Z}$ $\quad f(n)=b$ HAS 0 o 1 SOL

$3n-2 = b$ HAS 0 o 1 SOL

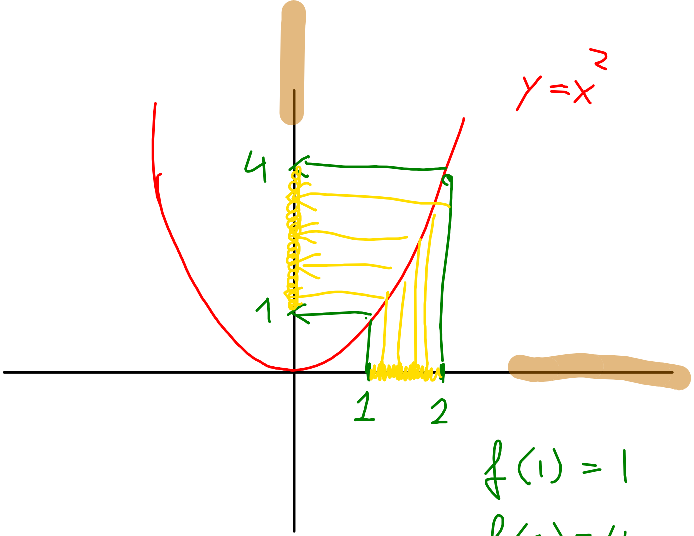
$$n = \frac{b+2}{3}$$

IF $b+2$ IS DIV. BY 3
1 SOL

$b+2$ IS NOT
0 SOL

$\Downarrow$

f IS INJ



$f:\mathbb{R}\to\mathbb{R}$

$x \mapsto x^2$

NOT INJ

NOT SURJ

NOT INV

$f(1)=1$

$f(2)=4$

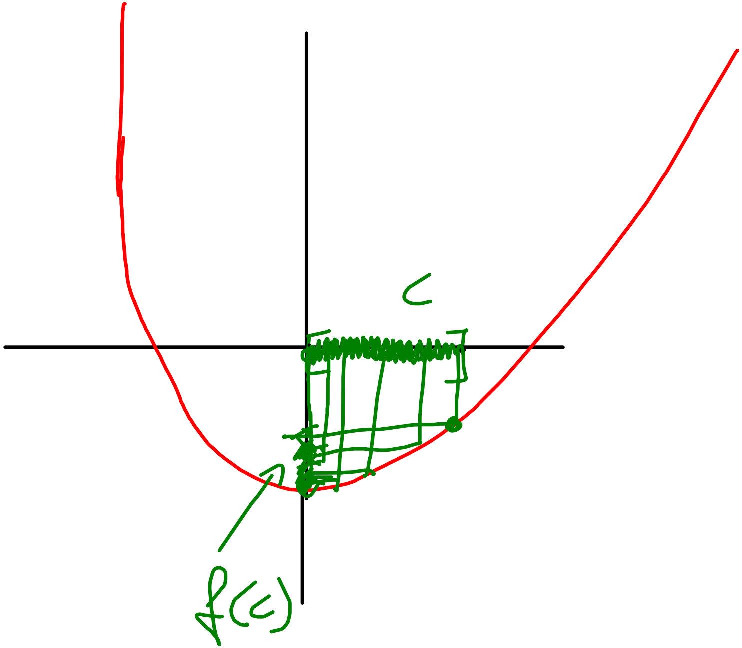
$f([1,2]) = [1,4]$

DEF $f:A\to B$ $\quad C\subseteq A$

$f(C)=\{f(a)\mid a\in C\}$ IMAGE OF C (W.R.T. f)

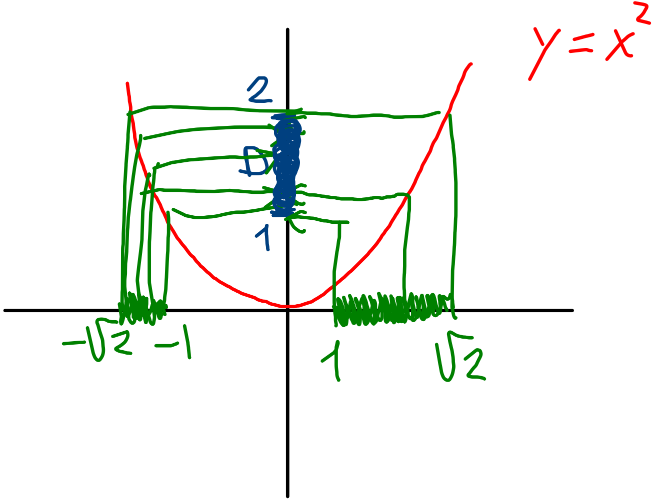
Ex

$f: \mathbb{R} \to \mathbb{R}$



$f: \mathbb{R} \to \mathbb{R}$

$x \to x^2$

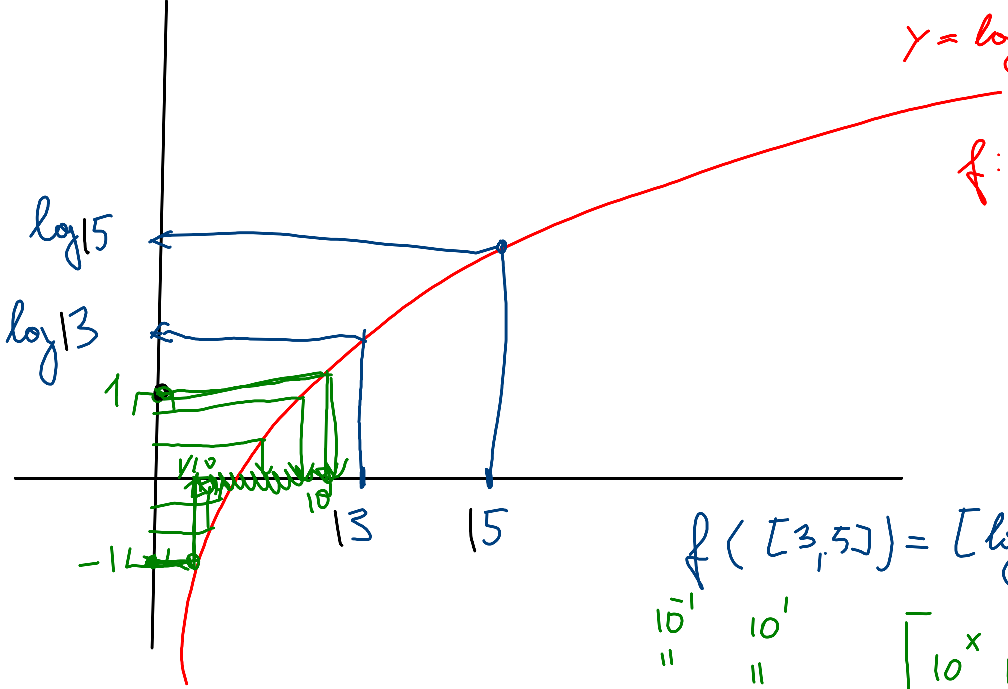


DEF $f: A \to B$ $\quad D \subseteq B$

COUNTERIMAGE$(D) = \{a \in A \mid f(a) \in D\}$

$\| \; f^{-1}(D)$ EVEN IS f IS NOT INVERTIBLE

$y = \log x$

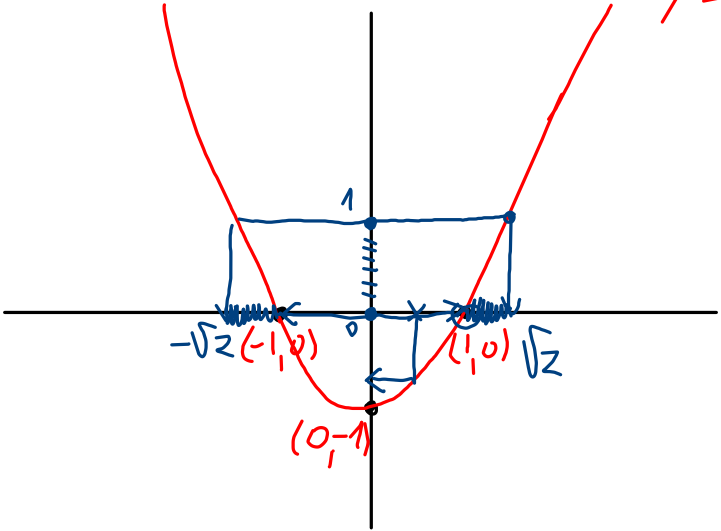


$$f : \mathbb{R}^+ \to \mathbb{R}$$
$$x \mapsto \log x$$

$$f([3,5]) = [\log 3, \log 5]$$

$$\text{COUNTERIMAGE}([-1,1]) = \left[\underset{\overset{\Vert}{10^{-1}}}{\frac{1}{10}},\ \underset{\overset{\Vert}{10^{1}}}{10}\right]$$

$$\boxed{\begin{array}{l} 10^x \text{ IS THE} \\ \text{INVERSE OF} \\ \log x \end{array}}$$



$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto x^2 - 1$

$f(0) = -1 \notin [0,1]$

$x^2 - 1 = 0$

$\Downarrow$

$\boxed{x = \pm 1}$

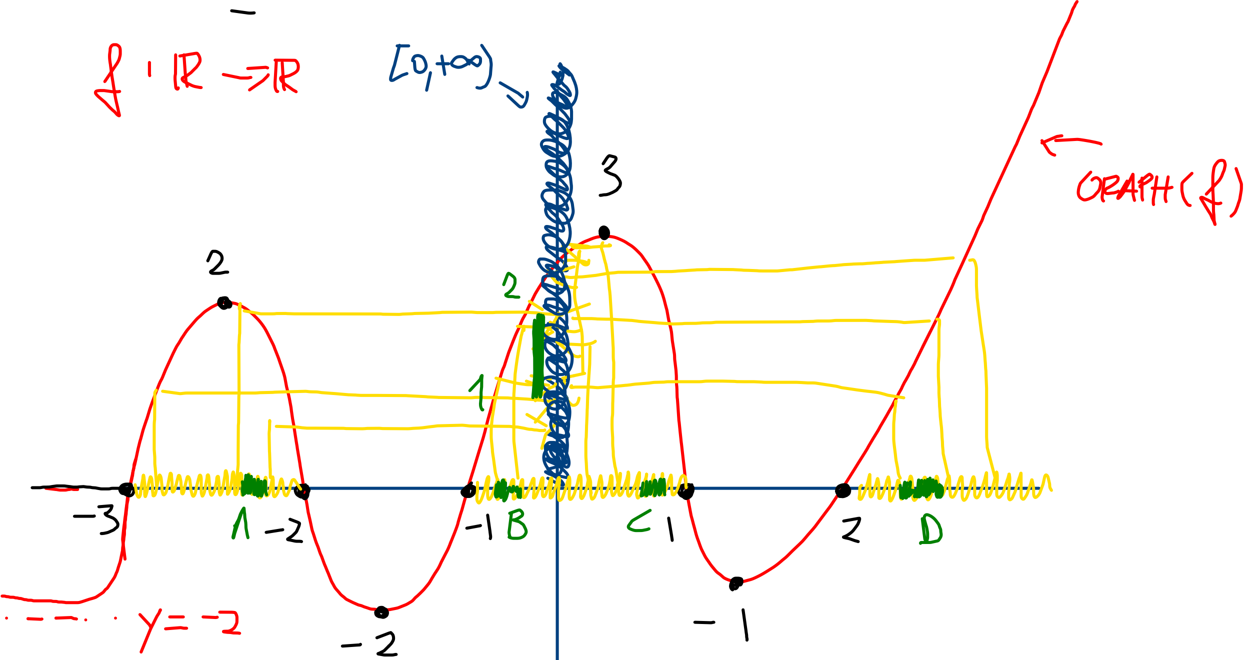
$x^2 - 1 = 1$

$x^2 = 2$

$\boxed{x = \pm\sqrt{2}}$

COUNTERIMAGE $([0,1]) = [-\sqrt{2}, \sqrt{2}]$

$= [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$



COUNTERIMAGE$([0,+\infty)) = [-3,-2] \cup [-1,1] \cup [2,+\infty)$

COUNTERIMAGE$([1,2])$ = GREEN PARTS A, B, C, D

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

$$\mathbb{N} \times \mathbb{N} \ni (3, 9)$$

$$\mathbb{N} \times \mathbb{N} \times \mathbb{N} \ni (1, 0, 7)$$

$$\underbrace{\mathbb{N} \times \mathbb{N} \times \cdots\cdots \mathbb{N}}_{\text{11 times}} \ni \overbrace{(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)}^{\text{11 places}}$$