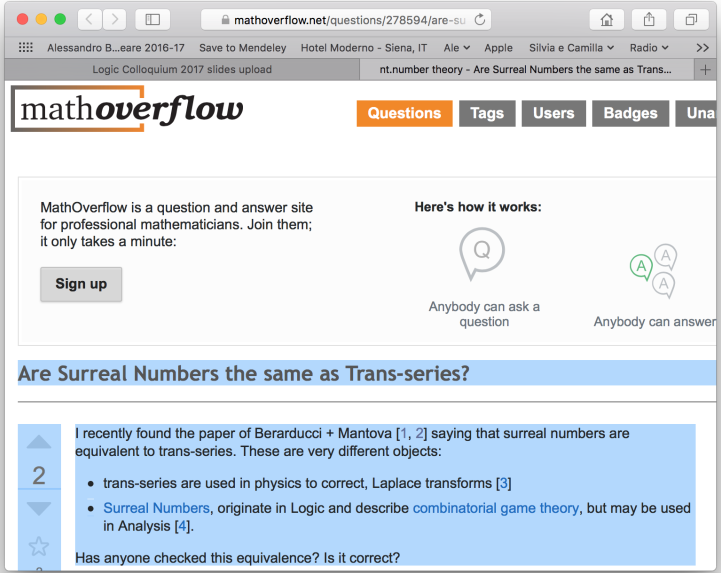
Surreal differential calculus and transseries

Alessandro Berarducci

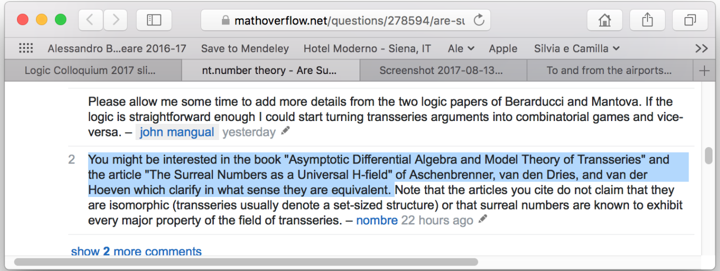
University of Pisa

Stockholm, 15 August 2017

Mathoverflow question



Mathoverwlow answer



Please allow me some time to add more details from the two logic papers of Berarducci and Mantova. If the logic is straightforward enough I could start turning transseries arguments into combinatorial games and vice-versa. – john mangual yesterday

2 You might be interested in the book "Asymptotic Differential Algebra and Model Theory of Transseries" and the article "The Surreal Numbers as a Universal H-field" of Aschenbrenner, van den Dries, and van der Hoeven which clarify in what sense they are equivalent. Note that the articles you cite do not claim that they are isomorphic (transseries usually denote a set-sized structure) or that surreal numbers are known to exhibit every major property of the field of transseries. – nombre 22 hours ago

show 2 more comments

References I

[AvdDvdH15] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven. The surreal numbers as a universal H-field. *ArXiv 1512.02267*, pages 1–17, dec 2015.

[BM15] Alessandro Berarducci and Vincenzo Mantova. Surreal numbers, derivations and transseries. *arXiv:1503.00315. To appear in: Journal of the European Mathematical Society*, pages 1–47, 2015.

[BM17] Alessandro Berarducci and Vincenzo Mantova. Transseries as germs of surreal functions. *arXiv 1703.01995*, pages 1–44, 2017.

[KS05] Salma Kuhlmann and Saharon Shelah. κ-bounded exponential-logarithmic power series fields. *Annals of Pure and Applied Logic*, 136(3):284–296, nov 2005.

References II

[vdDMM97] Lou van den Dries, Angus Macintyre, and David Marker. Logarithmic-Exponential Power Series. *Journal of the London Mathematical Society*, 56(3):417–434, dec 1997.

Hahn fields

- Let $(G, <, \cdot, 1)$ be an abelian ordered group.
- The Hahn field $\mathbb{R}((G))$ consists of series $\sum_{i<\alpha} r_i g_i$ where $\alpha \in \mathbf{On}$, $(g_i : i < \alpha)$ is decreasing in G and $r_i \in \mathbb{R}^*$.
- $\mathbb{R}((x^{\mathbb{Z}}))$ = Laurent series (with $x > \mathbb{R}$)
- If G is divisible, $\mathbb{R}((G))$ is a real closed field. Ex: $\mathbb{R}((x^{\mathbb{Q}}))$
- The Puiseux series $\bigcup_{d\in\mathbb{N}} \mathbb{R}((x^{\mathbb{Z}/d}))$ are contained in $\mathbb{R}((x^{\mathbb{Q}}))$.
- $\mathbb{R}((G))$ is maximal: it has no extensions with the same value group G and residue field $\mathbb{R}$.

Summability

- A sequence $(f_i : i \in I)$ in $\mathbb{R}((G))$ is summable if each $g \in G$ appears in finitely many f_i and the union of the supports of the f_i's is a reverse well ordered subset of G.

- In this case we can define $f = \sum_{i \in I} f_i$ as the unique element of $\mathbb{R}((G))$ such that for all $g \in G$, the coefficient $f_g \in \mathbb{R}$ is given by $\sum_{i \in I} (f_i)_g$.

- Dominated convergence fails: $\sum_{i \in I} h_i$ may not exist even if $|h_i| \leq |f_i|$ and $\sum_i f_i$ exists.

Defects: no integrals or exp

- The Puiseux series admit a natural derivation but they are not closed under integrals (antiderivatives): $\int \frac{1}{x} = \log(x)$ is not a Puiseux series.
- They do not admit an exp function: $\exp(x)$ should be bigger than x^n $\forall n \in \mathbb{N}$, but there is not such a Puiseux series.
- $\mathbb{R}((G))$ never admits an exp making it a model of $T_{\exp} = Th(\mathbb{R}_{\exp})$ [KS05].
- The "transseries" overcome these defects, and were instrumental in Écalle's solution of Dulac's problem (a weakening of Hilbert's 16th).
- We shall approach the transseries via the surreal numbers.

Restricted Hahn fields

- Let κ be either **On** or a regular cardinal with $\kappa^{<\kappa} = \kappa$.
- Let $|G| = \kappa$ and let $\mathbb{R}((G))_{sm} \subset \mathbb{R}((G))$ consist of the series of length $< \kappa$.
- For suitable G, it is possible to make $\mathbb{R}((G))_{sm}$ into a model of $T_{\exp}$ [KS05].
- We can write
  - $(\mathbb{R}((G))_{sm}^{>0}, \cdot) = G \cdot \mathbb{R}^{>0} \cdot (1 + o(1))$; represent x as $rg(1+\varepsilon)$.
  - $(\mathbb{R}((G))_{sm}, +) = \mathbb{J} \oplus \mathbb{R} \oplus o(1)$, where $\mathbb{J} := \mathbb{R}((G^{>1}))_{sm}$.
- In this case, log must take $(1 + o(1))$ to $o(1)$, $\mathbb{R}^{>0}$ to $\mathbb{R}$, and G to a direct summand of $\mathcal{O}(1) := \mathbb{R} \oplus o(1)$, not necessarily equal to $\mathbb{J}$.

Conway's field **No** of surreal numbers

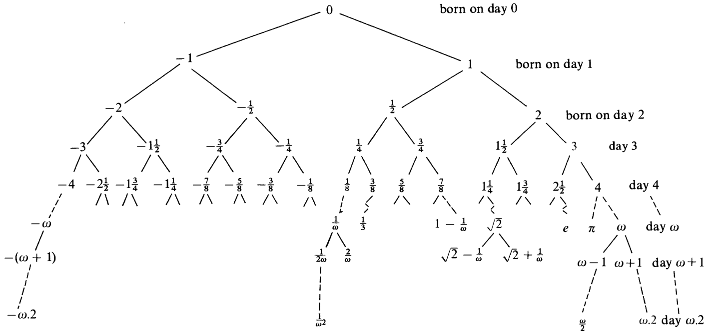


FIG. 0. When the first few numbers were born.

Normal form

- The surreal numbers **No** have the form $\mathbb{R}((G))_{sm}$. The group of monomials $G \subset \mathbf{No}^{>0}$ is a proper class, but we only take series $\sum_{i<\alpha} r_i g_i$ whose lenght is a SET
- There is a natural isomorphism $x \mapsto \omega^x$ from $(\mathbf{No}, +)$ to $(G, \cdot) \subset (\mathbf{No}^{>0}, \cdot)$.
- Thus $G = \omega^{\mathbf{No}} \subset \mathbf{No}$ and

  $$\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))_{sm},$$

  so we can represent $x \in \mathbf{No}$ as

  $$\sum_{i<\alpha} r_i \omega^{x_i}$$

  with $\alpha \in \mathbf{On}$, $r_i \in \mathbb{R}^*$, $x_i \in \mathbf{No}$.
- This extends Cantor's normal form for ordinals: $\alpha = \omega^{\alpha_1} n_1 + \ldots + \omega^{\alpha_k} n_k$

Surreal log

- Start with a chain isomorphism $h : \mathbf{No} \to \mathbf{No}^{>0}$ with $h(x) \prec \omega^x$.
- Let $\log(\omega^{\omega^x}) = \omega^{h(x)}$ and more generally
$$\log(\omega^{\sum_i r_i \omega^{x_i}}) = \sum_i \omega^{h(x_i)} r_i$$
This defines log on $G = \omega^{\mathbf{No}}$.
- We extend it to $\mathbf{No}^{>0}$ by
$$\log(r\omega^x(1+\varepsilon)) = \log(r) + \log(\omega^x) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \varepsilon^n$$
- This makes $\mathbf{No}$ into a model of $T_{\exp}$.
- Normal form: since $\omega^{\mathbf{No}} = \exp(\mathbb{J})$, every $x \in \mathbf{No}$ can be written as
$$\sum_{i<\alpha} r_i e^{\gamma_i}$$
with $\gamma_i \in \mathbb{J} \subseteq \mathbf{No}$.

Derivation

- We have seen that
$$\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))_{sm} = \mathbb{J} \oplus \mathbb{R} \oplus o(1)$$
- On the other hand $e^{\mathbb{J}} = \omega^{\mathbf{No}}$ so we also have
$$\mathbf{No} = \mathbb{R}((e^{\mathbb{J}}))_{sm}$$
- Thus every $x \in \mathbf{No}$ can be uniquely written either in the form
$$\sum_{i<\alpha} r_i \omega^{x_i} \in \mathbb{R}((\omega^{\mathbf{No}}))_{sm}$$
with $x_i \in \mathbf{No}$, or in the form
$$\sum_{i<\alpha} r_i e^{\gamma_i} \in \mathbb{R}((e^{\mathbb{J}}))_{sm}$$
with $\gamma_i \in \mathbb{J}$.
- [BM15]: There is a derivation ∂ on $\mathbf{No}$ such that $\partial\omega = 1$ and
$$\partial\left(\sum_{i<\alpha} r_i e^{\gamma_i}\right) = \sum_{i<\alpha} r_i e^{\gamma_i} \partial\gamma_i$$

Transseries

- Omega-series: Let $\mathbb{R}\langle\langle\omega\rangle\rangle$ be the smallest subfield of **No** containing ω and closed under exp, log and all sums of summable sequences. Ex. $\sum_{n\in\mathbb{N}} \omega^n \frac{\log(\omega)}{\exp_n(\omega)}$. On this subfield (a proper class) the derivation is unique.
- Transseries: Let $\mathbb{R}((\omega))^{LE} \subset \mathbb{R}\langle\langle\omega\rangle\rangle$ be the set of all $f \in$ **No** which can be obtained from $\mathbb{R}(\omega)$ by finitely many applications of $\sum$,exp,log.
- $\omega^n = \exp(n\log(\omega))$ is obtained in 3 steps (independent of n).
- $\sum_n n!\omega^{-1-n}\exp(\omega) = \int \frac{\exp(\omega)}{\omega}$ is a transseries.
- $\sum_{n\in\mathbb{N}} \log_n(\omega)$ is an omega-series, not a transseries.
- [BM17]: There is a natural isomorphism between $\mathbb{R}((\omega))^{LE}$, as defined above, and the LE-series of [vdDMM97].

Hardy fields

- A Hardy field is a field germs at $+\infty$ of functions $f \in C^1(\mathbb{R}, \mathbb{R})$ closed under differentiation. Examples:
  - $\mathbb{R}(x)$;
  - Hardy L-functions, given by terms involving $+, \cdot, \exp, \log$ and constants;
  - Germs of functions definable in $(\mathbb{R}, +, \cdot, \exp)$.
- The natural derivation on a Hardy field is compatible with the order: if $f >$ ker(∂, then $\partial f > 0$.
- [AvdDvdH15]: Every Hardy field embeds in $(\mathbf{No}, \partial)$ as a differential field.
- $\mathbf{No} \equiv \mathbb{R}((\omega))^{LE}$ as differential fields [AvdDvdH15]; both closed under integrals (anti-derivatives).

Composition

- [BM17] There is a composition operator $\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ satisfying the following conditions for all $f, g \in \mathbb{R}\langle\langle\omega\rangle\rangle$ and $x \in \mathbf{No}^{>\mathbb{R}}$:
  - If $f = \sum_{i<\alpha} r_i e^{\gamma_i}$, then $f \circ x = \sum_{i<\alpha} r_i e^{\gamma_i \circ x}$;
  - If $f, g \in \mathbb{R}\langle\langle\omega\rangle\rangle$, then $f \circ g \in \mathbb{R}\langle\langle\omega\rangle\rangle$;
  - $(f \circ g) \circ x = f \circ (g \circ x)$;
  - $f \circ \omega = f$ and $\omega \circ x = x$.
- The idea is to substitute x for ω in the expression for $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$ and evaluate the resulting expression, but the proof of summability is long and complex.
- Example:

$$\sum_{n\in\mathbb{N}} \log_n(\omega) \circ \sum_{n\in\mathbb{N}} \log_n(\omega) = \sum_{n\in\mathbb{N}} \log_n(\sum_{i\in\mathbb{N}} \log_i(\omega))$$

is a well defined surreal number (in fact, an omega-series).

Derivation and composition

- There is a nice interaction between ∂ and $\circ$.
  - Chain rule:
  $$\partial(f \circ g) = (\partial f \circ g) \cdot \partial g$$
  - Limit formula:
  $$\partial f \circ x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(f \circ (x + \varepsilon) - f \circ x)$$
  - Analyticity: for small $\varepsilon \in \mathbf{No}$,
  $$f \circ (x + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{1}{n!}(\partial^n f \circ x)\varepsilon^n$$
  namely $\hat{f}(x) := f \circ x$ defines a surreal analytic germ $\hat{f} : \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$.
- Conjecture: **No** equipped with all the $\hat{f}$ for $f \in \mathbb{R}\langle\langle\omega\rangle\rangle$ is tame.

A negative result

- The derivation $\partial : \mathbf{No} \to \mathbf{No}$ in [BM15] is not compatible with a composition $\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$.
- I am going to show that if there is a compatible composition, then there is a proper class of elements λ with derivative 1, contradicting the fact that $\ker(\partial) = \mathbb{R}$ is a SET.
- Let $\partial \ell_\omega = \frac{1}{\prod_{n \in \mathbb{N}} \ell_n}$ where $\ell_n = \log_n(\omega)$.
- Let λ be a "log-atomic" number with $\lambda > \exp_n(\omega)\ \forall n \in \mathbb{N}$.
- By [BM15] $\partial \lambda = \prod_n \log_n(\lambda)$. Now,

$$\begin{aligned}
\partial(\ell_\omega \circ \lambda) &= (\partial \ell_\omega \circ \lambda) \cdot \partial \lambda \\
&= \left(\frac{1}{\prod_n \ell_n} \circ \lambda \right) \cdot \partial \lambda \\
&= \left(\frac{1}{\prod_n \log_n(\lambda)} \right) \cdot \partial \lambda = 1
\end{aligned}$$