Infinite λ-calculus and non-sensible models *

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Abstract

We define a model of $\lambda\beta$-calculus which is similar to the model of Böhm trees, but it does not identify all the unsolvable lambda-terms. The role of the unsolvable terms is taken by a much smaller class of terms which we call mute. Mute terms are those zero terms which are not β-convertible to a zero term applied to something else. We prove that it is consistent with the $\lambda\beta$-calculus to simultaneously equate all the mute terms to a fixed arbitrary closed term. This allows us to strengthen some results of Jacopini and Venturini Zilli concerning easy λ-terms. Our results depend on an infinitary version of λ-calculus. We set the foundations for such a calculus, which might turn out to be a useful tool for the study of non-sensible models of λ-calculus.

Dedicated to the memory of Roberto Magari

1 Introduction

Our aim is to define a new model of λ-calculus in which two λ-terms are identified if they have the same "asymptotic behaviour", namely they approach the same limit by repeated β-reductions. Such an idea is already present in the notion of "Böhm tree" (see [6, 2]) but it is not fully exploited in the sense

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that the aymptotic behaviour of the unsolvable terms is completely ignored. The Böhm tree of a λ-term, is a kind of infinite unfolding of the λ-term with respect to β-reduction. Consider for instance the Turing fixed point combinator $\mathbf{Y_t} \equiv QQ$ where $Q \equiv \lambda x, y.y(xx)$ (as usual $\equiv$ between λ-terms is syntactic identity up to renaming of bound variables, whilst $=$ denotes β-convertibility). The Böhm tree $BT(\mathbf{Y_t})$ is obtained as a limit of the ω-sequence of β-reductions $\mathbf{Y_t} \to \lambda y.y(QQ) \to \lambda y.y(y(QQ)) \to \ldots$, namely $BT(\mathbf{Y_t})$ is the "completely unfolded" infinite λ-term $\lambda y.y(y(y(\ldots)))$. Böhm trees are more often depicted as trees, but we prefer to think of them as infinite λ-terms. For the reader's convenience we recall the formal definition of Böhm tree in section 2. Since the Böhm tree of an unsolvable λ-term is defined to be $\perp$ (where $\perp$ is a special symbol staying for "bottom", or "undefined", or "empty"), Böhm trees do not distinguish among unsolvable terms.

The importance of Böhm trees both for the proof-theory and for the semantics of lambda-calculus needs not be stressed. Let us just recall that two λ-terms are equal in the Plotkin model $P\omega$ if and only if they have the same Böhm tree (see [11, 2]). Moreover two λ-terms are equal in the Scott model D_∞ if and only if they have the same "$\beta\eta$-Böhm tree" (see [11, 24, 2]).

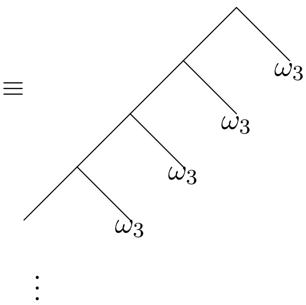
The fact that the Böhm tree of an unsolvable term is $\perp$ can be a drawback: it means that Böhm trees give no information on the inner structure of the unsolvable terms. It has been argued that the unsolvable terms correspond to the notion of "undefined" in λ-calculus (see [2]), and therefore one does not need to look inside them. However some recent papers [12, 13, 4] have suggested that some unsolvable terms can have an operational meaning and therefore one should take as undefined elements a smaller set of terms than the unsolvables, for instance the *zero terms* (Statman) or the *easy terms* [14, 15, 16, 25]. Using a result of Visser [23] Statman proved (see [3]) that one can take any Π_1 set of terms closed under β-conversion to represent the undefined value of a partial recursive function. Topological models of λ-calculus where not all the unsolvable are identified are studied in [10, 21].

With these motivations in mind we extend non-trivially the notion of Böhm tree to a large class of unsolvable terms. We do so by introducing what we call the *infinite $\beta\perp$-normal form* of a λ-term. The definition of the infinite $\beta\perp$-normal form is very natural: we just apply the idea of infinite unfolding also to the unsolvable terms. Consider for instance the unsolvable term $\Omega_3 \equiv \omega_3\omega_3$ where $\omega_3 \equiv \lambda x.xxx$. We have the ω-sequence of β-reductions

$\Omega_3 \to \Omega_3\omega_3 \to \Omega_3\omega_3\omega_3 \to \ldots$

such a sequence "converges" to the infinite β-reduction $\Omega_3 \rightarrow_\infty NF^\infty(\Omega_3)$, where $NF^\infty(\Omega_3)$ is the unique infinite term satisfying the syntactic identity $NF^\infty(\Omega_3) \equiv NF^\infty(\Omega_3)\omega_3$. This can be depicted as a tree with binary application nodes:

$NF^\infty(\Omega_3) \equiv$



The notion of convergence here used is convergence with respect to the usual topology of infinite trees plus the stronger requirement that the depth of the redexes reduced in an infinite β-reduction must go to infinity (such a notion is called strong-convergence in [17] in the context of term-rewriting systems for first order infinite terms). Unlike what happens for finite lambda-terms, the above example shows that there are terms, like $NF^\infty(\Omega_3)$, which are in normal form (as they have no β-redexes) and yet they begin neither with a λ-abstraction nor with a variable.

The main result of this paper is that the infinite $\beta\bot$-normal forms are, like the Böhm trees, a model of the $\lambda\beta$-calculus. Such a model is not *sensible*, i.e. not all the unsolvable are identified. What plays the role of the unsolvable terms is a much smaller class of terms which we call *mute*. We recall that a *zero term* is a term which cannot by reduced to an abstraction term $\lambda x.T$. Mute terms are then defined as those zero terms which cannot be reduced to a variable or to a zero term applied to some other term. If A is mute we set by convention $NF^\infty(A) = \bot$. For instance $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ is mute, whilst Ω_3 is a zero term but it is not mute. The idea is that mute terms have a totally indefined operational behaviour.

To prove our main result the crucial lemma is to show that infinite $\beta\bot$-normal forms behave well under substitutions, i.e. we must show:

$$NF^\infty(A[x := B]) = NF^\infty(NF^\infty(A)[x := NF^\infty(B)]) \qquad (*)$$

A similar problem arises if one deals with Böhm trees, and in that context it is usually handled by using finite approximations to infinite trees and proving some kind of continuity theorem. Here we propose a different approach. While in the usual treatment of Böhm trees one deals exclusively with infinite λ-terms which are in normal form, we take the more liberal view of considering a full fledged infinite λ-calculus in which we allow arbitrary infinite λ-terms and infinite reductions among them. This extra freedom allows us to state and prove some general results about infinite λ-calculus of which $(*)$ is an immediate consequence. More precisely we formulate two versions of infinite λ-calculus, an infinite $\lambda\beta$-calculus and an infinite $\lambda\beta\bot$-calculus. The infinite $\lambda\beta$-calculus is not Church-Rosser and not normalizing (with respect to infinite β-reductions), but if a finite term has an infinite β-normal form, then it is unique. The infinite $\beta\bot$-calculus behaves better: it is Church-Rosser, at least for reductions starting from a finite term, and normalizing. So a finite term A has one and only one $\beta\bot$-normal form, which coincides with what we have called $NF^{\infty}(A)$. The equality $(*)$ follows at once since the two sides coincide with the (unique) infinite $\beta\bot$-normal form of $A[x := B]$.

In our approach finite approximations are not explicitly used but they are somehow implicit in the notion of infinite reduction.

In section 14 we prove, that mute terms are much more "undefined" than the unsolvable terms in the sense that it is consistent with the $\lambda\beta$-calculus to simultaneously equate all mute terms to an arbitrary closed term (not necessarily mute). In particular each mute term is *easy*, i.e. it can be consistently equated to every closed term. The proof consists in defining a suitable Church-Rosser extension of λ-calculus. The method of proving consistency results via Church-Rosser extensions was used by Mitschke (see [2] Section 15.3, [19]) and by Intrigila [12] in a more complex situation. The method is improved in [4] where the use of infinite Böhm trees is introduced. A general overview is given in [5].

The fact that all mute terms can be simultaneously equated to an arbitrary closed fixed term is a very strong property which is not shared by the class of the easy terms, or even by the smaller class of the closed recurrent zero terms: for instance $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ and $\Omega\mathbf{I}$ (where $\mathbf{I} \equiv \lambda x.x$) are two closed recurrent zero terms which cannot be simultaneously equated to $\lambda xy.x$.

B. Intrigila [12] showed that the class of the easy terms is not semantically stable in the sense that if we equate all the closed easy terms to Ω, then the rules of the $\lambda\beta$-calculus force us to equate an easy term to a non-easy one. Our main result shows instead that the class of all mute terms is stable, in the sense that it is possible to have a model in which all mute terms are identified without having equalities between mute and non-mute terms.

We finish the paper by comparing mute terms with another class of previously studied easy terms, namely closed recurrent zero terms studied in [15] (A is recurrent if every reduct of A can be reduced to A). Neither class is included in the other but every closed recurrent zero term is a *strong zero term of finite degree* (see section 15), namely it is β-convertible to a term of the form $AM_1 \dots M_n$ with A mute. We show that every strong zero term of finite degree is easy. (This is an immediate consequence of the fact that mute terms are easy and that if A is easy, then AM is also easy.) We thus obtain a strenghtening of the fact that closed recurrent zero terms are easy.

By our results the problem of which zero terms are easy, is reduced to the case of the strong zero terms of infinite degree. Here both possibilities can happen and some difficult problems remain: for instance it is still not known whether $\mathbf{Y_t}\Omega_3$ is easy (see [15, 16, 4]). We hope that our results will shed some light on the elusive nature of easy-terms.

We finish this introduction with the following two remarks, which point out the limitations of the infinite $\beta\bot$-normal forms. First it is clear that there is no "Böhm out technique" for the infinite $\beta\bot$-normal forms. Secondly note that while the unsolvable terms form a Π_1 set, the mute terms form prima facie a more complicated set, namely a Π_2 set (complete?). Moreover, unlike the case of Böhm trees, there seems to be no general algorithm to compute higher and higher finite approximations to an infinite $\beta\bot$-normal form, although in many special cases, for instance the case of Ω_3, this is possible.

For related work in the area of term rewriting systems and infinite first order terms see the last section.

Notation: A *context* $C[\ ]$ is a term containing some occurrences of a special constant "hole". $C[B]$ is the term obtained by replacing all the occurrences of the holes with the term B. With the notation $A[x := B]$ we indicate as usual the result of substituting all the free occurrences of x in A with B after renaming the bound variables of A in such a way that the free variables of B do not become bound in $A[x := B]$. The difference between substitutions and contexts is that the free variables of B might become bound in $C[B]$ but not in $A[x := B]$.

A *notion of reduction* is an arbitrary binary relation on lambda-terms. We consider also notion of reductions ρ different from β-reduction. $\to_\rho$ denotes one-step ρ-reductions, i.e. the closure of ρ under substitutions and contexts (see [2] Definition 3.1.5, p. 51). $\Rightarrow_\rho$ is the reflexive and transitive closure of $\to_\rho$ and $\to_{=\rho}$ is the reflexive closure of $\to_\rho$. If we omit subscripts we mean β-reduction. The sign $\equiv$ between terms stands for syntactical identity up to renaming of bound variables (α-conversion). The sign $=$ between two terms stands for provable equality in some theory (in most of the cases $=$ is β-convertibility).

Added in proofs: An infinitary version of λ-calculus has beed independently introduced by Kennaway, Klop, Sleep and Van de Vries in a recent manuscript [18]. Our approach is different since: 1) we do not equate all the unsolvable closed terms; 2) we allow infinite terms of the form $(((\ldots)A_2)A_1)A_0$ (infinitely many parenthesis). In this paper the Church-Rosser theorem for infinite $\beta\bot$-reductions is only proved for the class of those infinite terms B which arise from finite terms, in the sense that there is an infinite reduction $A \to_{\beta\bot\infty} B$ with A a finite term. Later investigations in collaboration with B. Intrigila showed that the Church-Rosser property holds even without this restriction (see also [5]).

2 Infinite λ-terms and normal forms

We identify λ-terms with their parsing trees, so we write λ-terms either in linear form or in tree form. This is convenient when we consider infinite λ-terms. An infinite λ-term is defined as a finite or infinite rooted tree such that each leaf is labeled by a variable and the inner nodes are either binary "application nodes", or unary "abstraction nodes", in which case they have a label of the form λx where x is a variable. For later purposes we expand the language with a constant $\bot$ which can then appear as a label of a leaf. Unless otherwise stated "term" means "infinite λ-term" possibly containing some occurrences of $\bot$. Finite λ-terms are special cases of infinite λ-terms. We have seen examples of infinite λ-terms in the introduction, for instance $BT(\mathbf{Y_t})$ or $NF^\infty(\Omega_3)$. When we write terms in linear form we follow the usual convention of left-associativity, thus ABC is $(AB)C$ etc. This corresponds to a tree whose root has left-son AB and right-son C. The term $\lambda x.A$ is identified with the tree with root λx having the son A.

The Böhm tree of a (finite) λ-term A is an infinite λ-term $BT(A)$ (possibly containing some occurrences of $\bot$), defined as follows.

- if there is a (multistep) β-reduction of the form

  $A \Rightarrow \lambda x_1, \ldots, x_n.x_i A_1 \ldots A_k$,

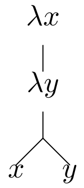
  then

$BT(A) \equiv \lambda x_1, \ldots, x_n . x_i BT(A_1) \ldots BT(A_k)$.

- if A is *unsolvable*, i.e. there is no reduction of the above form, then $BT(A) = \perp$.

Notice that a finite term in normal form coincides with its Böhm tree. It can be shown that the map $A \mapsto BT(A)$ is single valued, namely it does not depend on the non-deterministic choice of the β-reductions in the definition of $BT(A)$.

Our way of depicting Böhm trees is different from the usual one, for instance we write $\lambda x, y.xy$ as



This change of notation is justified by the extension of the notion of Böhm tree that we want to do. We recall that a *zero term* is a term which cannot be reduced (by a multiple β-reduction) to an abstraction term, i.e. to a term of the form $\lambda x.A$ (it then follows that a zero term is not β-convertible to an abstraction term). Note that a variable is a zero term. The key property of zero terms is that if A is a zero term, M is an arbitrary term, and $AM \Rightarrow B$, then B has the form $A'M'$ with $A \Rightarrow A'$ and $M \Rightarrow M'$. The infinite $\beta\perp$-normal form $NF^\infty(A)$ of a term A is defined as follows.

- If $A \Rightarrow x$ for some variable x, then $NF^\infty(A) \equiv x$.
- If $A \Rightarrow \lambda x.B$ for some B, then $NF^\infty(A) \equiv \lambda x.NF^\infty(B)$.
- If $A \Rightarrow BC$ where B is a zero term, then $NF^\infty(A) \equiv NF^\infty(B)NF^\infty(C)$.
- In the remaining cases we say that A is *mute* and we set $NF^\infty(A) \equiv \perp$.

Thus a term is mute if and only if it is a zero term and it is not β-reducible to a variable or to a zero term applied to some other term.

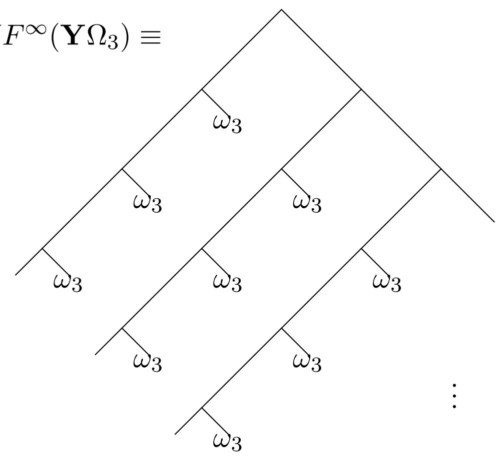
The fact that $A \mapsto NF^\infty(A)$ is single valued will follow from the Church-Rosser theorem for infinite $\beta\perp$-reductions (starting from a finite term) to be proved later.

Mute terms can be characterized as follows. We say that a term is in *top normal form* (or is *a* top normal form) if it is either a variable, or an abstraction term $\lambda x.T$, or a term of the form BC where B is a zero term. We say that A *has a top normal form* if there is a β-reduction $A \Rightarrow A'$ with A' in top normal form. It follows from the definitions that a term A is mute iff it has no top normal form. (If we work in ordinary $\lambda\beta$-calculus a term is in top normal form if and only if it can never be reduced to a β-redex, however this characterization fails if we extend the language with a new constant $\perp$. According to our definition $\perp$ is mute.)

The name top normal form is justified by the remark that any β-reduct of a term in top normal form is in top normal form. Since to be a zero term is an undecidable property, to *be* in top normal form is also undecidable (and to *have* a top normal form is even more complex). The intuitive reason for the undecidability is that, unlike what happens for head normal forms (see [2]), in order to recognize if a term is in top normal form, it does not suffice to look at the first level of its tree-representation, it is sometimes necessary to look at the whole term.

Example 2.1 We have already seen in the introduction the infinite normal form of Ω_3. Now let $\mathbf{Y_t}\Omega_3$ be the Curry fixed point of Ω_3. Then $\mathbf{Y_t}\Omega_3 \rightarrow \Omega_3(\mathbf{Y_t}\Omega_3)$ and it can be shown that $NF^\infty(\mathbf{Y_t}\Omega_3) \equiv NF^\infty(\Omega_3)NF^\infty(\mathbf{Y}\Omega_3)$. In tree-form this means:

$NF^\infty(\mathbf{Y}\Omega_3) \equiv$



It is easy to see that the infinite normal form of a term is "finer" than its Böhm tree, in the sense that $NF^\infty(A)$ can be obtained from $BT(A)$ by replacing all the $\perp$'s in $BT(A)$ by suitable terms (possibly containing other $\perp$'s). This depends on the fact that every term in head-normal form is in top normal form but not conversely, for instance Ω_3 has no head-normal form.

Notice that an infinite $\beta\perp$-normal form can have infinitely many initial abstractions, consider for instance $NF^\infty(K^\infty)$ where K^∞ is the Turing fixed point of $K \equiv \lambda x, y.x$. Then $NF^\infty(K^\infty) \equiv \lambda x_1, x_2, x_3, x_4, \ldots$. On the other hand since K^∞ is unsolvable, the Böhm tree of K^∞ is $\perp$.

3 Infinite $\lambda\beta$-calculus

So far the notion of infinite $\beta\perp$-normal form that we have defined in the previous section is not associated to any notion of reduction. Moreover we have defined $NF^\infty(A)$ only for a finite term A. In the sequel of this paper we will fill these gaps by defining an infinite $\lambda\beta$-calculus (in this section) and an infinite $\lambda\beta\perp$-calculus (in later sections).

First notice that β-reduction $\to$ is defined for infinite lambda terms in exactly the same way as for finite ones, namely $(\lambda x.A)B \to A[x := B]$ (as usual $\to$ is closed under substitutions and contexts). Having defined β-reduction we can extend the notion of zero term, mute term, etc. to the infinite terms. Therefore the definition of $NF^\infty(A)$ makes sense also for infinite terms.

To obtain the infinite $\lambda\beta$-calculus we will define a notion of infinite β-reduction $\to_\infty$.

Definition 3.1 Given two terms A and B, we say $A \equiv_n B$ if A and B coincide up to the n-th level of their tree-representation. More precisely:

1. $A \equiv_0 B$ iff A and B have the same root.
2. $A \equiv_{n+1} B$ iff A and B have the same root and each immediate subterm of A is in relation $\equiv_n$ to the corresponding immediate subterm of B.

If n is negative we make the convention that $A \equiv_n B$ holds for every A and B.

Note that the root of a term, determines in particular whether the term is an application term (of the form BC), or an abstraction term (of the form $\lambda x.U$), or a variable. In the last two cases the root specifies also which is the corresponding variable.

As an example we have: $AB \not\equiv_0 \lambda x.U \equiv_0 \lambda x.V \not\equiv_0 \lambda y.Q$ where x and y are different variables. Note that $A \equiv B$ iff $A \equiv_n B$ for every n.

Since terms are trees, they possess a natural topology and a notion of limit.

Definition 3.2 Let $< A_n \mid n \in \omega >$ be a sequence of terms. We say $\lim < A_n > \equiv A$ if A is a term, and $\forall k \exists n \forall m \geq n \;\; A_m \equiv_k A$.

Definition 3.3 The *depth* of a specific occurrence of a subterm A in B, is defined as the length of the path connecting the root of B to the root of A in the tree-representation of B.

Example 3.4 The depth of A in A is 0. The depth of A in $\lambda x.A$ is 1. The depth of A in $\lambda x.AB$ and in $\lambda x.BA$ is 2. The depth of A in $\lambda x.ABC$ is 3.

Definition 3.5 The depth of a β-reduction $A \to B$ is defined as the depth of the redex being contracted. The depth of a multistep β-reduction $A \Rightarrow B$ is the minimum of the depths of the one-step β-reductions of which the multistep reduction is composed. The empty reduction has infinite depth.

Remark 3.6 If the reduction $A \Rightarrow B$ has depth n, then $A \equiv_{n-1} B$.

We define infinite β-reduction $A \to_\infty B$ as follows.

Definition 3.7 Let $\sigma\colon A_0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \ldots$ be an infinite sequence of β-reductions such that the depth of the reduction $A_i \Rightarrow A_{i+1}$ tends to infinity with i. Then by the previous remark $B \equiv \lim < A_i >$ exists and we set by definition $A_0 \to_\infty B$ via the reduction σ. We also say that the ω-sequence σ *converges*.

So an ω-sequence of reductions is an infinite reduction if and only if it converges. Note that by our convention empty-reductions have infinite depth. It follows that $A \Rightarrow B$ implies $A \to_\infty B$.

Definition 3.8 The depth of an infinite reduction $A \to_\infty B$ given by $A \equiv A_0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \ldots \to_\infty B$, is defined as the mimimum of the depths of the reductions $A_i \Rightarrow A_{i+1}$.

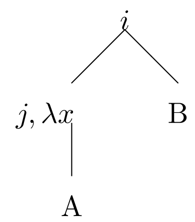
Remark 3.9 Let $\omega \equiv \lambda x.xx$. The infinite sequence of β-reductions $\omega\omega \to \omega\omega \to \omega\omega \to \ldots$ does not converge because the depth of redexes does not go to infinity. However we still have $\omega\omega \to_\infty \omega\omega$ (with another reduction) because empty reductions have infinite depth .

4 Residuals under infinite β-reductions

The residuals of a subterm under a (finite) β-reduction can be defined as in the case of finite lambda-terms. A simple way of doing this is to introduce labels. We do this in some detail in order to extend it to infinite reductions. We make a distinction between "residuals" and "extended residuals". The latter will be used in section 11.

Labeled terms are terms in which some (possibly all) subterm occurrences have a label (we can take the natural numbers as set of labels). Since terms are trees and subterms are subtrees, we can consider a labeled term as a tree in which some nodes are labeled by a natural number. Substitutions are defined for labeled terms as follows: $A[x := B]$ is obtained by erasing all the labels attached to the free occurrences of x in A and then replacing all such free occurrences with the term B (with all its labels). So if a free occurrence of x has a label, then it looses its label in the substitution $x[x := B]$. (This ensures that there are no overlappings of labels when a substitution $A[x := B]$ occurs.)

Having defined $A[x := B]$ for labeled terms we can define β-reduction for labeled terms as usual: $(\lambda x.A)B \to A[x := B]$. We can depict this in tree-form:



$\to_\beta \; A[x := B]$

Note that the labels i and j are lost during the reduction. Clearly if B has a label, then $A[x := B]$ has as many copies of that label as the number of free occurrences of x in A (which loose their labels).

We now define the *residuals* of a specific occurrence of a subterm $A \subset B$ under a β-reduction $B \to C$. This is done as follows: give a label n to the given occurrence of A in B (and to nothing else). Perform the β-reduction $B \to C$ and look for the set of all subterms of C with label n. These subterms (actually subterm occurrences) are the residuals of A. From the above picture we see that if a redex is contracted, it has no residuals.

We now extend these notions to infinite reductions. If $A \to_\infty B$ via the sequence of reductions $A \equiv A_0 \to A_1 \to A_2 \to \ldots$, and if some subterms of A are labeled, then the labeling of A induces a unique labeling of the A_i's.

Moreover since the depth of the reductions $A_i \to A_{i+1}$ tends to infinity, there is a unique labeling of B such that the A_i's converge to B as labeled terms. It thus make sense to define the residuals of a subterm of A under an infinite β-reduction.

Extended residuals are defined exaclty as residuals except that we add the further clause that if $A \to A'$, then A' is an extended residual of A. A precise definition can be obtained by modifying the above picture in such a way that the label i is not lost but it is adjoined to the root of $A[x := B]$ in addition to the other labels possibly present. In this process the labels can cumulate, so unlike what happens for residuals, two subterms can have the same extended residual.

5 Failure of the Church-Rosser property for infinite $\lambda\beta$-calculus

The next example shows that the Church-Rosser property fails for infinite β-reductions.

Example 5.1 Let $I \equiv \lambda x.x$, $\omega \equiv \lambda x.xx$ and $Q \equiv \lambda x.I(xx)$. We have the reductions

$$\begin{array}{ccccccc} QQ & \to & I(QQ) & \to & I(I(QQ)) & \to_\infty & I(I(I(I(\ldots)))) \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ \omega\omega & \to & \omega\omega & \to & \omega\omega & \ldots & ? \end{array}$$

but there is no reduction, whether finite or infinite, from $I(I(I(I(\ldots))))$ (infinitely many I's) to $\omega\omega$.

The term QQ responsible for the failure of the Church-Rosser property is mute. Later we will see that mute terms are the only responsible for the failure of the Church-Rosser property in the sense that if we add a rule sending all the mute terms to $\perp$, then the Church-Rosser property for infinite reductions is restored (at least if we start from a finite term).

6 Projections of β-reductions

For finite λ-calculus given two reductions σ and ρ starting from the same term, there is a well known manner to define in a canonical way a reduction σ/ρ called the *projection* of σ over ρ. We investigate up to what extent this can be extended to infinite $\lambda\beta$-calculus. We will see that there are limitations. We start by recalling the finite case.

Definition 6.1 Let A be a finite term. Consider two β-reductions $\sigma: A \to B$, and $\rho: A \to C$ and suppose that Δ is the redex contracted by σ. Define $\sigma/\rho: C \Rightarrow D$ as the multistep β-reduction obtained by reducing from left to right all the residuals of Δ under ρ.

Proposition 6.2 *If A is a finite term and $B \overset{\rho}{\leftarrow} A \overset{\sigma}{\to} C$, then ρ/σ and σ/ρ end up in the same term D, so that we have the following diagram*

$$\begin{array}{ccc} A & \overset{\rho}{\to} & B \\ \sigma\downarrow & & \Downarrow \sigma/\rho \\ C & \overset{\rho/\sigma}{\Rightarrow} & D \end{array}$$

Proof. It is instructive to recall the proof. If the two redexes being reduced in $B \leftarrow A \to C$ are disjoint or coincide, the result is obvious. If they are nested one inside the other, then, after removing the outer context, we are in one of the following two cases.

Case 1.

$$\begin{array}{ccc} (\lambda x.A)B & \to & A[x := B] \\ \downarrow & & \Downarrow \\ (\lambda x.A)B' & \to & A[x := B'] \end{array}$$

where the vertical reductions are induced by a given β-reduction $B \to B'$.

Case 2.

$$\begin{array}{ccc} (\lambda x.A)B & \to & A[x := B] \\ \downarrow & & \downarrow \\ (\lambda x.A')B & \to & A'[x := B] \end{array}$$

where the vertical reductions are induced by a given β-reduction $A \to A'$.
QED

Definition 6.3 The diagram appearing in the statment of the above proposition is called the *elementary* diagram determined by σ and ρ, and the reduction σ/ρ is called the *projection* of σ over ρ (similarly for ρ/σ).

Note that an elementary diagram can split on at most one side, i.e. either σ/ρ or ρ/σ is a one-step or empty β-reduction. To define the projection of a multistep reduction, we need to recall the notion of development.

Definition 6.4 A finite or infinite sequence of reductions $A \to A_1 \to A_2 \to \ldots$ is called a *development* if it reduces only residuals of redexes occurring in A.

Theorem 6.5 *(see [2]) If A is finite, then all developments starting from A are finite.*

Proposition 6.6 *(see [2]) If A is finite, and $B \overset{\rho}{\Leftarrow} A \overset{\sigma}{\Rightarrow} C$ are multistep β-reductions, we can define in a canonical way the projections σ/ρ and ρ/σ so that we have:*

$$\begin{array}{ccc} A & \overset{\rho}{\Rightarrow} & B \\ \sigma \Downarrow & & \Downarrow \sigma/\rho \\ C & \overset{\rho/\sigma}{\Rightarrow} & D \end{array}$$

Proof. (Sketch) The projection σ/ρ is defined by putting many elementary diagrams side to side until they "fill the rectangle" and yield the desired reductions. However since elementary diagrams can split, we need finiteness of developments to see that this process terminates after finitely many steps. QED

Since the Church-Rosser property fails for infinite $\lambda\beta$-calculus there is no hope to define σ/ρ for arbitrary infinite β-reductions σ and ρ. However in some special cases this can be done.

Definition 6.7 If A is an infinite term and $B \overset{\rho}{\leftarrow} A \overset{\sigma}{\to} C$ are single step reductions, we define σ/ρ as the possibly infinite β-reduction which is obtained by reducing in some fixed order the residuals under ρ of the redex Δ contracted in $A \overset{\sigma}{\to} C$ (the precise order is not important: the residuals are all disjoint and we can reduce them in any order obtaining an infinite sequence of reductions in which the depth tends to infinity).

The following example shows that σ/ρ can be infinite even if σ and ρ are finite.

Example 6.8 Let $T \equiv \lambda x.x(x(x\ldots))$ (infinitely many x's) and let Δ be a β-redex with contractum Δ'. Then we have the diagram

$$\begin{array}{ccc} T\Delta & \overset{\sigma}{\to} & T\Delta' \\ \rho \downarrow & & \downarrow \rho/\sigma \\ \Delta(\Delta(\Delta\ldots)) & \overset{\sigma/\rho}{\to}_{\infty} & \Delta'(\Delta'(\Delta'\ldots)) \end{array}$$

It is easy to see that σ/ρ is well defined as an infinite β-reduction in the sense that the depth of the redexes which are reduced tends to infinity.

Another special case in which we can define σ/ρ is the following.

Definition 6.9 If A_0 is a finite term and σ is an ω-sequence of multistep β-reductions $A_0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \ldots$, and ρ is a multistep β-reduction $A_0 \Rightarrow B_0$, we define the ω-sequence of reductions $\sigma/\rho\colon B_0 \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \ldots$ in such a way that each $B_i \Rightarrow B_{i+1}$ is the projection of $A_i \Rightarrow A_{i+1}$ under ρ. We thus obtain

$$\begin{array}{rccccccc} \sigma\colon & A_0 & \Rightarrow & A_1 & \Rightarrow & A_2 & \Rightarrow & \ldots \\ & \Downarrow & & \Downarrow & & \Downarrow & & \\ \sigma/\rho\colon & B_0 & \Rightarrow & B_1 & \Rightarrow & B_2 & \Rightarrow & \ldots \end{array}$$

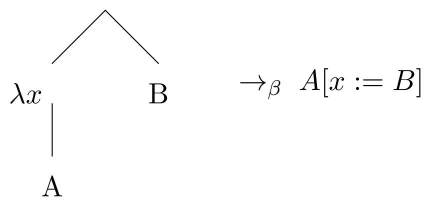
It might happen that σ converges to an infinite reduction $\sigma\colon A \to_\infty \lim < A_i >$, while the ω-sequence σ/ρ does not converge to an infinite reduction (as the depth of $B_i \Rightarrow B_{i+1}$ might not go to infinity). This is exactly what happens in the example showing the failure of the Church-Rosser property. However if σ/ρ *does converge* to an infinite reduction, then $\sigma/\rho\colon B_0 \to_\infty \lim < B_i >$ is called the *projection of σ over ρ*.

7 Depth of redexes

All terms in this section are assumed to be finite. We investigate how the relation $\equiv_n$ behaves in connection with β-reduction. The general flavour of the results of this section is that the depth of a subterm cannot decrease too much under a finite β-reduction (although it can increase by an arbitrary amount). The next proposition shows that deep subterms have deep residuals.

Proposition 7.1 *If T' is a residual of T in a β-reduction $A \to B$, then the depth of T' in B is greater or equal than the depth of T in A minus* 2.

Proof. Clear from the tree-representation of β-reduction:



In the above picture the root of A goes up from depth 2 on the left to depth 0 on the right (provided that A is not a variable in which case it has no residuals). A case analysis shows that all the other subterms do not decrease in depth by more than two. This clearly holds even if the reduction takes place inside a context. QED

Note that the depth of B can increase by an arbitrary amount under the above depicted β-reduction (although it cannot decrease by more than 1).

Definition 7.2 Given a term A a *position* in A is a finite sequence of ternary digits which tell us whether to go down, left or right in the tree-representation of A. Two reductions $A \to A'$ and $B \to B'$ are called *similar* if they are obtained by reducing two redexes Δ in A and Δ' in B which occur at the same position.

Note that similar reductions have the same depth.

Proposition 7.3 *If $A \equiv_n B$ and the two reductions $A \to A'$ and $B \to B'$ are similar, then $A' \equiv_{n-2} B'$.*

Proof. By Proposition 7.1. QED

We now consider what happens to the depth of a subterm in an elementary diagram.

Proposition 7.4 *Consider two β-reductions $B \overset{\rho}{\leftarrow} A \overset{\sigma}{\to} C$. We have:*

1. *If σ is strictly deeper than ρ, then ρ and ρ/σ are similar.*
2. *The depth of ρ/σ is greater or equal than the depth of ρ minus* 2.
3. *If σ and ρ have depth $\geq n$, then also the reductions σ/ρ and ρ/σ have depth $\geq n$.*

Proof. By a simple case analysis. QED

8 Transitivity of infinite β-reductions

We prove that if $A \to_\infty B \to_\infty C$, then $A \to_\infty C$. We begin with a preliminary result.

Lemma 8.1 *If $A \to_\infty B$ and $B \to C$, then $A \to_\infty C$.*

Proof. Let $A \to_\infty B$ be given by the sequence of β-reductions $A \equiv B_0 \to B_1 \to B_2 \to \ldots \to_\infty B$. This means that the depth of the reductions $\sigma_i \colon B_i \to B_{i+1}$ tends to infinity as $i \to \infty$, and $\lim < B_i > \equiv B$. If we choose n big enough we can write:

$$\begin{array}{ccccccccc} A & \Rightarrow & B_n & \to & B_{n+1} & \to & B_{n+2} & \to & \ldots \to_\infty & B \\ & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ & & C_n & \to_\infty & C_{n+1} & \to_\infty & C_{n+2} & \to_\infty & \ldots & C \end{array}$$

where all the vertical reductions are similar in the sense of Definition 7.2 and the horizontal ones, with the exception of $A \Rightarrow B_n$, are deeper than the vertical ones. It suffices to show that $C_n \to_\infty C$. It is clear that the reductions $C_i \to C_{i+1}$ have a depth tending to infinity with i (since those of the upper row do), but the resulting "reduction" from C_n to C has length $\omega \cdot \omega$ (at most) instead of ω. If A is finite, then all the reductions from C_i to C_{i+1} are finite (for $i \geq n$), and we have $C_n \to_\infty C$. If A is infinite, we have to reorder the $\omega \cdot \omega$-sequence of reductions in order to get an equivalent converging sequence of length ω. This can be done as follows. Let $\Delta \equiv (\lambda x.S)T$ be the redex reduced in $B_n \to C_n$. Since for $i \geq n$ the reductions $B_i \to B_{i+1}$ are deeper than the reductions $B_i \to C_i$, Δ has one and only one residual $\Delta_i \equiv (\lambda x.S_i)T_i$ in each B_i and C_i is obtained from B_i by replacing Δ_i with its contractum $S_i[x := T_i]$. The reason why the reduction $C_i \to_\infty C_{i+1}$ can be infinite is that, although T_{i+1} is certainly obtained from T_i by a one-step or empty β-reduction, there might be infinitely many occurrences of T_i inside $S_i[x := T_i]$. However an important point to notice is that the various reductions $T_i \to_{=\beta} T_{i+1}$ (for the various i's and the various occurrences of T_i) do not affect and are not affected in any way by the surrounding context in the sense that:

1) nothing is substituted inside the T_i's by the outer context (since the reductions $B_i \to B_{i+1}$ have a depth bigger than the one at which Δ_i occurs);

2) No T_i is the first half of a redex which is reduced somewhere in the above diagram (i.e. in the above diagram there are no reductions of redexes of the form $(\lambda x.U)V$ where $\lambda x.U$ is one of the T_i's).

It follows that the reductions affecting the various occurrences of the T_i's and the outer context are independent of each other and they can be performed in any order. All these reductions can then be reordered in a converging ω-sequence yielding the desired reduction $C_n \to_\infty C$. QED

Theorem 8.2 *If $A \to_\infty B \to_\infty C$, then $A \to_\infty C$.*

Proof. Let $A \to_\infty B$ be given by $A \equiv A_0 \to A_1 \to A_2 \to \dots \to_\infty B$, and let $B \to_\infty C$ be given by $B \equiv B_0 \to B_1 \to B_2 \to \dots \to_\infty C$. By a repeated application of the proof of the previous lemma we can construct a diagram of the form

$$\begin{array}{ccccccccc}
A & \Rightarrow & D_0 & \dots & & & & \to_\infty & B_0 \\
 & & \downarrow & & & & & & \downarrow \\
 & & C_1 & \Rightarrow & D_1 & \dots & & \to_\infty & B_1 \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & C_2 & \Rightarrow & D_2 & \to_\infty & B_2 \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & C_3 & \to_\infty & B_3 \\
 & & & & & & & \vdots & \downarrow_\infty \\
 & & & & & & & & C
\end{array}$$

where each vertical reduction $D_i \to C_{i+1}$ is similar to $B_i \to B_{i+1}$ and the depth of the reductions $C_i \Rightarrow D_i \to_\infty B_i$ is bigger or equal than the depth of $B_{i-1} \to B_i$ mimus two. It follows that the depth of the reductions $C_i \Rightarrow D_i$ and $D_i \to C_{i+1}$ tends to infinity with i and therefore $A \to_\infty \lim < C_i > \equiv \lim < B_i > \equiv C$. QED

9 Unicity of infinite β-normal forms

If a (possibly infinite) term A has no β-redexes, we say that *A is in β-normal form*, written $A \in NF$. We say that *A has an infinite β-normal form* if there is an infinite β-reduction $A \to_\infty A'$ with A' in β-normal form. Some terms, for instance the mute terms, do not have an infinite β-normal form. In this section we prove that if a finite term A has an infinite β-normal form, then it has a unique infinite β-normal form.

The next proposition shows that we can bound the depth of a development just looking at the starting term. The proof is easy and left to the reader.

Proposition 9.1 *If A is a term and $\mathcal{F}$ is a set of redex occurrences in A each of which has depth $\geq n$, then every development $A \Rightarrow B$ which reduces only the residuals of the redexes in $\mathcal{F}$ has depth $\geq n$.*

Lemma 9.2 *If A is a finite term and $C \leftarrow A \rightarrow_\infty B$ with $B \in NF$, then $C \rightarrow_\infty B$.*

Proof. Since A is a finite term, we can use Definition 6.9 to contruct a diagram of the form

$$\begin{array}{ccccccccccc} A & \equiv & B_0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & \dots & \rightarrow_\infty & B \\ & & \downarrow & & \Downarrow & & \Downarrow & & & & \\ C & \equiv & C_0 & \Rightarrow & C_1 & \Rightarrow & C_2 & \Rightarrow & \dots & & \end{array}$$

where each subdiagram

$$\begin{array}{ccc} B_i & \rightarrow & B_{i+1} \\ \Downarrow & & \Downarrow \\ C_i & \Rightarrow & C_{i+1} \end{array}$$

is defined by taking projections. All the vertical reductions in this diagram are developments since they contract only the residuals of the redex Δ contracted in $A \rightarrow C$. Since $B \equiv \lim < B_i >$ is a normal form (hence it has no redexes), the minimal depth of the residuals of Δ in B_i tends to infinity with i, and therefore also the depth of the reduction $B_i \Rightarrow C_i$ tends to infinity by Proposition 9.1. But then $\lim < C_i >$ exists and it is equal to $\lim < B_i >$, i.e. $\lim < C_i > \equiv B$.

Since the depths of $B_i \Rightarrow B_{i+1}$ and $B_i \Rightarrow C_i$ tend to infinity with i, by Proposition 7.4 also the depth of the reductions $C_i \Rightarrow C_{i+1}$ tends to infinity, hence $C_0 \rightarrow_\infty B$. QED

The notion of zero term has been defined in terms of finite β-reductions. The next lemma and the more general result of Lemma 10.5, show that zero terms are well behaved with respect to infinite reductions.

Lemma 9.3 *If A is finite and $A \rightarrow_\infty B \in NF$, then A is a zero term if and only if B is a zero term.*

Proof. If A is not a zero term, then there is an abstraction term $\lambda x.T$ such that $A \Rightarrow \lambda x.T$. Since $B \in NF$ by the previous lemma $\lambda x.T \rightarrow_\infty B$. But then B must be of the form $\lambda x.T'$ and therefore B is not a zero term.

Conversely if B is not a zero term, then being a normal form it must be an abstraction term. Since $A \rightarrow_\infty B$, in this reduction there is an intermediate finite step $A \Rightarrow A' \to_\infty B$ where A' coincides with B up to depth 1. Hence A' is an abstraction term and A is not a zero term. QED

Lemma 9.4 *Let B be a normal form and for every i let $B_i \to_\infty B$ be a reduction of depth $\geq i$ with B_i a finite term. If for each i we have a reduction $B_i \to_\infty C_i$, then the depth of the reductions $B_i \to_\infty C_i$ tends to infinity with i.*

Proof. It is enough to show that the depth of $B_i \to_\infty C_i$ is greater than 0 if i is big enough, for then we can apply the same reasoning to the subterms of B_i, B and C_i.

Case 1. For some i, B_i is a variable or an abstraction term $\lambda x.U$. Then the result is clear: every multistep β-reduction starting from B_i has depth > 0 (if we work in the expanded language with $\perp$ we can treat $\perp$ as a free variable, since so far we have not introduced any special rule concerning $\perp$).

Case 2. B_i is an application term, say $B_i \equiv U_i V_i$. We can assume $i > 0$, so the depth of $B_i \to_\infty B$ is greater than 0. It then follows that B is of the form UV with $U_i \to_\infty U$ and $V_i \to_\infty V$. Clearly U is a zero term and a normal form, otherwise UV cannot be a normal form. Since $U_i \to_\infty U$, by Lemma 9.3 U_i is also a zero term. It follows that any reduction starting from $U_i V_i$ has depth > 0. QED

Theorem 9.5 *If A is finite, $B\,{}_\infty\!\leftarrow A \to_\infty C$ and B, C are normal forms, then $B \equiv C$.*

Proof. Let $A \to_\infty B$ be given by $A \equiv B_0 \to B_1 \to B_2 \to \ldots \to_\infty B$. Since C is a normal form, by Lemma 9.2 the reduction $B_0 \to_\infty C$ induces reductions $B_n \to_\infty C$ for every n and we can construct a diagram

$$\begin{array}{ccccccccc} B_0 & \to & B_1 & \to & B_2 & \to & \ldots & \to_\infty & B \\ \downarrow_\infty & & \downarrow_\infty & & \downarrow_\infty & & & & \\ C & \equiv & C & \equiv & C & \equiv & \ldots & & \end{array}$$

In order to show that $B \equiv C$ it suffices to show that the depth of the reductions $B_i \to_\infty C$ tends to infinity. Here we use the fact that B is a normal form and Lemma 9.4. QED

10 Head reductions

We recall the well known notion of head-redex and internal redex and we extend it to infinite terms. An important warning is that an infinite term can have a redex without having a left-most redex.

If a term T is of the form $T \equiv \lambda x_1, \dots, x_k.\Delta M_1 \dots M_n$ where $k, n \geq 0$ and Δ has the form $(\lambda x.A)B$, then we say that the displayed occurrence of Δ in T is the *head-redex* of T. An *internal redex* is a redex which is not the head-redex. An *head-reduction* is a β-reduction $A \Rightarrow_h B$ where only head redexes are reduced. An *internal reduction* is a β-reduction $A \Rightarrow_i B$ where only internal redexes are reduced.

Infinite internal reductions $A \to_{i\infty} B$ are defined in the obvious way and we will see that if $A \to_{i\infty} B$, then A is a top normal form if and only if B is such.

The main result of this section is that if a (possibly infinite) term has a top normal form, then it can be reduced to a top normal form by an head reduction. For finite terms this would be an easy consequence of the standardization theorem of Curry and Feys that says that internal reductions can always be postponed, but for infinite terms the problem is more delicate because if we try to postpone a one-step internal reduction, then we might generate an infinite internal reduction (see Example 6.8).

Let us recall the situation for finite terms:

Theorem 10.1 *If A is finite, then every β-reduction $A \Rightarrow B$ can be factored as an head-reduction followed by an internal reduction.*

Proof. See [2] Lemma 11.4.6. p. 299. QED

The next lemma says in particular that we can posptone a single internal reduction after an head reduction at the expense of getting an infinite internal reduction. The result holds for infinite terms. Note that if $U \to_i V$, then U has an head redex if and only if V has an head redex.

Lemma 10.2 *If $U \to_i V$ and U has an head redex, then by taking projections we can construct a diagram of the form*

$$\begin{array}{ccc} U & \to_i & V \\ \downarrow_h & & \downarrow_h \\ V' & \to_{i\infty} & W \end{array}$$

Proof. Define $V' \to_{i\infty} W$ as the converging sequence of reductions obtained by reducing in some fixed order the residuals of the redex Δ contrated in $U \to_i V$. Example 6.8 gives an idea of the general situation. QED

Lemma 10.3 *Any diagram $C\ {}_h\!\!\leftarrow A \to_\infty B$ can be extended to a diagram $C \to_\infty B\ {}_{=h}\!\!\leftarrow B$.*

Proof. Let $A \equiv A_0$ and $C \equiv C_0$. The result follows by constructing the diagram:

$$\begin{array}{ccccccccc} A_0 & \to & A_1 & \to & A_2 & \to & \dots & \to_\infty & B \\ \downarrow_h & & \downarrow_{=h} & & \downarrow_{=h} & & & & \downarrow_{=h} \\ C_0 & \to_\infty & C_1 & \to_\infty & C_2 & \to_\infty & \dots & \to_\infty & D \end{array}$$

where each square is defined by taking projections and $B_i \to_{=h} C_i$ is the empty reduction if and only if the sequence $B_0 \to \dots \to B_i$ contains some head reductions. Reasoning as in Lemma 8.1 we get $C_0 \to_\infty D$. QED

We can now extend to infinite terms a result which is well known for finite terms:

Theorem 10.4 *If A is not a zero term, then there exists an head-reduction of the form $A \Rightarrow_h \lambda x.T$.*

Proof. Given a reduction of the form $\sigma: A \Rightarrow \lambda x.S$, we prove by induction on its length that there exists a term T and an head-reduction $A \Rightarrow_h \lambda x.T$ such that the length of $A \Rightarrow_h \lambda x.T$ is less or equal than the length of $A \Rightarrow \lambda x.S$.

If σ consists of a (possibly empty) head-reduction $A \Rightarrow_h B$ followed by a non-empty internal reduction $B \Rightarrow_i \lambda x.S$, then B has the form $\lambda x.T$ for some T and therefore we can take ρ to be $A \Rightarrow_h B$.

Otherwise σ can be factored as $A \Rightarrow U \to_i V \to_h W \Rightarrow_i \lambda x.S$. Since V has an head redex and $U \to_i V$ is an internal reduction, U has an head-redex. Hence there is V' and an head reduction $U \to_h V'$. By Lemma 10.2 we can write $A \Rightarrow U \to_h V' \to_{i\infty} W \Rightarrow_i \lambda x.S$. But then V' has the form $\lambda x.S'$ for some S' and therefore we can apply the induction hypothesis to $A \Rightarrow U \to_h V'$. QED

We can now strengthen Lemma 9.3 as follows:

Lemma 10.5 *If $A \to_\infty B$ is an infinite β-reduction, then A is a zero term if and only if B is a zero term.*

Proof. If $B \Rightarrow \lambda x.T$, then $A \to_\infty \lambda x.T$ by transitivity, hence there is T' such that $A \Rightarrow \lambda x.T'$. This shows that if B is not a zero term, then A is not a zero term.

Conversely if A is not a zero term, then by Theorem 10.4 there is an head-reduction of the form $A \Rightarrow_h \lambda x.S$. By Lemma 10.3 there is a term D and reductions $B \Rightarrow_h D$ and $\lambda x.S \to_\infty D$. Hence D has the form $\lambda x.S'$ and B is not a zero term. QED

Corollary 10.6 *If $A \to_{i\infty} B$, then A is a top normal form if and only if B is a top normal form.*

Proof. The only non-trivial case is when A and B are application terms. But then we can write $A \equiv UV$ and $B \equiv U'V'$ with $U \to_\infty U'$ and $V \to_\infty V'$. By Lemma 10.5 U is a zero term if and only if U' is such. Hence A is in top normal form if and only if B is in top normal form. QED

We can now prove:

Theorem 10.7 *If a term A has a top normal form, then there exists an head-reduction $A \Rightarrow_h T$ with T in top normal form.*

Proof. Given a reduction $\sigma\colon A \Rightarrow B$ with B in top normal form, we prove by induction on the its length that there exists an head-reduction $\rho\colon A \Rightarrow T$ with T in top normal form and the length of $A \Rightarrow T$ less or equal than the length of $A \Rightarrow B$.

If σ consists of a (possibly empty) head-reduction $A \Rightarrow_h B'$ followed by a non-empty internal reduction $B' \Rightarrow_i B$, then by Corollary 10.6 B' is a top normal form and therefore we can take ρ to be $A \Rightarrow_h B'$.

Otherwise σ can be decomposed as $A \Rightarrow U \to_i V \to_h W \Rightarrow_i B$. Since V has an head redex, U has an head-redex. Hence there is V' and an head reduction $U \to_h V'$. By Lemma 10.2 we can write $A \Rightarrow U \to_h V' \to_{i\infty} W \Rightarrow_i B$. By Corollary 10.6 (applied twice), V' is a top normal form and therefore we can apply the induction hypothesis to $A \Rightarrow U \to_h V'$. QED

11 Substitution instances of zero terms

All terms and contexts in this section are possibly infinite terms and contexts in the expanded language with $\perp$. A substitution instance of a zero term is not necessarily a zero term, take for instance a zero term of the form $xT_1 \dots T_k$ where x is a variable. In this section we characterize those terms A such that every substitution instance of A is a zero term and we prove some related results.

Definition 11.1 We say that M is *not active* in the β-reduction $C[M] \Rightarrow T$ if no redex contracted in this reduction is of the form $(\lambda x.A)B$ where $\lambda x.A$ is an extended residual of M. We say that M is *not touched* in the β-reduction $C[M] \Rightarrow T$ if no redex contracted in this reduction is of the form $(\lambda x.A)B$ where $\lambda x.A$ is a subterm of an extended residual of M.

Note that "not touched" implies "not active". If a term is not touched in a reduction, then its residuals coincide with its extended residuals and they are just substitution instances of the term itself.

If M is not active in a β-reduction we can replace everywhere M and its "maximal" extended residulas by a free variable x still getting a valid reduction, where a maximal extended residual is an extended residual which is not properly included in any other one. Thus we have:

Lemma 11.2 *If $C[M] \Rightarrow T[M_1, \dots, M_n]$ where $M_1, \dots, M_n$ are all the maximal extended residuals of M and M is not active, then $C[x] \Rightarrow T[x, \dots, x]$ where x is free in $C[x]$.*

The above lemma holds a fortiori if M is not touched.

Definition 11.3 We say that M *goes to the head* in a reduction $C[M] \Rightarrow T$, if the given reduction contains an intermediate step of the form $C[M] \Rightarrow \lambda y_1 \dots y_n.M'Q_1 \dots Q_k \Rightarrow T$ where $n, k \geq 0$ and M' is an extended residual of M.

Remark 11.4

1. In any head reduction $C[M] \Rightarrow_h T$, M and its maximal extended residuals can be replaced by a free variable for as long as M does not go to the head.

2. From 1. it follows that if M goes to the head in $C[M] \Rightarrow_h T$, then there exists a reduction of the form $C[x] \Rightarrow_h \lambda y_1 \dots y_n.xT_1 \dots T_k$ where x is free in $C[x]$ and $n, k \geq 0$.

Lemma 11.5 *Let M be an arbitrary term and let x be free in $A[x]$. We have:*

1. *If $A[M]$ is a zero term, so is $A[x]$.*
2. *If $A[M]$ is a top normal form, so is $A[x]$.*
3. *If $A[M]$ has a top normal form, so does $A[x]$.*

Proof. If $A[x]$ is not a zero term there exists a reduction of the form $A[x] \Rightarrow \lambda y.S$. Substituting M for x in this reduction we obtain a reduction of the form $A[M] \Rightarrow \lambda y.S'$. This proves 1. For point 2. the only non trivial case is when the top normal form $A[M]$ is an application $U[M]V[M]$ with $U[M]$ a zero term. By point 1. $U[x]$ is a zero term, hence $A[x] \equiv U[x]V[x]$ is a top normal form. To prove point 3. we use Theorem 10.7. Suppose that there is a reduction $A[M] \Rightarrow T$ with T a top normal form. We can assume that $A[M] \Rightarrow T$ is an head reduction. If M goes to the head in this reduction, then by Remark 11.4 there is a reduction of the form $A[x] \Rightarrow xT_1 \dots T_k$, hence $A[x]$ has a top normal form. If M does not go to the head, we can replace M and all its maximal extended residuals by x getting again a top normal form of $A[x]$ (by point 2.). QED

Corollary 11.6 *If $A[x]$ is mute, so is $A[M]$ for every M (x free in $A[x]$).*

We would like to prove some partial converses of the above results. We need restrictions on M that will ensure that M behaves like a free variable in any reduction.

Definition 11.7 A term A is a *strong zero term*, if it is a zero term and there is no reduction of the form $A \Rightarrow xT_1 \dots T_k$ where x is a variable and $k \geq 0$.

Note that any mute term is a strong zero term. In particular $\perp$ is a strong zero term.

Lemma 11.8 *If x is free in $A[x]$ and $A[x]$ is a strong zero term, then $A[B]$ is a strong zero term for every B.*

Proof. First we prove that $A[B]$ is a zero term. If it is not, then by Theorem 10.4 there exists an head reduction of the form $A[B] \Rightarrow_h \lambda y.S$. If B goes to the head in this reduction, then $A[x] \Rightarrow_h \lambda y_1 \ldots y_n.xT_1 \ldots T_k$ for some $n, k \geq 0$ and some $T_1, \ldots, T_k$. This is absurd since $A[x]$ is a strong zero term. Otherwise B is not active in the given head reduction and therefore its maximal extended residuals can be replaced by x yielding $A[x] \Rightarrow_h \lambda y.S'$ for some S', which is again a contradiction since $A[x]$ is a zero term.

It remains to show that $A[B]$ is a strong zero term. If it is not then there exists an head reduction of the form $A[B] \Rightarrow_h yS_1 \ldots S_k$. If B goes to the head we reach a contradiction as above. Otherwise B is not touched and we can replace its maximal extended residuals by x getting a reduction from the term $A[x]$ to a term beginning with a variable, which is absurd since $A[x]$ is a strong zero term. QED

Corollary 11.9 *A term A is a strong zero term if and only if every substitution instance of A is a zero term.*

Proof. By the above lemma if A is a strong zero term, then any substitution instance of A is a strong zero term. Conversely assume that every substitution instance of A is a zero term. Then certainly A itself is a zero term. If it is not a strong zero term, there is a reduction of the form $A \Rightarrow xT_1 \ldots T_k$. By substituting x in this reduction by $U_1^{k+1} \equiv \lambda y_1, \ldots, y_{k+1}.y_{k+1}$, we obtain $A[x := U_1^{k+1}] \Rightarrow \lambda y_{k+1}.y_{k+1}$ contradicting the fact that any substitution instance of A is a zero term. QED

Corollary 11.10 *1. Any extended residual of a strong zero term is a strong zero term.*

2. A strong zero term is not active in any β-reduction.

Proof. 2. follows from 1. To prove 1. notice that any β-reduct and any substitution instance of a strong zero term is a strong zero term. Hence every extended residual U' of a strong zero term U is a strong zero term. QED

We can now prove a converse to Lemma 11.5

Lemma 11.11 *Let M be a strong zero-term and let x be free in $A[x]$. Suppose $A[x] \neq_\beta x$. We have:*

1. *$A[x]$ is a zero-term if and only if $A[M]$ is a zero-term.*

2. *$A[x]$ is a top normal form if and only if $A[M]$ is a top normal form.*

3. *$A[x]$ has a top normal form if and only if $A[M]$ has a top normal form.*

Proof. We have already proved one direction of the double implication for an arbitrary term M. Let us prove the other direction.

1. If $A[M]$ is not a zero-term, there is a reduction of the form $A[M] \Rightarrow \lambda y.C[M_1, \ldots, M_n]$ where $M_1, \ldots, M_n$ are all the occurrences of maximal extended residuals of M ($\lambda y.C[M_1, \ldots, M_n]$ cannot be an extended residual of M because M is a strong zero term). Replacing all the maximal extended residuals of M by x we obtain a reduction $A[x] \Rightarrow \lambda y.C[x, \ldots, x]$, thus $A[x]$ is not a zero-term.

2. The only interesting case is when $A[x]$ has the form $U[x]V[x]$ and therefore $A[M] \equiv U[M]V[M]$. By the part 1. $U[x]$ is a zero term if and only if $U[M]$ is such. Hence $A[x]$ is a top normal form if and only if $A[M]$ is such.

3. Suppose $A[x] \Rightarrow T[x]$ where $T[x]$ is a top normal form. Then $A[M] \Rightarrow T[M']$ where M' is a substitution instance of M, hence a strong zero term (by Corollary 11.9). By assumption $T[x] \neq_\beta x$. The result now follows from part 2. QED

Corollary 11.12 *If M_1 and M_2 are mute, then $A[M_1]$ is mute if and only if $A[M_2]$ is mute.*

Proof. If $A[M_1]$ is mute, then $A[x] =_\beta x$ or $A[x]$ is mute. In both cases $A[M_2]$ is mute. QED

12 Infinite $\lambda\beta\bot$-calculus

We define a $\bot$-*redex* as a mute term different from $\bot$. We define $\bot$-reduction $\rightarrow_\bot$ as the reduction generated (under substitutions and contexts) by $A \rightarrow_\bot \bot$ for every $\bot$-redex A. Since mute terms are closed under substitutions, a generic $\bot$-reduction has the form $C[A] \rightarrow_\bot C[\bot]$ where A is a mute term different form $\bot$. A term is a $\bot$-*normal form* is it has not $\bot$-redexes and it is a β-*normal form* if it has no β-redexes. Since every $\bot$-redex contains a β-redex, β-normal forms are also $\bot$-normal forms and we can simply speak about normal forms without further qualifications. Note that $\perp\perp$ and $\lambda x.\perp$ are normal forms.

$\beta\perp$-reduction is defined in the obvious way: $A \to_{\beta\perp} B$ if and only if either $A \to_\beta B$ or $A \to_\perp B$. The infinite reductions $\to_{\infty\perp}$ and $\to_{\infty\beta\perp}$ are defined in a completely similar way than infinite β-reductions $\to_\infty$ (also called $\to_{\beta\infty}$).

The notion of *residual* can be extended to $\perp$-reductions by defining $\perp$-reductions with labels in the following way:

$$(A)^n \to_\perp \perp$$

where A is a mute term with label n (the above reduction can take place inside a context). So A and all its subterms have no residuals in the reduction $A \to_\perp \perp$.

We want to define σ/ρ and ρ/σ for β $\perp$-reductions. For one step reductions we can give the following definition.

Definition 12.1 If $\rho\colon A \to_{\beta\perp} B$ and $\sigma\colon A \to_{\beta\perp} C$ are one-step β $\perp$-reductions, $\rho/\sigma\colon C \to_{\beta\perp\infty} D$ is the reduction which reduces all the residuals of σ in C in some fixed order (they are disjoint and if there are infinitely many of them, their depth must tend to infinity).

Lemma 12.2 *If σ and ρ are one-step β $\perp$-reductions starting from the same term, then σ/ρ and ρ/σ end up in the same term.*

Proof. We need three facts:

1) mute terms are closed under substitutions;

2) a mute term cannot be of the form $\lambda x.T$.

3) if U and $C[U]$ are mute, then $C[\perp]$ is mute (by Corollary 11.12).

Fact 1) is used to construct the following diagram (where U is the $\perp$-redex):

$$\begin{array}{ccc} (\lambda x.A[U])B & \to_\beta & A[U][x:=B] \\ \downarrow_\perp & & \downarrow_\perp \\ (\lambda x.A[\perp])B & \to_\beta & A[\perp][x:=B] \end{array}$$

Fact 2) is used to ensure that the reduction of a $\perp$-redex U does not destroy those β-redexes that are not contained in U, and so it allows us to postpone a β-reduction after a $\perp$-reduction when this is necessary to construct the appropriate diagram.

Fact 3) allows us to construct the following diagram (where U is the inner $\perp$-redex and $C[U]$ is the outer M-redex):

$$\begin{array}{ccc} C[U] & \to_\perp & C[\perp] \\ \downarrow_\perp & & \downarrow_\perp \\ \perp & \equiv & \perp \end{array}$$

Form these special cases the result easily follows. QED

To extend these notions to multistep reductions we need the notion of $\beta\perp$-development.

Definition 12.3 A *$\beta\perp$-development* is a finite or infinite sequence of $\beta\perp$-reductions $A_0 \to_{\beta\perp} A_1 \to_{\beta\perp}\to \ldots$ which reduce only the residuals of some redexes of A.

As in the case of β-reduction it can be shown that if A is a finite term, then every $\beta\perp$-development $A \equiv A_0 \to_{\beta\perp} A_1 \to_{\beta\perp}\to \ldots$ is finite ($\perp$-reduction is a very simple collapsing reduction, so it does not pose any problem for the finiteness of developments).

It follows that if we start from a finite term, we can define σ/ρ also for multistep $\beta\perp$-reductions σ and ρ as we did for β-reductions.

Lemma 12.4 *If A is finite and $\rho\colon A \Rightarrow_{\beta\perp} B$ and $\sigma\colon A \Rightarrow_{\beta\perp} C$ are multistep $\beta\ \perp$-reductions, there is a canonical way of defining a term D and two multistep $\beta\perp$-reductions $\rho/\sigma\colon C \Rightarrow_{\beta\perp} D$ and $\sigma/\rho\colon B \Rightarrow_{\beta\perp} D$.*

In particular $\beta\perp$-reduction for finite terms is Church-Rosser. For infinite terms there might be problems to define σ/ρ as in the case of β-reduction.

Theorem 12.5 $\to_{\beta\perp\infty}$ *is transitive.*

Proof. Completely analogous to the proof of the transitivity of $\to_\infty$. QED

The next theorem says that a finite term has at most one infinite $\beta\perp$-normal form:

Theorem 12.6 *If A is finite, $B\ {}_{\beta\perp\infty}\!\leftarrow A \to_{\beta\perp\infty} C$ and B, C are normal forms, then $B \equiv C$.*

Proof. Exaclty as the proof of Theorem 9.5. In fact in that theorem we used only few properties of β-reductions which continue to hold for $\beta\perp$-reduction: namely the fact that we can define projections σ/ρ for reductions among finite terms, and the fact that the depth of residuals under one-step reductions decreases by 2 at most (see section 3). Such properties clearly hold for $\beta\perp$-reductions. QED

Unlike the case of infinite $\lambda\beta$-calculus, for infinite $\lambda\beta\perp$-calculus we also have existence of infinite normal forms.

Theorem 12.7 *$NF^\infty(A)$ is a normal form of A with respect to infinite $\beta\perp$-reduction.*

Proof. It is clear from the definitions that $NF^\infty(A)$ is a normal form. It is also clear that by "unfolding" the definition of $NF^\infty(A)$ we obtain an infinite $\beta\perp$-reduction $A \rightarrow_{\beta\perp\infty} NF^\infty(A)$. QED

Thus for infinite $\beta\perp$-calculus we have both existence and unicity of infinite normal forms of finite terms.

Corollary 12.8 *The Church-Rosser theorem for infinite $\beta\perp$-calculus holds, provided we restrict to finitely generated terms (where B is finitely generated if there is a finite term A and a reduction $A \rightarrow_{\beta\perp\infty} B$)*[1].

Proof. The set of finitely generated terms is closed under $\rightarrow_{\beta\perp\infty}$ by transitivity of $\rightarrow_{\beta\perp\infty}$. The Church-Rosser theorem follows at once from the existence and the unicity of infinite normal forms of finite terms. QED

13 A new model of lambda-calculus

We recall that a model of λ-calculus can be defined as a pair $(X, \cdot)$ where X is a non-empty set and $\cdot$ is a binary operation on X together with a semantic map $(A, \rho) \mapsto [[A]]_\rho$ which associates to every (finite) lambda-term A and every map $\rho: V \rightarrow X$, where V includes the free variables of A, an element $[[A]]_\rho \in X$ in such a way that the following conditions are satisfied:

1. $[[x]]_\rho = \rho(x)$ if x is a variable.

[1] Added in proofs: later research with B. Intrigila has shown that the restriction to finitely generated terms is not necessary.

2. $[[(AB)]]_\rho = [[A]]_\rho \cdot [[B]]_\rho$.

3. $[[\lambda x.A]]_\rho \cdot b = [[A]]_{\rho[x:=b]}$.

4. If for all $d \in X$ we have $[[A]]_{\rho[x:=d]} = [[B]]_{\rho[x:=d]}$, then $[[\lambda x.A]]_\rho = [[\lambda x.B]]_\rho$.

5. $[[A]]_\rho = [[A]]_\tau$ if ρ and τ agree on the free variables of A.

6. $[[A]]_\rho = [[B]]_\rho$ if A and B differ only by a renaming of bound variables.

Definition 13.1 We define a model of lambda-calculus as follows. Take X to be the set of all normal forms of (finite) lambda-terms with respect to $\rightarrow_{\beta\bot\infty}$. For $A, B \in X$ define $A \cdot B$ as $NF^\infty(AB)$. This is well defined by Corollary 12.8. Finally define $[[A]]_\rho$ as the infinite normal form of the term A_ρ obtained by performing the substitution ρ on the term A.

Theorem 13.2 *The triple* $(X, \cdot, [[\ ]])$ *defined above, is a model of lambda-calculus.*

Proof. The crucial condition to verify is the third. It suffices to show that for every two finite terms A and B, $NF^\infty(A[x := B]) \equiv NF^\infty(NF^\infty(A)[x := NF^\infty(B)])$. This follows by the unicity of infinite normal forms for finitely generated terms, the transitivity of $\rightarrow_{\beta\bot\infty}$, and the fact that $A[x := B] \rightarrow_{\beta\bot\infty} NF^\infty(A)[x := NF^\infty(B)]$. QED

We call $NF^\infty(\Lambda)$ the model defined above. This model identifies all the mute terms and more generally all the terms with the same infinite $\beta\bot$-normal form. For instance let A be a (finite) normal form and let X be a (finite) λ-term such that $X \Rightarrow XA$. Then the infinite $\beta\ \bot$-normal form of X is uniquely determined by $NF^\infty(X) \equiv (((\ldots)A)A)A$ (infinitely many A's). It follows that all the terms X such that $X \Rightarrow XA$ are identified in the model $NF^\infty(\Lambda)$. On the other hand B. Intrigila remarked, in a private communication, that the model $NF^\infty(\Lambda)$ does not equate all the terms with $X = XA$. In fact one can construct many such terms which are in (finite) normal form (see [7]). This is an example of how the model discriminates between β-reduction and β-convertibility.

14 It is consistent to equate all mute terms to an arbitrary closed term

All terms and reductions in this section are finite. We prove that it is consistent with the $\lambda\beta$-calculus to simultaneously equate all the mute terms to a fixed arbitrary closed term M, i.e. the theory $\lambda\beta + \{M = U \mid U$ is mute $\}$ is consistent. We prove this by defining a Church-Rosser notion of reduction which sends all the mute terms to M. The main lemma is the following.

Lemma 14.1 *If U is mute and $C[\ ]$ is a context such that $C[U]$ is mute, then either $C[x] \Rightarrow x$ where x is a variable not in $C[\ ]$, or for every closed term N, $C[N]$ is mute.*

Proof. If $C[N]$ is not mute, then there exists an head reduction $C[N] \Rightarrow_h Q[N_1, \ldots, N_n]$ where $Q[N_1, \ldots, N_n]$ is a top normal form and we have displayed all the occurrences of maximal extended residuals $N_1, \ldots, N_n$ of N. If in this reduction N goes to the head, then there are terms $T_1, \ldots, T_k$ ($n \geq 0$) and an head-reduction $C[x] \Rightarrow_h xT_1 \ldots T_k$ where x is a variable. It follows that $C[U] \Rightarrow_h U'T_1^* \ldots T_k^*$ where $U', T_1^*, \ldots, T_k^*$ are substitution instances of $U, T_1, \ldots, T_k$. U' is mute by Corollary 11.6. Since $C[U]$ and U' are mute, $k = 0$. Thus $C[x] \Rightarrow x$.

On the other hand if N does not go to the head in the head reduction $C[N] \Rightarrow_h Q[N_1, \ldots, N_n]$, then N and all its extended residuals can be replaced by x yielding $C[x] \Rightarrow Q[x, \ldots, x]$. It follows that $C[U] \Rightarrow Q[U_1, \ldots, U_n]$ where each U_i is a substitution instance of U. Since $Q[N_1, \ldots, N_n]$ is a top normal form by Lemma 11.5 $Q[x_1, \ldots, x_n]$ is a top normal form. Since $U_1, \ldots, U_n$ are strong zero terms, by Lemma 11.11 also $Q[U_1, \ldots, U_n]$ is a top normal form. This is absurd because $C[U] \Rightarrow Q[U_1, \ldots, U_n]$ and $C[U]$ is mute. QED

Definition 14.2 Fix an arbitrary closed term M. Define M-reduction by: $C[U] \rightarrow_M C[M]$ for every mute term U and every context $C[\ ]$ with exactly one hole. We say that U *is the M-redex* of the given M-reduction.

Since mute terms are closed under substitutions and β-reductions, every substitution instance of an M-redex is an M-redex.

The main difficulty with M-reduction is that there seems to be no sensible way of defining projections σ/ρ for M-reductions. The next lemma gives a partial substitute.

Lemma 14.3 *Any diagram $C\ {}_M\!\!\leftarrow A \to_M B$ can be extended to a diagram of one of the following forms:*

$$\begin{array}{ccc} A & \to_M & B \\ \downarrow_M & & \Downarrow_\beta \\ C & \equiv & D \end{array} \qquad \begin{array}{ccc} A & \to_M & B \\ \downarrow_M & & \equiv \\ C & \Rightarrow_\beta & D \end{array} \qquad \begin{array}{ccc} A & \to_M & B \\ \downarrow_M & & \downarrow_{=M} \\ C & \to_{=M} & D \end{array}$$

Proof. Note that the first two diagrams are one the transpose of the other. If the two M-redexes are disjoint or coincide the proof is trivial (and we obtain the third kind of diagram). Consider the case of nested M-redexes U and $C[U]$. By symmetry we can assume that $C[U]$ is the one reduced in $A \to_M B$. Since U and $C[U]$ are mute, by Lemma 14.1 either $C[x] \Rightarrow x$, where x is a variable not in $C[\ ]$, or $C[M]$ is mute. In the first case $C[M] \Rightarrow_\beta M$ and we obtain the following diagram:

$$\begin{array}{ccc} C[U] & \to_M & C[M] \\ \downarrow_M & & \Downarrow_\beta \\ M & \equiv & M \end{array}$$

The result follows by writing $A \equiv A_1[C[U]]$ and inserting the four terms of the above diagram inside the context $A_1[\ ]$.

In the second case $C[M]$ is mute and we obtain an instance of the third kind of diagram:

$$\begin{array}{ccc} C[U] & \to_M & C[M] \\ \downarrow_M & & \downarrow_M \\ M & \equiv & M \end{array}$$

QED

Definition 14.4 The notion of *residual* is defined for M-reductions $C[U] \to_M C[M]$ by stipulating that if $H[U]$ is a subterm of $C[M]$ properly containing (the given occurrence of) U, then $H[M]$ is the residual of $H[U]$ under the given M-reduction. U itself has no residuals.

Lemma 14.5 *Any diagram $C\ {}_M\!\!\leftarrow A \to_\beta B$ can be extended to a diagram of the form:*

$$\begin{array}{ccc} A & \to_\beta & B \\ \downarrow_M & & \Downarrow_M \\ C & \to_{=\beta} & D \end{array}$$

Proof. Let U be the M-redex reduced by $A \to_M C$. Since mute terms are closed under substitutions, every residual of an M-redex under a β-reduction is an M-redex. So we can define $B \Rightarrow_M D$ as the multistep β-reduction which reduces all the residuals of U from left to right.

Since U is a zero term, it is not the first part of a β-redex. It follows that M-reductions do not destroy β-redexes, unless the β-redex is contained in the M-redex. Thus either $C \equiv D$ or we can define $C \to_\beta D$ as the one-step β-reduction which reduces the residual of the β-redex of $A \to_\beta B$ under the M-reduction $A \to_M C$. QED

Corollary 14.6 *$\Rightarrow_{\beta M}$ satisfies the weak Church-Rosser property, i.e. given reductions $C\ {}_{\beta M}\!\!\leftarrow A \to_{\beta M} B$, there is a term D and reductions $C\ {}_{\beta M}\!\!\Rightarrow D \Leftarrow_{\beta M} B$.*

From what we have proved so far the Church-Rosser property for $\Rightarrow_{\beta M}$ does not follow because a priori one can imagine diagrams such as:

$$\begin{array}{ccccccc}
\cdot & \to_M & \cdot & \to_M & \cdot & \to_M & \cdot \\
\downarrow_M & & \downarrow_\beta & & \downarrow_\beta & & \downarrow_\beta \\
\cdot & \equiv & \cdot & \to_M & \cdot & \Rightarrow_M & \cdot \\
\downarrow_M & & \downarrow_M & & \equiv & & \equiv \\
\cdot & \equiv & \cdot & \Rightarrow_\beta & \cdot & \Rightarrow_M & \cdot \\
\downarrow_M & & \downarrow_M & & \Downarrow_M & & \\
\cdot & \equiv & \cdot & \Rightarrow_\beta & \cdot & & \cdot
\end{array}$$

Fig. 1

This would cause troubles because to complete the diagram we must find a common reduct for the two multi-steps M-reduction on the lower-right corner, and these two M-reductions are, a priori, longer than those we started with.

To avoid the occurrence of such diagrams we need the following.

Definition 14.7 We say that a β-reduction $\sigma\colon A \Rightarrow_\beta B$ is *collapsing*, written $\sigma\colon A \to_c B$, if there exist two contexts $C[\ ]$ and $H[\ ]$ with exactly one hole, and a closed term N, such that $A \equiv C[H[N]]$, $B \equiv C[N]$, and the β-reduction $\sigma\colon C[H[N]] \Rightarrow C[N]$ is induced by a β-reduction $H[x] \Rightarrow_\beta x$ where x is not in $H[\ ]$.

We say that $H[N]$ (or better the pair N, $H[\ ]$) is the *c-redex* of the reduction $\sigma\colon A \to_c B$.

As the notation $A \to_c B$ suggests, we are going to treat $\to_c$ as a one-step reduction, even if it actually consists of several β-reductions. When we write $\Rightarrow_c$ we mean a sequence of $(\to_c)$-reductions (possibly with different $H[\,]$'s). The motivation for introducing collapsing reductions is the following:

Remark 14.8 The β-reductions mentioned in Lemma 14.3 are collapsing.

Remark 14.9 If in the definition of collapsing reductions we allow $H[\,]$ to have several holes, we obtain an equivalent definition (but $C[\ ]$ is always assumed to have exactly one hole). In fact even if $H[\,]$ has several holes, in the reduction $H[x] \Rightarrow x$ only one specific occurrence of x has a residual so we can redefine $H[\,]$ in such a way that only that occurrence is placed inside the hole.

The above remark will be repeatedly used (without explicite note) in the following way: in order to prove a result of the form: "if some reduction σ is collapsing, then some some other reduction σ' is also collapsing", we can assume for σ the one-hole definition, and for σ' the several-holes definition.

Note that, unlike arbitrary β-reductions, collapsing reductions do not duplicate subterms (each subterm has zero or one residual under a collapsing reduction). As a consequence we have:

Lemma 14.10 *Any diagram $C \ {}_c\!\leftarrow A \to_\beta B$, can be extended to a diagram of the form:*

$$\begin{array}{ccc} A & \to_\beta & B \\ \downarrow_c & & \Downarrow_c \\ C & \to_{=\beta} & D \end{array}$$

(The key point to observe is that $C \to_{=\beta} D$ is a one-step or empty reduction.)

Proof. Let $\Delta \equiv (\lambda y.S)T$ be the β-redex contracted in $A \to_\beta B$, and let $H[N]$ be the c-redex of $A \to_c C$. So we have $H[x] \Rightarrow x$ for some variable x not in $H[\,]$ and we can write $A \equiv A_1[H[N]] \to_c A_1[N] \equiv C$ for some context $A_1[\,]$ with exactly one hole.

Case 1. Suppose that $\Delta \subset N$ (here we are implicitly using the assumption that the contexts in the definition of collapsing reduction have exaclty one hole: so it is clear which occurrence of N we refer to). By contracting Δ we obtain a reduction $N \to_\beta N'$. Thus we can write:

$$\begin{array}{ccccc} A & \equiv & A_1[H[N]] & \rightarrow_\beta & A_1[H[N']] \\ & & \downarrow_c & & \downarrow_c \\ & & A_1[N] & \rightarrow_\beta & A_1[N'] \end{array}$$

which gives the desired result.

Case 2. Suppose that $N \subset \Delta \subset H[N]$ and $N \not\equiv \Delta$.

N cannot be the "$\lambda x.S$" part of the redex $\Delta \equiv (\lambda y.S)T$, because otherwise $H[N] \equiv H_1[NT]$ for some context $H_1[\ ]$ such that $H_1[xT] \Rightarrow x$, which is impossible since T cannot be erased without erasing x as well. So N is contained either in S or in T. Suppose $N \subset T$. Then $\Delta \equiv (\lambda y.S)T_1[N]$ and $H[N] \equiv H_1[(\lambda y.S)T_1[N]]$ for some contexts $T_1[\ ]$ and $H_1[\ ]$ (with one hole) such that $H_1[(\lambda y.S)T_1[x]] \Rightarrow x$. It then follows (form the unicity of normal forms for β-reduction) that $H_1[S[y := T_1[x]]] \Rightarrow x$. Hence, by Remark 14.9, $H_1[S[y := T_1[N]] \rightarrow_c N$ and we have:

$$\begin{array}{ccccc} A & \equiv & A_1[H_1[(\lambda y.S)T_1[N]]] & \rightarrow_\beta & A_1[H_1[S[y := T_1[N]]]] \\ & & \downarrow_c & & \downarrow_c \\ & & A_1[N] & \rightarrow_\beta & A_1[N] \end{array}$$

The case in which $N \subset S$ is similar.

Case 3. Suppose that $\Delta \subset H[N]$ and Δ is disjoint form N. Then $\Delta \subset H[\ ]$ and by contracting Δ we obtain a context $H'[\ ]$ with $H'[x] \Rightarrow x$. The desired result follows.

Case 4. Suppose that $H[N] \subset \Delta$ and $\Delta \not\equiv H[N]$. Since $H[x] \Rightarrow x$, if $H[N] \equiv \lambda y.S$, then the only possibility is that $H[N] \equiv N \equiv \lambda y.S$. In this case $A \rightarrow_c C$ is the empty reduction and there is nothing to prove.

So we can assume that $H[N]$ is contained either in S or in T. Suppose $H[N] \subset T$. Then $\Delta \equiv (\lambda y.S)T_1[H[N]]$ for some context $T_1[\ ]$ with one hole. So we can write:

$$\begin{array}{ccccc} A & \equiv & A_1[(\lambda y.S)T_1[H[N]]] & \rightarrow_\beta & A_1[S[y := T_1[H[N]]]] \\ & & \downarrow_c & & \Downarrow_c \\ & & A_1[(\lambda y.S)T_1[N]] & \rightarrow_\beta & A_1[S[y := T_1[N]]] \end{array}$$

This time the vertical reduction on the right is $\Rightarrow_c$ rather than $\rightarrow_c$ because there are several occurrences of $H[\ ]$ being erased.

Case 5. If Δ and $H[N]$ are disjoint the result is trivial. QED

Lemma 14.11 *$\Rightarrow_\beta$ and $\Rightarrow_M$ commute, i.e any diagram of the form $C\ {}_M\!\Leftarrow A \Rightarrow_\beta B$ can be extended to a diagram of the form:*

$$\begin{array}{ccc} A & \Rightarrow_\beta & B \\ \Downarrow_M & & \Downarrow_M \\ C & \Rightarrow_\beta & D \end{array}$$

Proof. By induction on the length of $A \Rightarrow_\beta B$ and Lemma 14.5. QED

Corollary 14.12 *$\Rightarrow_\beta$ and $\Rightarrow_{\beta M}$ commute.*

Proof. Clear form the Church-Rosser property of $\Rightarrow_\beta$ and the previous lemma. QED

Lemma 14.13 *Any diagram $C\ {}_M\!\!\leftarrow A \rightarrow_c B$ can be extended to a diagram of the form:*

$$\begin{array}{ccc} A & \rightarrow_c & B \\ \downarrow_M & & \downarrow_{=M} \\ C & \rightarrow_{=c} & D \end{array}$$

Proof. Let U be the M-redex contracted in $A \rightarrow_M C$ and let $H[N]$ be the c-redex of $A \rightarrow_c B$.

Case 1. Suppose $U \subset N$. Then we can reason as in Case 1 of Lemma 14.10 with U instead of Δ.

Case 2. Suppose that $N \subset U \subset H[N]$ with $N \not\equiv U$. We can then write $U \equiv U_1[N]$ and $H[N] \equiv H_1[U_1[N]]$ for some contexts $U_1[\ ]$ and $H_1[\ ]$ with one hole, such that $H_1[U_1[x]] \Rightarrow x$, where x is a fresh variable. Since $U_1[N]$ is mute, it is in particular a zero term. Hence $U_1[x]$ is also a zero term. But then the only possibility to have $H_1[U_1[x]] \Rightarrow x$ is that $U_1[x] \Rightarrow x$ and $H_1[x] \Rightarrow x$. It follows that N is mute because $U_1[N] \Rightarrow N$. Thus we have:

$$\begin{array}{ccc} H_1[U_1[N]] & \rightarrow_c & H_1[N] \\ \downarrow_M & & \downarrow_M \\ H_1[M] & \equiv & H_1[M] \end{array}$$

and the result follows.

Case 3. Suppose that $U \subset H[N]$ and U is disjoint from N. Then we can write $H[N] \equiv H_1[U, N]$ for some context $H_1[\ ,\ ]$ such that $H_1[U, x] \Rightarrow x$. Since U is mute, it behaves as a free variable in any reduction and therefore it cannot give any contribution to this reduction. Hence $H_1[M, x] \Rightarrow x$ and $H_1[M, N] \rightarrow_c N$. Thus we have:

$$
\begin{array}{ccc}
H_1[U,N] & \to_c & N \\
\downarrow_M & & \equiv \\
H_1[M,N] & \to_c & N
\end{array}
$$

and the result follows.

Case 4. Suppose $H[N] \subset U$ and $H[N] \not\equiv U$. We can write $U \equiv U_1[H[N]]$. Since $U_1[H[N]]$ is mute and $U_1[H[N]] \Rightarrow U_1[N]$, $U_1[N]$ is mute. Hence we have:

$$
\begin{array}{ccc}
U_1[H[N]] & \to_c & U_1[N] \\
\downarrow_M & & \downarrow_M \\
M & \equiv & M
\end{array}
$$

and we are done. QED

Lemma 14.14 *Any diagram $C\ {}_c\!\Leftarrow A \Rightarrow_{\beta M} B$ can be extended to a diagram of the form:*

$$
\begin{array}{ccc}
A & \Rightarrow_{\beta M} & B \\
\Downarrow_c & & \Downarrow_c \\
C & \Rightarrow_{\beta M} & D
\end{array}
$$

where the length of $C \Rightarrow_{\beta M} D$ is less or equal than the length of $A \Rightarrow_{\beta M} B$.

Proof. By induction on the length of $A \Rightarrow_{\beta M} B$ by Lemma 14.13 and Lemma 14.10. QED

Lemma 14.15 *Any diagram $C\ {}_M\!\Leftarrow A \Rightarrow_{\beta M} B$ can be extended to a diagram of the form:*

$$
\begin{array}{ccc}
A & \Rightarrow_{\beta M} & B \\
\Downarrow_M & & \Downarrow_{\beta M} \\
C & \Rightarrow_{\beta M} & D
\end{array}
$$

Proof. By induction on the length of $A \Rightarrow_{\beta M} B$. If the first step of the the reduction $A \Rightarrow_{\beta M} B$ is a β-reduction we can use Lemma 14.11 and then we apply the induction hypothesis. If the first reduction of $A \Rightarrow_{\beta M} B$ is an M-reduction, then there are three cases corresponding to the three diagrams of Lemma 14.3. In the first case (using Remark 14.8) we can write:

$$\begin{array}{ccccc}
A & \to_M & B' & \Rightarrow_{\beta M} & B \\
\downarrow_M & & \downarrow_c & & \Downarrow_c \\
C' & \equiv & C' & \Rightarrow_{\beta M} & E \\
\Downarrow_M & & \Downarrow_M & & \\
C & \equiv & C & &
\end{array}$$

where $C' \Rightarrow_{\beta M} E$ is obtained using Lemma 14.14 and has length less or equal than the length of $B' \Rightarrow_{\beta M} B$. We can now find a common reduct to the diagram $C_M \Leftarrow C' \Rightarrow_{\beta M} E$ by applying the induction hypothesis.

The other two cases are similar. QED

Lemma 14.16 $\Rightarrow_{\beta M}$ *satisfies the Church-Rosser property.*

Proof. By Lemma 14.15 and Corollary 14.12, $\Rightarrow_{\beta M}$ commutes both with $\Rightarrow_\beta$ and with $\Rightarrow_M$, hence with $\Rightarrow_{\beta M}$. QED

Theorem 14.17 *For every closed term M, the theory $T = \lambda\beta + \{M = U | U \text{ is mute}\}$ is consistent.*

Proof. If T derives a contradiction, say $T \vdash 0 = 1$ (where 0 and 1 are the Church numerals for zero and one), then $0 =_{\beta M} 1$ where $=_{\beta M}$ is the transitive closure of $\Rightarrow_{\beta M}$. This is absurd since $\Rightarrow_{\beta M}$ is Church-Rosser and 0 and 1 are normal forms. QED

15 Zero terms of finite degree are easy

All terms and reductions in this section are finite. We recall that an *easy* term is a term U such that for every closed term M the theory $\lambda\beta + \{M = U\}$ is consistent. In the previous section we have shown that the class of mute terms has a much stronger property: all mute terms can be simultaneously equated to an arbitrary closed term. In this section we prove that all strong zero term (in particular all closed zero terms) of finite "degree" are easy, thus strengthening some results of [15].

Definition 15.1 Let U be a strong zero term. We say that U has degree 0 if it is mute (i.e. it is not β-convertible to VM with V a zero term). We say that U has degree $n+1$ if it is β-convertible to a term of the form VM where V is a strong zero term of degree n. We say that U has infinite degree in the remaining cases.

Lemma 15.2 *([15, 16]) If U is easy, then for every M, UM is easy.*

Proof. If $UM = Q$ is inconsitent, then so is $U = \mathbf{K}Q$ where $\mathbf{K} \equiv \lambda x, y.x$. QED

Since mute terms are easy it follows:

Theorem 15.3 *All strong zero terms of finite degree are easy.*

This is a strengthening of a theorem of [15] which says that all recurrent closed zero terms are easy, where "recurrent" is defined as follows:

Definition 15.4 A term A is *recurrent* if whenever $A \Rightarrow B$, there is a reduction $B \Rightarrow A$.

Lemma 15.5 *Every recurrent closed zero term A is a strong zero terms of finite degree (hence it is easy).*

Proof. Let k be the number of subterms of A. If A is not of finite degree, there is $n > k$ and a reduction $A \Rightarrow BT_1 \dots T_n$ where B is a zero term and each $T_i \in \Lambda_0$. But then every reduct of $BT_1 \dots T_n$ has the form $B'Q_1 \dots Q_n$ so it has at least n subterms, and therefore cannot coincide with A, contradicting the fact that A is recurrent. QED

The problem of classifying the closed zero terms which are easy has thus been reduced to the case of those of infinite degree. Ω_3 is an example of a zero term of infinite degree which is not easy. In [12] there is an example of an easy term of infinite degree. The fixed point $\mathbf{Y}_t\Omega_3$ has infinite order and it is not known to be easy (see [15, 16]). In [4] it is shown that $\mathbf{Y}_t\Omega_3$ can be consistently equated to every closed normal form.

16 Related work

Infinite reductions and unicity of normal forms are considered in [9] and [17] in the context of term rewriting systems for infinite first order terms. I was not initially aware of this fact, which was pointed out to me by M. Venturini Zilli. In particular the notion of infinite β-reduction, based on the assumption that in an infinite β-reduction the depth or redexes tends to infinity, correspond exaclty to the notion of strongly converging reduction in [17]. The use of residues under an infinite reduction can also be found there.

It should be noted however that we work with infinite terms with lambda-abstractions, while [9] and [17] work with infinite first order terms, i.e. they do not allow binding of variables. The notion of non-top-terminating in [9] resembles very closely the notion of mute. In [17] we find the idea that such terms, which are there called terms without head-normal form, are meaningless from an operational point of view. This idea is there formalized in a Church-Rosser theorem "up to hypercollapsing terms" (Theorem 7.4). In [17] we also find a counterexample to the infinite Church-Rosser property for combinatory logic. In [9] there is a theorem about the unicity of $\perp$-normal forms for $\perp$-converging semi-$\perp$-confluent rewrite systems (Theorem 8). This is reminiscent of our theorem on uniqueness of infinite β-normal forms. However λ-calculus seems to be neither $\perp$-converging nor semi-$\perp$-confluent if we extend the notions in the natural way.

Acknowledgements. The original motivation for this research comes from the previous paper with Benedetto Intrigila [4]. In that paper we left open the question, still not solved, of whether $\mathbf{Y_t}\Omega_3$ is easy. During our attempts to solve the problem Intrigila suggested to expand the language of λ-calculus with a new constant δ together with the rule $\delta \to \delta\omega_3$. The idea was that the operational behaviour of Ω_3 is fully described by δ. The infinite normal form $NF^\infty(\Omega_3)$ defined in this paper achieves a similar effect without expanding the language.

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References

[1] Baeten and Boerboom, Ω *can be anything it shouldn't be*, Indag. Math. **41** (1979), 111–120.

[2] H. P. Barendregt, The Lambda Calculus, Studies in Logic 103, North-Holland, Amsterdam 1984.

[3] H. P. Barendregt, *Representing 'undefined' in lambda calculus*, Theoretical pearls, J. Functional Programming **2 (3)**, (1992) 367–374.

[4] A. Berarducci and B. Intrigila, *Some new results on easy lambda-terms*, Theoretical Computer Science **121** (1993) 71–88.

[5] A. Berarducci and B. Intrigila, *Church Rosser λ-theories, infinite λ-calculus and consistency problems*, manuscript.

[6] C. Böhm, *Alcune proprietà delle forme $\beta - \eta$-normali nel $\lambda - K$-calcolo*, Pubblicazioni dell'Istituto per le Applicazioni del Calcolo, n. 696, Roma (1968), 19 pp.

[7] C. Böhm and B. Intrigila, *The Ant-Lion paradigm for strong normalization*, Information and computation **114** (1994), 30–49.

[8] H. B. Curry and R. Feys, Combinatory Logic, Vol. I, North-Holland, Amsterdam 1958.

[9] N. Dershowitz, S. Kaplan, D. A. Plaisted, *Rewrite, Rewrite, Rewrite, Rewrite, Rewrite, ...*, Theoretical Computer Science **83** (1991).

[10] F. Honsell and S. Ronchi della Rocca, *An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus*, Journal of Computer and System Sciences **45, 1** (1992), 49–75.

[11] J. M. E. Hyland, *A syntactic characterization of the equality in some models of the λ-calculus*, J. London Math. Soc. **(2), 12** (1976), 361–370.

[12] B. Intrigila, *A problem on easy terms in λ-calculus*, Fundamenta informaticae, **XV.1** (1991).

[13] B. Intrigila, *Some results on numeral systems in λ-calculus*, to appear in the Notre Dame Journal of Formal Logic.

[14] G. Jacopini, A condition for identifying two elements of whatever model of combinatory logic, in: C. Böhm (Editor) Springer Lecture Notes in Computer Science *37*, 1975.

[15] G. Jacopini and M. Venturini Zilli, *Equating for Recurrent Terms of λ-calculus and Combinatory Logic*, Quaderni dello IAC, serie III, N.85, Roma (1978), 3–14.

[16] G. Jacopini and M. Venturini Zilli, *Easy terms in the lambda- calculus*, Fundamenta informaticae, **VIII.2** (1985).

[17] R. Kennaway, J. W. Klop, R. Sleep, F-J de Vries, *Transfinite reductions in orthogonal term rewriting systems*, preprint 1993, to appear in: Information and computation.

[18] R. Kennaway, J. W. Klop, R. Sleep, F-J de Vries, *The infinite lambda calculus* Λ^∞, manuscript (1994).

[19] G. Mitschke, *λ-Kalkül, δ-conversion und axiomatische Rekursionstheorie*, Preprint Nr. 274, Technische Hochschule, Darmstadt. Fachbereit Mathematik, (1976), 77 pp.

[20] R. Nakajima, *Infinite normal forms for the λ-calculus*, Proceedings of the Symposium held in Rome, March 25-27, 1975 "λ-Calculus and Computer Science Theory" (eds. Goos and Hartmanis), Springer Lecture Notes in Computer Science, **37** (1975), 62–82.

[21] D. Park, *The Y-combinator in Scott's lambda calculus models*, Theory of computation, Report n. 13, University of Warwick, Dept. of Comput. Sci. (1976).

[22] D. Scott, *The lattice of flow diagrams, Semantics of Algorithmic languages*, Springer Lecture Notes in Mathematics, **188** (1971), 311–366.

[23] A. Visser, Numerations λ-calculus and arithmetic, To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism (J. P. Seldin and J. R. Hindley eds.), Academic Press, London 1980, 259–284

[24] C. P. Wadsworth, *The relation between computational and denotational properties for Scott's D_∞ models of the lambda-calculus*, SIAM J. Comput. **5** (1976), 488–521.

[25] C. Zylberajch, Sintaxe et semantique de la facilitè en lambda calcul, These de doctorat, Universite de Paris VII, 1991.