On the homotopy type of definable groups

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Let $M$ be a first order structure. Assume $M$ big and sufficiently saturated.

Let $X$ be a definable set (in $M$), let $E \subset X \times X$ be a type-definable equivalence relation of bounded index and put on $X/E$ the logic topology: a subset of $X/E$ is closed if its preimage in $X$ is type-definable.

Then $X/E$ is a compact Hausdorff space (Lascar-Pillay 2001).
The group case

\[ M = \text{an o-minimal expansion of a real closed field.} \]

\[ G = \text{a definable group (in } M). \]

\[ G^0 \triangleleft G \text{ is the infinitesimal subgroup of } G, \text{ namely the “smallest type-definable subgroup of bounded index”}. \text{ (Assume } M \text{ sufficiently saturated.)} \]

**Thm.** (Berarducci, Otero, Petezil, Pillay 2005) \( G/G^0 \) with the “logic topology” is a compact real Lie group

In particular \( G/G^0 \) is locally contractible.
Compact domination

Let $G$ be a definably compact definable group.

**Thm.** (Hrushovksi-Pillay 2007) The image in $G/G^{00}$ of a nowhere dense definable subset of $G$ has Haar measure zero.

**Cor.** Each generic subset of $G$ contains a coset of $G^{00}$.

**Cor.** (Ber. JSL 2009) $G^{00}$ is a countable decreasing intersection of definable open sets definably homeomorphic to cells.
Let $X$ be a definable set, let $E \subset X \times X$ be a type-definable equivalence relation of bounded index and put on $X/E$ the logic topology. In the rest of this talk we make the following assumption.

**Assumption.**

(1) For all $y \in X/E$ the equivalence class $y/E \subset X$ is a decreasing intersection $\bigcap_{i \in \mathbb{N}} C_i$ of definable sets $C_i$ definably homeomorphic to open cells.

(2) $X/E$ is locally contractible.

Note that $X/E = G/G^{00}$ satisfies these assumptions.
There is no natural way of associating to a definable path $\gamma: [0, 1]^M \to X$ a path $\eta: [0, 1]^\mathbb{R} \to X/E$.

However we show that this can be done up to homotopy.

**Def.**

$\pi_{\text{def}}(X, \Gamma) := \text{definable homotopy classes of definable paths in } X \text{ with endpoints in } \Gamma \subset X$.

$\pi(X/E, \Lambda) := \text{homotopy classes of paths in } X/E \text{ with endpoints in } \Lambda \subset X/E$. 
Comparison theorem

Let $\tau : X \to X/E$ be the projection.

**Thm.** (Ber.-Mamino 2009) There is a *unique* morphism of groupoids $\tau_* : \pi^{\text{def}}(X, X) \to \pi(X/E, X/E)$ with the following properties:

1. $\tau_*$ maps the definable homotopy class of a definable path with endpoints $x, y$, to the homotopy class of a path with endpoints $\tau(x), \tau(y)$ respectively.

2. For any open $O \subseteq X/E$, and for any $[a] \in \pi^{\text{def}}(X, X)$ such that $\text{Im}(a) \subseteq \tau^{-1}(O)$, there is a path $b$ in $X/E$ such that $\text{Im}(b) \subseteq O$ and $\tau_*([a]) = [b]$.

Moreover if $\Gamma$ is a subset of $X$ such that $\tau|_\Gamma$ is injective, then the restriction of $\tau_*$ to $\pi^{\text{def}}(X, \Gamma)$ is an isomorphism onto $\pi(X/E, \tau(\Gamma))$. 
In particular:

**Thm.** (Ber.-Mamino 2009)

1. (Global form) $\pi_1^\text{def}(X) \cong \pi_1(X/E)$.

2. (Local form) If $O$ is an open connected subset of $X/E$, then $\pi_1^\text{def}(\tau^{-1}(O)) \cong \pi_1(O)$.

Point (1) gives a new proof (using compact domination) of:

**Thm.** (Edmundo-Otero 2004) $\pi_1^\text{def}(G) \cong \pi_1(G/G^{00})$.

Point (2) shows in particular that the preimage of a simply connected open subset $O$ of $G/G^{00}$ is a definably simply connected (or-definable) subset of $G$. 
Conjecture.

1. $\pi_n^{\text{def}}(X) \cong \pi_n(X/E)$.

2. $X/E$ determines the definable homotopy type of $X$.

In the sequel we consider the case when $X/E = G/G^{00}$. 
The group case

Let $G$ be a definably compact definable group.

**Thm.** (Ber.-Mamino-Otero 2009)

1. If $G$ is abelian, $\pi_n^{\text{def}}(G) = 0$ for all $n \geq 2$.
2. If $G$ is abelian, $G/G^{00}$ determines $G$ up to definable homotopy.
3. In general $\pi_n^{\text{def}}(G) = \pi_n(G/G^{00})$.

Point 3 is a reduction to the abelian case using:

**Thm.** (Hrushowski, Peterzil, Pillay 2008) Every definably compact group $G$ is an almost direct product $A \times_{\Gamma} S$ of a definable abelian subgroup $A$ and a definable semisimple subgroup $S$.
The general case

**Thm.** (Ber.-Mamino 2009)

1. If $G$ is semisimple, the isomorphism type of $G/G^{00}$ determines the definable isomorphism type of $G$.
2. In general if $\psi: G/G^{00} \cong G'/G'^{00}$, then $G$ is definably homotopy equivalent to $G'$.

In order to prove 2, we write $G = A \times_{\Gamma} S$, $G' = A' \times_{\Gamma'} S'$ ($A, A'$ abelian, $S, S'$ semisimple), and observe that $\psi$ induces isomorphisms $A/A^{00} \cong A'/A'^{00}$, $\Gamma \cong \Gamma'$, $S/S^{00} \cong S'/S'^{00}$. We are then lead to the following lifting problem:
Lifting problem

Let $A, A'$ be definably connected definably compact abelian groups, and let $\Gamma < A$ and $\Gamma' < A'$ be isomorphic finite subgroups. Given a continuous map $g : A/\Gamma \to A'/\Gamma'$, we look for a continuous map $f : A \to A'$ sending 1 to 1 and making the following diagram commute.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow p & & \downarrow p' \\
A/\Gamma & \xrightarrow{g} & A'/\Gamma'
\end{array}
\]

In general $f$ may not exist. However it does exist if there are no obstructions coming from the fundamental groups. Thanks to the work on the fundamental groupoid we know that this will be the case if there is a Lie isomorphism $A/A^{00} \cong A'/A'^{00}$ sending $\Gamma A^{00}$ to $\Gamma' A'^{00}$. 
Let $G$ be a definably compact definably connected definable group.

The (o-minimal) universal cover $\tilde{G}$ of $G$ can be identified with the subset of $\pi^\text{def}(G, G)$ consisting of all the definable homotopy classes of paths starting at $1 \in G$.

The group operation of $\tilde{G}$ is defined by $[a] \ast [b] := [a + a(1) \cdot b]$ where $+$ is concatenation and $a(1)$ is the endpoint of $a$.

**Thm.** The morphism $\tau^G_* : \pi^\text{def}(G, G) \rightarrow \pi(G/G^{00}, G/G^{00})$ induces, by restriction, a morphism $\rho^G : \tilde{G} \rightarrow \tilde{G}/G^{00}$ whose kernel is isomorphic to $G^{00}$ (via the map $[a] \mapsto a(1)$).
Cor. (Ber-Mamino 2009) Let $G$ be a definably compact definably connected definable group. Given an extension of Lie groups $f : H \to G/G^00$ with a finite kernel, there is a definable group extension $\pi : H \to G$ of $G$ such that $H/H^00 \cong H$ (as coverings of $G/G^00$). We thus obtain a commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\varphi} & H \\
\downarrow{\pi} & & \downarrow{f} \\
G & \xrightarrow{\tau} & G/G^00
\end{array}
\]

where $\varphi : H \to H$ is the composition of the projection $H \to H/H^00$ with the isomorphism $H/H^00 \cong H$. 
Let $G, G'$ be definably connected definably compact groups.

**Thm.** (Baro 2009) $G$ and $G'$ are definably homotopy equivalent if and only if $G/G^{00}$ and $G'/G'^{00}$ are homotopy equivalent.
http://www.uam.es/personal_pdi/ciencias/ebaro/articulos


