2.8 Lemma *If a limit ordinal α is not a cardinal, then* $\mathrm{cf}(\alpha) < \alpha$. □

As a corollary, we have, for all limit ordinals α,

$$\mathrm{cf}(\alpha) = \alpha \text{ if and only if } \alpha \text{ is a regular cardinal.}$$

2.9 Lemma *For every limit ordinal α,* $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.

Proof. Let $\vartheta = \mathrm{cf}(\alpha)$. Clearly, ϑ is a limit ordinal, and $\mathrm{cf}(\vartheta) \leq \vartheta$. We have to show that $\mathrm{cf}(\vartheta)$ is not smaller than ϑ. If $\gamma = \mathrm{cf}(\vartheta) < \vartheta$, then there exists an increasing sequence of ordinals $\langle \nu_\xi \mid \xi < \gamma \rangle$ such that $\lim_{\xi \to \gamma} \nu_\xi = \vartheta$. Since $\vartheta = \mathrm{cf}(\alpha)$, there exists an increasing sequence of ordinals $\langle \alpha_\nu \mid \nu < \vartheta \rangle$ such that $\lim_{\nu \to \vartheta} \alpha_\nu = \alpha$. Then the sequence $\langle \alpha_{\nu_\xi} \mid \xi < \gamma \rangle$ has length γ and $\lim_{\xi \to \gamma} \alpha_{\nu_\xi} = \alpha$. But $\gamma < \vartheta$, and we reached a contradiction, since ϑ is supposed to be the least length of an increasing sequence with limit α. □

2.10 Corollary *For every limit ordinal α, $\mathrm{cf}(\alpha)$ is a regular cardinal.* □

Exercises

2.1 $\mathrm{cf}(\aleph_\omega) = \mathrm{cf}(\aleph_{\omega+\omega}) = \omega$.

2.2 $\mathrm{cf}(\aleph_{\omega_1}) = \omega_1$, $\mathrm{cf}(\aleph_{\omega_2}) = \omega_2$.

2.3 Let α be the cardinal number defined in the proof of Lemma 2.6. Show that $\mathrm{cf}(\alpha) = \omega$.

2.4 Show that $\mathrm{cf}(\alpha)$ is the least γ such that α is the union of γ sets of cardinality less than $|\alpha|$.

2.5 Let $\aleph_\alpha$ be a limit cardinal, $\alpha > 0$. Show that there is an increasing sequence of *alephs* of length $\mathrm{cf}(\aleph_\alpha)$ with limit $\aleph_\alpha$.

2.6 Let κ be a limit cardinal, and let $\lambda < \kappa$ be a regular infinite cardinal. Show that there is an increasing sequence $\langle \alpha_\nu \mid \nu < \mathrm{cf}(\kappa) \rangle$ of cardinals such that $\lim_{\nu \to \mathrm{cf}(\kappa)} \alpha_\nu = \kappa$ and $\mathrm{cf}(\alpha_\nu) = \lambda$ for all ν.

3. Exponentiation of Cardinals

While addition and multiplication of cardinals are simple (due to the fact that $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta =$ the greater of the two), the evaluation of cardinal exponentiation is rather complicated. Here, we do not give a complete set of rules (in fact, in a sense, the general problem of evaluation of κ^λ is still open), but prove only the basic properties of the operation κ^λ. It turns out that there is a difference between regular and singular cardinals.

First, we investigate the operation $2^{\aleph_\alpha}$. By Cantor's Theorem, $2^{\aleph_\alpha} > \aleph_\alpha$; in other words,

(3.1) $$2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$$

Let us recall that Cantor's *Continuum Hypothesis* is the conjecture that $2^{\aleph_0} = \aleph_1$. A generalization of this conjecture is the *Generalized Continuum Hypothesis*:

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad \text{for all } \alpha.$$

As we show, the Generalized Continuum Hypothesis greatly simplifies the cardinal exponentiation; in fact, the operation κ^λ can then be evaluated by very simple rules.

The Generalized Continuum Hypothesis can be neither proved nor refuted from the axioms of set theory. (See the discussion of this subject in Chapter 15.)

Without assuming the Generalized Continuum Hypothesis, there is not much one can prove about $2^{\aleph_\alpha}$ except (3.1) and the trivial property:

(3.2) $$2^{\aleph_\alpha} \leq 2^{\aleph_\beta} \quad \text{whenever} \quad \alpha \leq \beta.$$

The following fact is a consequence of König's Theorem.

3.3 Lemma *For every α,*

(3.4) $$\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha.$$

Thus $2^{\aleph_0}$ cannot be $\aleph_\omega$, since $\mathrm{cf}(2^{\aleph_\omega}) = \aleph_0$, but the lemma does not prevent $2^{\aleph_0}$ from being $\aleph_{\omega_1}$. Similarly, $2^{\aleph_1}$ cannot be either $\aleph_{\omega_1}$ or $\aleph_\omega$ or $\aleph_{\omega+\omega}$, etc.

Proof. Let $\vartheta = \mathrm{cf}(2^{\aleph_\alpha})$; ϑ is a cardinal. Thus $2^{\aleph_\alpha}$ is the limit of an increasing sequence of length ϑ, and it follows (see the proof of Lemma 2.3 for details) that

$$2^{\aleph_\alpha} = \sum_{\nu < \vartheta} \kappa_\nu,$$

where each κ_ν is a cardinal smaller than $2^{\aleph_\alpha}$. By König's Theorem (where we let $\lambda_\nu = 2^{\aleph_\alpha}$ for all $\nu < \vartheta$), we have

$$\sum_{\nu<\vartheta} \kappa_\nu < \prod_{\nu<\vartheta} 2^{\aleph_\alpha}$$

and hence $2^{\aleph_\alpha} < (2^{\aleph_\alpha})^\vartheta$. Now if ϑ were less than or equal to $\aleph_\alpha$, we would get

$$2^{\aleph_\alpha} < (2^{\aleph_\alpha})^\vartheta \leq (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha \cdot \aleph_\alpha} = 2^{\aleph_\alpha},$$

a contradiction. □

The inequalities (3.1), (3.2), and (3.4) are the only properties that can be proved for the operation $2^{\aleph_\alpha}$ if the cardinal $\aleph_\alpha$ is regular. If $\aleph_\alpha$ is singular, then various additional rules restraining the behavior of $2^{\aleph_\alpha}$ are known. We prove one such theorem here (Theorem 3.5); in Chapter 11 we prove Silver's Theorem (Theorem 4.1): If $\aleph_\alpha$ is a singular cardinal of cofinality $\mathrm{cf}(\aleph_\alpha) \geq \aleph_1$, and if

3.5 Theorem *Let $\aleph_\alpha$ be a singular cardinal. Let us assume that the value of $2^{\aleph_\xi}$ is the same for all $\xi < \alpha$, say $2^{\aleph_\xi} = \aleph_\beta$. Then $2^{\aleph_\alpha} = \aleph_\beta$.*

Note that it is implicit in the theorem that $\aleph_\beta$ is greater than $\aleph_\alpha$. For instance, if we know that $2^{\aleph_n} = \aleph_{\omega+5}$ for all $n < \omega$, then $2^{\aleph_\omega} = \aleph_{\omega+5}$.

Proof. Since $\aleph_\alpha$ is singular, there exists, by Lemma 2.3, a collection $\langle \kappa_i \mid i \in I\rangle$ of cardinals such that $\kappa_i < \aleph_\alpha$ for all $i \in I$, and $|I| = \aleph_\gamma$ is a cardinal less than $\aleph_\alpha$, and $\aleph_\alpha = \sum_{i\in I} \kappa_i$. By the assumption, we have $2^{\kappa_i} = \aleph_\beta$ for all $i \in I$, and also $2^{\aleph_\gamma} = \aleph_\beta$, so

$$2^{\aleph_\alpha} = 2^{\sum_{i\in I}\kappa_i} = \prod_{i\in I} 2^{\kappa_i} = \prod_{i\in I} \aleph_\beta = \aleph_\beta^{\aleph_\gamma} = (2^{\aleph_\gamma})^{\aleph_\gamma} = 2^{\aleph_\gamma} = \aleph_\beta.$$

□

We now approach the problem of evaluating $\aleph_\alpha^{\aleph_\beta}$, where $\aleph_\alpha$ and $\aleph_\beta$ are arbitrary infinite cardinals. First, we make the following observation.

3.6 Lemma *If $\alpha \le \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.*

Proof. Clearly, $2^{\aleph_\beta} \le \aleph_\alpha^{\aleph_\beta}$. Since $\aleph_\alpha \le 2^{\aleph_\alpha}$, we also have

$$\aleph_\alpha^{\aleph_\beta} \le (2^{\aleph_\alpha})^{\aleph_\beta} = 2^{\aleph_\alpha\cdot\aleph_\beta} = 2^{\aleph_\beta}$$

because $\aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}$. □

When trying to evaluate $\aleph_\alpha^{\aleph_\beta}$ for $\alpha > \beta$, we find the following useful.

3.7 Lemma *Let $\alpha \ge \beta$ and let S be the set of all subsets $X \subseteq \omega_\alpha$ such that $|X| = \aleph_\beta$. Then $|S| = \aleph_\alpha^{\aleph_\beta}$.*

Proof. We first show that $\aleph_\alpha^{\aleph_\beta} \le |S|$. Let S' be the set of all subsets $X \subseteq \omega_\beta \times \omega_\alpha$ such that $|X| = \aleph_\beta$. Since $\aleph_\beta \cdot \aleph_\alpha = \aleph_\alpha$, we have $|S'| = |S|$. Now every function $f : \omega_\beta \to \omega_\alpha$ is a member of the set S' and hence $\omega_\alpha^{\omega_\beta} \subseteq S'$. Therefore, $\aleph_\alpha^{\aleph_\beta} \le |S|$.

Conversely, if $X \in S$, then there exists a function f on ω_β such that X is the range of f. We pick one f for each $X \in S$ and let $f = F(X)$. Clearly, if $X \ne Y$ and $f = F(X)$ and $g = F(Y)$, we have $X = \operatorname{ran} f$ and $Y = \operatorname{ran} g$, and so $f \ne g$. Thus F is a one-to-one mapping of S into $\omega_\alpha^{\omega_\beta}$, and therefore $|S| \le \aleph_\alpha^{\aleph_\beta}$. □

We are now in a position to evaluate $\aleph_\alpha^{\aleph_\beta}$ for regular cardinals $\aleph_\alpha$, under the assumption of the Generalized Continuum Hypothesis.

3.8 Theorem *Let us assume the Generalized Continuum Hypothesis. If $\aleph_\alpha$ is a regular cardinal, then*

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_\alpha & \text{if } \beta < \alpha, \\ \aleph_{\beta+1} & \text{if } \beta \ge \alpha. \end{cases}$$

Proof. If $\beta \ge \alpha$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta} = \aleph_{\beta+1}$ by Lemma 3.6. So let $\beta < \alpha$ and let $S = \{X \subseteq \omega_\alpha \mid |X| = \aleph_\beta\}$. By Lemma 3.7, $|S| = \aleph_\alpha^{\aleph_\beta}$. By Theorem 2.2(a), every $X \in S$ is a bounded subset of ω_α. Thus, let $B = \bigcup_{\delta<\omega_\alpha} \mathcal{P}(\delta)$ be the collection of all bounded subsets of ω_α. We will show that $|B| \le \aleph_\alpha$; as $S \subset B$, it then follows that $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$.

Since $B = \bigcup_{\delta<\omega_\alpha} \mathcal{P}(\delta)$, we have

$$|B| \le \sum_{\delta<\omega_\alpha} 2^{|\delta|}.$$

However, for every cardinal $\aleph_\gamma < \aleph_\alpha$, we have $2^{\aleph_\gamma} = \aleph_{\gamma+1} \le \aleph_\alpha$ and so $2^{|\delta|} \le \aleph_\alpha$ for every $\delta < \omega_\alpha$, and we get

$$|B| \le \sum_{\delta<\omega_\alpha} 2^{|\delta|} \le \sum_{\delta<\omega_\alpha} \aleph_\alpha = \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha.$$

□

We prove a similar (but a little more complicated) formula for singular $\aleph_\alpha$, but first we need a generalization of Lemma 3.3.

3.9 Lemma *For every cardinal $\kappa > 1$ and every α, $\operatorname{cf}(\kappa^{\aleph_\alpha}) > \aleph_\alpha$.*

Proof. Exactly like the proof of Lemma 3.3, except that $2^{\aleph_\alpha}$ is replaced by $\kappa^{\aleph_\alpha}$. □

3.10 Theorem *Let us assume the Generalized Continuum Hypothesis. If $\aleph_\alpha$ is a singular cardinal, then*

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_\alpha & \text{if } \aleph_\beta < \operatorname{cf}(\aleph_\alpha), \\ \aleph_{\alpha+1} & \text{if } \operatorname{cf}(\aleph_\alpha) \le \aleph_\beta \le \aleph_\alpha, \\ \aleph_{\beta+1} & \text{if } \aleph_\beta \ge \aleph_\alpha. \end{cases}$$

Proof. If $\beta \ge \alpha$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta} = \aleph_{\beta+1}$. If $\aleph_\beta < \operatorname{cf}(\aleph_\alpha)$, then every subset $X \subseteq \omega_\alpha$ such that $|X| = \aleph_\beta$ is a bounded subset, and we get $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$ by exactly the same argument as in the case of regular $\aleph_\alpha$.

Thus let us assume that $\operatorname{cf}(\aleph_\alpha) \le \aleph_\beta \le \aleph_\alpha$. On the one hand, we have

$$\aleph_\alpha \le \aleph_\alpha^{\aleph_\beta} \le \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

On the other hand, $\operatorname{cf}(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$ by Lemma 3.9, and since $\aleph_\beta \ge \operatorname{cf}(\aleph_\alpha)$, we have $\operatorname{cf}(\aleph_\alpha^{\aleph_\beta}) \ne \operatorname{cf}(\aleph_\alpha)$, and therefore $\aleph_\alpha^{\aleph_\beta} \ne \aleph_\alpha$. Thus necessarily $\aleph_\alpha^{\aleph_\beta} = \aleph_{\alpha+1}$. □

If we do not assume the Generalized Continuum Hypothesis, the situation becomes much more complicated. We only prove the following theorem.

3.11 Hausdorff's Formula *For every α and every β,*

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}.$$

Proof. If $\beta \geq \alpha + 1$, then $\aleph_{\alpha+1}^{\aleph_\beta} = 2^{\aleph_\beta}$, $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$, and $\aleph_{\alpha+1} \leq \aleph_\beta \leq 2^{\aleph_\beta}$; hence the formula holds. Thus let us assume that $\beta \leq \alpha$. Since $\aleph_\alpha^{\aleph_\beta} \leq \aleph_{\alpha+1}^{\aleph_\beta}$ and $\aleph_{\alpha+1} \leq \aleph_{\alpha+1}^{\aleph_\beta}$, it suffices to show that $\aleph_{\alpha+1}^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.

Each function $f : \omega_\beta \to \omega_{\alpha+1}$ is bounded; i.e., there is $\gamma < \omega_{\alpha+1}$ such that $f(\xi) < \gamma$ for all $\xi < \omega_\beta$ (this is because $\omega_{\alpha+1}$ is regular and $\omega_\beta < \omega_{\alpha+1}$). Hence,

$$\omega_{\alpha+1}^{\omega_\beta} = \bigcup_{\gamma < \omega_{\alpha+1}} \gamma^{\omega_\beta}.$$

Now every $\gamma < \omega_{\alpha+1}$ has cardinality $|\gamma| \leq \aleph_\alpha$, and we have (by Exercise 1.6) $|\bigcup_{\gamma<\omega_{\alpha+1}} \gamma^{\omega_\beta}| \leq \sum_{\gamma<\omega_{\alpha+1}} |\gamma|^{\aleph_\beta}$. Thus

$$\aleph_{\alpha+1}^{\aleph_\beta} \leq \sum_{\gamma<\omega_{\alpha+1}} |\gamma|^{\aleph_\beta} \leq \sum_{\gamma<\omega_{\alpha+1}} \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}.$$

□

This theorem enables us to evaluate some simple cases of $\aleph_\alpha^{\aleph_\beta}$ (see Exercise 3.5).

An infinite cardinal $\aleph_\alpha$ is a *strong limit cardinal* if $2^{\aleph_\beta} < \aleph_\alpha$ for all $\beta < \alpha$.

Clearly, a strong limit cardinal is a limit cardinal, since if $\aleph_\alpha = \aleph_{\gamma+1}$, then $2^{\aleph_\gamma} \geq \aleph_\alpha$. Not every limit cardinal is necessarily a strong limit cardinal: If $2^{\aleph_0}$ is greater than $\aleph_\omega$, then $\aleph_\omega$ is a counterexample. However, if we assume the Generalized Continuum Hypothesis, then every limit cardinal is a strong limit cardinal.

3.12 Theorem *If $\aleph_\alpha$ is a strong limit cardinal and if κ and λ are infinite cardinals such that $\kappa < \aleph_\alpha$ and $\lambda < \aleph_\alpha$, then $\kappa^\lambda < \aleph_\alpha$.*

Proof. $\kappa^\lambda \leq (\kappa \cdot \lambda)^{\kappa\cdot\lambda} = 2^{\kappa\cdot\lambda} < \aleph_\alpha$. □

An uncountable cardinal number κ is *strongly inaccessible* if it is regular and a strong limit cardinal. (Thus every strongly inaccessible cardinal is weakly inaccessible, and, if we assume the Generalized Continuum Hypothesis, every weakly inaccessible cardinal is strongly inaccessible.) The reason why such cardinal numbers are called inaccessible is that they cannot be obtained by the

3.13 Theorem *Let κ be a strongly inaccessible cardinal.*
(a) If X has cardinality $< \kappa$, then $\mathcal{P}(X)$ has cardinality $< \kappa$.
(b) If each $X \in S$ has cardinality $< \kappa$ and $|S| < \kappa$, then $\bigcup S$ has cardinality $< \kappa$.
(c) If $|X| < \kappa$ and $f : X \to \kappa$, then $\sup f[X] < \kappa$.

Proof.
(a) κ is a strong limit cardinal.
(b) Let $\lambda = |S|$ and $\mu = \sup\{|X| \mid X \in S\}$. Then (by Theorem 2.2(a)) $\mu < \kappa$ because κ is regular, and $|\bigcup S| \leq \lambda \cdot \mu < \kappa$.
(c) By Theorem 2.2(b).

□

Exercises

3.1 If $2^{\aleph_\beta} \geq \aleph_\alpha$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.

3.2 Verify this generalization of Exercise 3.1: If there is $\gamma < \alpha$ such that $\aleph_\gamma^{\aleph_\beta} \geq \aleph_\alpha$, say $\aleph_\gamma^{\aleph_\beta} = \aleph_\delta$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\delta$.

3.3 Let α be a limit ordinal and let $\aleph_\beta < \operatorname{cf}(\aleph_\alpha)$. Show that if $\aleph_\xi^{\aleph_\beta} \leq \aleph_\alpha$ for all $\xi < \alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$. [*Hint:* If $X \subseteq \omega_\alpha$ is such that $|X| = \aleph_\beta$, then $X \subseteq \omega_\xi$ for some $\xi < \alpha$.]

3.4 If $\aleph_\alpha$ is strongly inaccessible and $\beta < \alpha$, then $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$. [*Hint:* Use Exercise 3.3.]

3.5 If $n < \omega$, then $\aleph_n^{\aleph_\beta} = \aleph_n \cdot 2^{\aleph_\beta}$. [*Hint:* Apply Hausdorff's formula n times.]

3.6 Prove that $\prod_{n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}$. [*Hint:* Let A_i ($i < \omega$) be mutually disjoint infinite subsets of ω. Then

$$\prod_{n<\omega} \aleph_n \geq \prod_{i<\omega}\Big(\prod_{n\in A_i} \aleph_n\Big) \geq \prod_{i<\omega}\Big(\sum_{n\in A_i} \aleph_n\Big) \geq \prod_{i<\omega} \aleph_\omega = \aleph_\omega^{\aleph_0}.$$

The other direction is easy.]

3.7 Prove that

$$\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}.$$

[*Hint:* $\aleph_\omega^{\aleph_1} = (\sum_{n<\omega} \aleph_n)^{\aleph_1} \leq (\prod_{n<\omega} \aleph_n)^{\aleph_1} = \prod_{n<\omega} \aleph_n^{\aleph_1} = \prod_{n<\omega}(\aleph_n \cdot 2^{\aleph_1}) = (\prod_{n<\omega} \aleph_n) \cdot (2^{\aleph_1})^{\aleph_0} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.]