FCS
Math: Functions
Additional exercises

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Exercise 1. *Is there an invertible function between the following sets? If the answer is affermative, give the explicit function, by formula or an explict description.* $A = \{n^2 \mid n \in \mathbb{N}\}$, $B = \{k^3 \mid k \in \mathbb{Z}\}$

The two sets to have the same cardinality, since $|A| = |\mathbb{N}|$, $|B| = |\mathbb{Z}|$ *and* $|\mathbb{N}| = |\mathbb{Z}|$. *We build the explicit function* $T : A \longrightarrow B$.

We have the functions

$$F: \begin{array}{ccc} \mathbb{N} & \longrightarrow & A \\ n & \mapsto & n^2 \end{array} \qquad G: \begin{array}{ccc} \mathbb{Z} & \longrightarrow & B \\ k & \mapsto & k^3 \end{array} \qquad H: \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{Z} \\ n & \mapsto & \begin{cases} -\frac{n}{2} & n \text{ is even} \\ \frac{n+1}{2} & n \text{ is odd} \end{cases} \end{array}$$

and their inverses

$$F^{-1}: \begin{array}{ccc} A & \longrightarrow & \mathbb{N} \\ n & \mapsto & \sqrt{n} \end{array} \qquad G^{-1}: \begin{array}{ccc} B & \longrightarrow & \mathbb{Z} \\ k & \mapsto & \sqrt[3]{k} \end{array} \qquad H^{-1}: \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{N} \\ n & \mapsto & \begin{cases} -2n & n < 0 \\ 2n-1 & n \geq 0 \end{cases} \end{array}$$

The first two functions and theirs inverses are easy to determine, and the last function and its inverse have been build in a previous exercise.

We have

$$A \xrightarrow{F^{-1}} \mathbb{N} \xrightarrow{H} \mathbb{Z} \xrightarrow{G} B$$

and we build

$$G \circ H \circ F^{-1}: \begin{array}{ccc} A & \longrightarrow & B \\ n & \mapsto & G(H(F^{-1}(n))) \end{array}$$

where

$$G(H(F^{-1}(n))) = G(H(\sqrt{n})) = G\left(\begin{cases} -\frac{\sqrt{n}}{2} & n \text{ is even} \\ \frac{\sqrt{n}+1}{2} & n \text{ is odd} \end{cases}\right) = \begin{cases} -\left(\frac{\sqrt{n}}{2}\right)^3 & n \text{ is even} \\ \left(\frac{\sqrt{n}+1}{2}\right)^3 & n \text{ is odd} \end{cases}$$

the function

$$T: \quad A \longrightarrow B$$
$$n \mapsto \begin{cases} -\left(\frac{\sqrt{n}}{2}\right)^3 & n \text{ is even} \\ \left(\frac{\sqrt{n}+1}{2}\right)^3 & n \text{ is odd} \end{cases}$$

is invertible since it is the composition of invertible functions. If we want the explicit inverse, we have

$$T^{-1} \equiv (G \circ H \circ F^{-1})^{-1} \equiv ((G \circ H) \circ F^{-1})^{-1} \equiv (F^{-1})^{-1} \circ (G \circ H)^{-1} \equiv F \circ H^{-1} \circ G^{-1}$$

and

$$\begin{aligned} F \circ H^{-1} \circ G^{-1}(n) &= F(H^{-1}(G^{-1}(n))) \\ &= F(H^{-1}(\sqrt[3]{n}) \\ &= F\left(\begin{cases} -2\sqrt[3]{n} & n < 0 \\ 2\sqrt[3]{n} - 1 & n \geq 0 \end{cases}\right) \\ &= \begin{cases} (-2\sqrt[3]{n})^2 & n < 0 \\ (2\sqrt[3]{n} - 1)^2 & n \geq 0 \end{cases} \end{aligned}$$

Exercise 2. *Is there an invertible function between the following sets? If the answer is affermative, give the explicit function, by formula or an explict description.* $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$, $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\} \subset \mathbb{R}^2$.

The two sets are described in the plane $\mathbb{R}^2$ *by the two circles of radius* 1 *and* 2. *The one-to-one correspondence between them is given by the map* F

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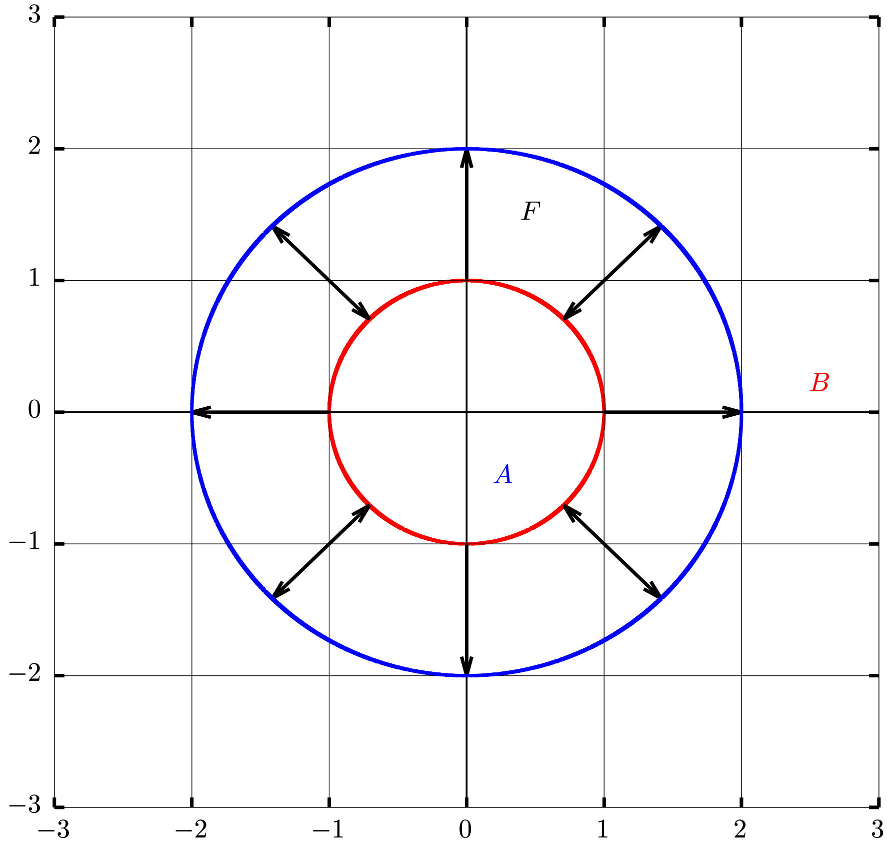
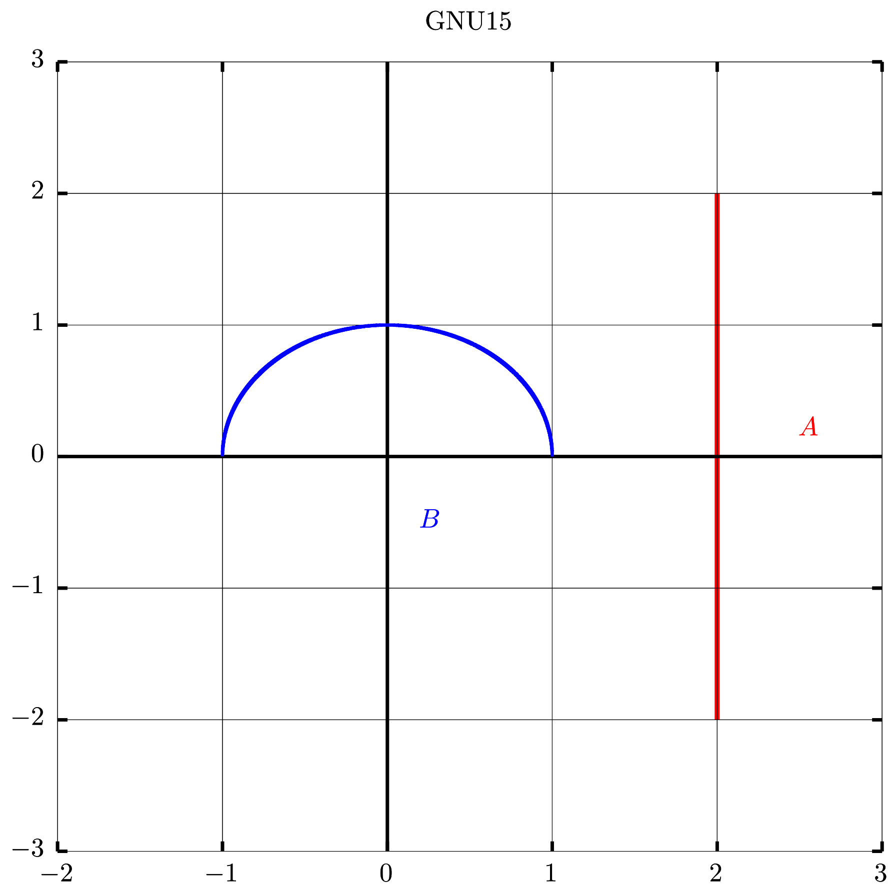
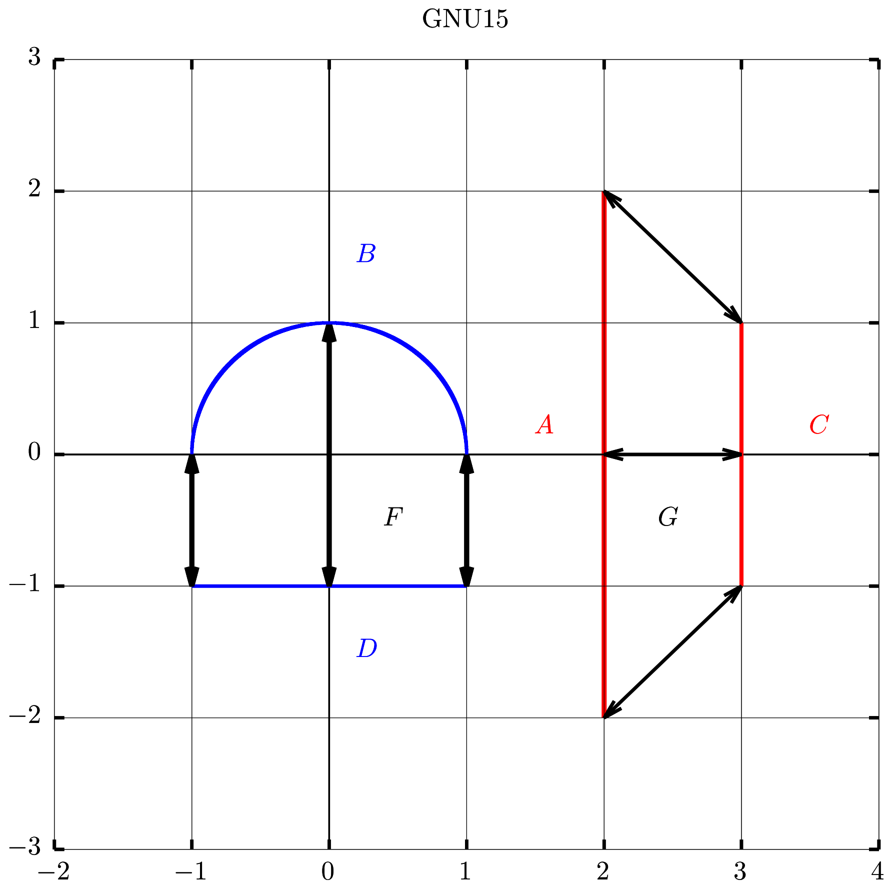


Exercise 3. *Do the subsets* A*,* B *in* $\mathbb{R}^2$ *have the same cardinality? [YES]*



We introduce the two sets C, D *as shown in the figure below. It is easy to see that* $\mid A \mid = \mid C \mid$ *and* $\mid B \mid = \mid D \mid$ *using the two one-to-one correspondences indicated as* G *and* F*, and* $\mid C \mid = \mid D \mid$ *since* C *is just a translation and rotation of* D*.*



Since $|A| = |C|$, $|C| = |D|$ *and* $|D| = |B|$ *we have* $|A| = |B|$.

Remark 1. *Note that the sets* A, B *are not the graph of any function - they fail the vertical line test. They are, though, subsets of* $\mathbb{R}^2$.