A CYCLE IS THE FUNDAMENTAL CLASS OF AN EULER SPACE

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Abstract. We prove that every cycle in a closed P.L. manifold $M$ can be regarded as the fundamental class of an Euler subpolyhedron of $M$.

Let $V$ be a compact real analytic manifold without boundary. It is a long-standing problem to see which $(\mathbb{Z}_2^\times)$-homology classes of $V$ can be represented as the fundamental class of an analytic subset of $V$ (and, in fact, it is conjectured that this is true for any homology class). The analogous problem arises with real algebraic manifolds, although in this case the general statement is false (even if $V$ is connected; see, for instance, [BT]).

D. Sullivan (in [S]) observed that every real analytic set can be regarded as an Euler space (see definition below); it is then natural to ask, first of all, if it is true that every homology class of a closed P.L. manifold $M$ can be represented as the fundamental class of an Euler subpolyhedron of $M$.

In this note we prove that this in fact happens: actually, we give a construction to add lower-dimensional simplexes to a cycle in $M$ until we get an Euler space (in $M$).

The techniques used are entirely elementary and involve merely P.L. transversality (as stated for example in [RS]) and combinatorial results on Euler spaces (see [A]).

We shall work in the P.L. category. For notations and definitions we refer to [RS]. All cycles and manifolds are intended unoriented and compact.

By an $n$-cycle $P$ we mean a polyhedron $P = \bigcup K$ such that

1. $n = \max\{\dim A, \text{ for } A \text{ a simplex of } K\}$,
2. each $(n - 1)$-simplex of $K$ is the face of an even number of simplexes of $K$.

By an $n$-cycle $P$ with boundary $\partial P$ we mean a pair of polyhedra $(P, \partial P) = \bigcup K, \partial K$ such that

1. $n = \max\{\dim A, \text{ for } A \text{ a simplex of } K\}$,
2. $\partial P$ is an $(n - 1)$-cycle,
3. each $(n - 1)$-simplex of $K \setminus \partial K$ is the face of an even number of $n$-simplexes of $K$,
4. each $(n - 1)$-simplex of $\partial K$ is the face of an odd number of $n$-simplexes of $K$.

A cycle (with boundary) in $M$ is a subpolyhedron of $M$ which is a cycle (with boundary).

A closed (P.L.) manifold is a compact (P.L.) manifold without boundary.

An Euler space is a polyhedron $P$ such that, for each $x \in P$, $\chi(\text{lk}(x, P)) \equiv 0 \pmod{2}$.

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An Euler pair is a pair of polyhedra \((P, Q)\) such that (1) \(\forall x \in P \setminus Q, \chi(\text{lk}(x, P)) \equiv 0 \pmod{2}\); (2) \(\forall x \in Q, \chi(\text{lk}(x, Q)) \equiv 0 \pmod{2}\); (3) \(\forall x \in Q, \chi(\text{lk}(x, P)) \equiv 1 \pmod{2}\).

REMARKS. (1) An Euler cycle is a cycle (without boundary).
(2) An Euler pair \((P, Q)\) is not, in general, a cycle with boundary (if \(\dim P = n\), \(Q\) may not be of dimension \(n - 1\)).
(3) Note that the definition of an \(n\)-cycle is slightly different from the usual one which requires also each simplex of \(K\) to be the face of an \(n\)-simplex of \(K\). However, a cycle as we defined it naturally carries a fundamental class (which is a cycle in the usual sense) as follows:

Let \(P = |K|\) be an \(n\)-cycle. The fundamental class \(\tilde{P}\) of \(P\) is the polyhedron obtained by taking all the \(n\)-simplexes of \(K\) (together with their faces). Note that, if \(P\) is connected, then \(\tilde{P} \approx P\) is a representative of the generator of \(H_n(P; \mathbb{Z}_2) \cong \mathbb{Z}_2\).

In order to show the kind of arguments used, we first prove an “abstract” version of the stated result, that is

**Theorem 1.** Let \(P\) be an \(n\)-cycle. Then there exists an Euler polyhedron \(P'\) such that \(P' \supset P\) and \(\dim(P' \setminus P) < n\).

**Proof.** Let \(P = |K|\) and assume that \(K = T(1)\), that is, \(K\) is the first barycentric subdivision of another triangulation \(T\) of \(P\). Set

\[Q = \{(A \in K : \chi(\text{lk}(A, K)) \equiv 1 \pmod{2})\}\]

\(Q = |H|\) is a subpolyhedron of \(P\) and \(\dim Q < n - 1\) (as \(P\) is a cycle).

(a) Assume \(\dim Q = 0\). Then \(Q\) consists of a finite number of points \(v_1, \ldots, v_h\) and \((P, Q)\) is an Euler pair. Let \(Z\) be the 1-skeleton of \(K\); then (for the properties of the barycentric subdivision) \(Z\) is a 1-cycle with boundary the 0-skeleton of \(H\), that is, \(Q\) itself (see \([A]\), Propositions 1 and 2, and the subsequent remark). Thus \(h\) is even and we can form \(P_1 = P \cup Q\), where \(\Gamma\) is any 1-cycle with boundary \(Q\).

(b) The general case. Let \(d = \dim Q (0 < d < n - 2)\). We prove first of all that \(Q = |H|\) is a \(d\)-cycle. Let \(A\) be a \((d - 1)\)-simplex of \(H\) and \(B_1, \ldots, B_h\) the set of \(d\)-simplexes of \(H\) such that \(B_i > A\). If \(C\) is a simplex of \(R = \text{lk}(A, K)\), then \(C \ast A \in K\) and \(\text{lk}(C, R) = \text{lk}(C \ast A, K)\) (here \(\ast\) denotes the join operation). Since \(\dim(C \ast A) = \dim C + d\), \(\chi(\text{lk}(C, R))\) is always even, except for the vertices \(v_1, \ldots, v_h\) such that \(v_i \ast A = B_i\). Then, by the case (a), \(h\) is even, which means that \(Q\) is a cycle.

Now we can form \(P_1 = P \cup Q\), where \(\Gamma\) is any \((d + 1)\)-cycle with boundary \(Q\), for example the cone on \(Q\). \(P_1\) is not necessarily an Euler space; however, if \(B\) is a \(d\)-simplex of \(H\), \(\text{lk}(B, P_1) = \text{lk}(B, P)\) \(\forall\) (odd number of points), so that \(Q_1 = \{A \in P_1 : \chi(\text{lk}(A, P_1)) \equiv 1\}\) is a subpolyhedron of dimension \(\leq (d - 1)\) in \(P_1\); by iterating the argument we obtain the required Euler space \(P'\).

Note that the hypothesis that \(P\) is a cycle is necessary; see, for example, the following Figure 1.

The difficulty which arises in the general case is essentially to prove that \(Q\) is now a boundary in the ambient manifold.
THEOREM 2. Let $M$ be a closed $m$-manifold and $P$ a cycle of dimension $n < m$ in $M$. Then there exists a subpolyhedron $P', P \subset P' \subset M$, such that $P'$ is an Euler space and $\dim(P' \setminus P) < n$.

PROOF. Let $Q$ be defined as in the previous theorem and $(L, K, H)$ be a triangulation of $(M, P, Q)$ which we assume, for the sake of simplicity, to be the first barycentric subdivision of another triangulation of $(M, P, Q)$ (see remark below).

CLAIM. $Q$ is a boundary in $P$.

(Note that this has already been proved in the case $\dim Q = 0$.) Let $d = \dim Q$; let $N$ be the simplicial neighbourhood of $H^{(1)}$ in $K^{(1)}$, $\bar{N}$ the boundary of $N$, $p: N \to Q$ the simplicial retraction and $\hat{p} = p|\bar{N}$. $(\bar{P} \setminus \bar{N}, \bar{N})$ is an Euler pair; therefore (again by [A, Proposition 1]), if $Z$ denotes the $(d+1)$-skeleton of $\bar{P} \setminus \bar{N}$ and $S$ denotes the $d$-skeleton of $\bar{N}$ (both with respect to $K^{(1)}$), we have that $Z$ is a $(d+1)$-cycle with boundary $S$. Let $f = \hat{p}|S$; $f$ is a simplicial map and we want to show that its degree is odd. Let $\sigma \in H^{(1)}$ be a $d$-simplex and $A \in H$ such that $\sigma \subset A$; we must prove that $\#\{\text{simplaxes in } f^{-1}(\delta)\} = \#\{\text{d-simplaxes in } \hat{p}^{-1}(\delta)\}$ is odd; as

$$\#\{B \in K: A < B\} = \#\{\text{simplaxes } C \text{ of } \lk(A, K)\}$$

$$\equiv \chi(\lk(A, K)) \equiv 1 \pmod{2},$$

it is enough to show that, for each $B > A$, $\#\{\text{d-simplaxes in } \hat{p}^{-1}(\delta) \cap \hat{B}\}$ is odd. Let $B > A$; then $B = A \ast C$ and $\hat{p}|\bar{N} \cap B: \bar{N} \cap B \to A$ is obtained by the pseudoradial projection from $C$. 

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Note that, if $\dim A = 0$, $\# \{\text{vertices of } \hat{p}^{-1}(A) \cap \hat{B} \} = 1$ and $\# \{\text{vertices of } \hat{p}^{-1}(A) \cap B \} = \# \{\text{vertices of } C \text{ in } K^{(1)} \} = 1 \pmod{2}$ (see Figure 2); while, if $\dim C = 0$ (so that $B$ is a cone over $A$ with vertex $C$), $\# \{d\text{-simplexes in } \hat{p}^{-1}(\hat{a}) \cap \hat{B} \} = \# \{d\text{-simplexes in } \hat{p}^{-1}(\hat{a}) \cap B \} = 1$ (see Figure 3). In general, if $\sigma = \hat{A} \ast \tau$, let $A'$
be the face of $A$ containing $\tau$ and $D = C \ast A'$. Then, if $\alpha$ is a \(d\)-simplex in \(p^{-1}(\delta) \cap \tilde{B}\), necessarily $\alpha = \tilde{B} \ast \gamma$, where $\gamma$ is a \((d - 1)\)-simplex in $p^{-1}(\tau) \cap D$ (see Figure 4). In order to conclude by induction, we have to show that also $\# \{d\text{-simplexes in } p^{-1}(\delta) \cap B\}$ is odd. But, if $C'$ varies over the faces of $C$, and $B' = A \ast C'$, then

\[
\# \{d\text{-simplexes in } p^{-1}(\delta) \cap B\} = \# \{d\text{-simplexes in } p^{-1}(\delta) \cap \tilde{B}\} + \sum_{C' \subset C} \# \{d\text{-simplexes in } p^{-1}(\delta) \cap \tilde{B}'\},
\]

By induction, all the terms of this sum are odd; moreover, their number equals $\# \{C': C' \subset C\} \equiv 1 \pmod{2}$. Thus $f: S \to Q$ is an odd degree map, so that the mapping cylinder $C_f$ is a \((d + 1)\)-cycle in $P$ with boundary $S \cup Q$ and $Q' = Z \cup_s C_f$ is the required cycle with boundary $Q$. This proves the claim.

In order to prove the theorem, it is enough now to put $Q'$ transverse to $P$ in $M$ relatively to $Q$ (see [RS, Theorem 5.3]). In this way we get a cycle $Q''$ in $M$ with boundary $Q$ such that $\dim(Q'' \cap P) = d + 1 + n - m = d$. Form $P_1 = P \cup Q Q''$; $P_1$ is an $n$-cycle in $M$ and, if $A$ is a $d$-simplex in $P_1$, then

\[
\text{lk}(A, P_1) = \begin{cases} 
\text{lk}(A, P) & \text{if } A \in Q, \\
\text{lk}(A, P) & \text{if } A \notin Q \cap P', \\
\text{lk}(A, P) & \text{if } A \in Q'' \cap P, \\
\text{lk}(A, Q'') & \text{if } A \in Q'' \cap P.
\end{cases}
\]

In each case $\chi(\text{lk}(A, P_1)) \equiv 0$, so that $Q_1 = \{A \in P_1 : \chi(\text{lk}(A, P_1)) \equiv 1\}$ has dimension $\leq (d - 1)$ and we can iterate the argument as before until we get an Euler space $P'$.

**Remark.** As regards the choice of the triangulation, what we need is only that the simplicial neighbourhood $N$ of $Q$ in $P$ (with respect to $K^{(1)}$) is in fact a regular neighbourhood; therefore, any triangulation $(K, H)$ such that $Q$ is full in $P$ would be enough (see [RS] for a definition of full).

**Corollary.** Every homology class $z \in H_n(M, \mathbb{Z}_2)$ can be represented as the fundamental class of an Euler subpolyhedron of dimension $n$ in $M$.

**Addendum.** With respect to the problem stated in the introduction (that is, to represent $\mathbb{Z}_2$-homology classes of a real algebraic manifold by algebraic subvarieties), since this paper was written we have proved the following (see [BD]):

For each $d \geq 11$, there exists a compact smooth manifold $V$ and a class $z \in H_{d-2}(V, \mathbb{Z}_2)$ such that, for any homeomorphism $h: V \to V'$ between $V$ and a real algebraic manifold $V'$, $h_*(z) \in H_{d-2}(V', \mathbb{Z}_2)$ cannot be represented by an algebraic subvariety of $V'$.
REFERENCES


[BD] R. Benedetti and M. Dedò, Counterexamples to representing homology classes by real algebraic subvarieties up to homeomorphism (to appear).