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Integral Representation of Local Functionals (*)

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Abstract. – *We study conditions under which a functional $F(u,B)$ admits an integral representation of the form*

$$F(u,B)=\int_B f\big(x,D^k u(x)\big)dx\ .$$

0. – Introduction.

When we study the relaxation $\overline{F}$ (with respect to a prescribed topology τ) of an integral functional F of the form

$$F(u)=\int_\Omega f\big(x,D^k u(x)\big)dx\ ,$$

the first question is whether $\overline{F}$ is an integral functional or not. The same question arises when we consider the Γ–limit of a sequence of integral functionals of the same form.

These problems have been successfully faced by considering relaxation and Γ–convergence of integral functionals of the form

$$F(u,B)=\int_B f\big(x,D^k u(x)\big)dx \tag{0.1}$$

and this lead to the following general problem: let $F: X\times\mathscr{B}\longrightarrow]-\infty,\infty]$ be given, where X is a k-th order Sobolev space $W^{k,p}(\Omega,\mathbb{R}^M)$ and $\mathscr{B}$ is the σ–field of all Borel subsets of Ω, under which conditions there exists a Borel function f which is lower semicontinuous in the second variable and satisfies (0.1) for all u and B?

A good survey of the question and a complete bibliography may be found in Buttazzo, chapters 2 and 4. In particular, this problem has been studied when $k=0$ (i.e. when $X=L^p(\Omega)$) and $k=1$ (i.e. when X is a first order Sobolev space) in

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BUTTAZZO and DAL MASO [1], [2] and HIAI. The result of the first paper is very refined and general. It was shown that when $X = L^p(\Omega, \mathbb{R}^M)$ $(1 \le p < \infty)$, every functional F admits an integral representation of the form

$$F(u,B) = \int_B f\big(x, u(x)\big)dx$$

provided the following conditions are fulfilled:

(i) $F(u,\cdot)$ is a measure absolutely continuous with respect to Lebesgue measure for all functions u,

(ii) F is local, i.e. $F(u,B) = F(v,B)$ whenever $u = v$ a.e. in B,

(iii) $F(\cdot,B)$ is lower semicontinuous in the strong topology of L^p for all Borel sets B,

(iv) $F(\bar{u},\Omega) < \infty$ for at least one function $\bar{u}$.

In the second paper a similar result was achieved for functionals defined on first order Sobolev space $W^{1,p}(\Omega,\mathbb{R}^M)$ $(1 \le p < \infty)$ which satisfy invariance condition

$$F(u,B) = F(u + c, B) \qquad \text{for all } u \in W^{1,p},\ B \in \mathscr{B} \text{ and } c \in \mathbb{R}^M,$$

provided some technical growth conditions are fulfilled.

In this paper we consider abstract functionals of the form $F : X \times \mathscr{B}(\Omega) \longrightarrow]-\infty,\infty]$ where X is k-th order Sobolev space of functions of the open set Ω and $\mathscr{B}(\Omega)$ is the σ–field of all Borel subsets of Ω. We want to point out that our technics are very general and allow us to deal with functionals defined on L^p or on every order Sobolev spaces as much.

In order to get a representation in the form (0.1), we consider functionals which satisfy invariance condition (0.6) and are measures with respect to the second variable (see Definition 0.3).

F is assumed to satisfy a suitable locality property (cf. Definition 0.3(ii)), i.e. $F(u,B) = F(v,B)$ whenever $u,v \in X$, $B \in \mathscr{B}$ and $D^k u = D^k v$ a.e. in B, and a suitable regularity property (cf. Definition 0.3(iii)) , i.e. $F(\cdot,B)$ is lower semicontinous in the strong topology of $W^{k,p}$ for every Borel set B. We shall discuss apart when these conditions may be replaced by milder ones.

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Basic Notation and Statements of Main Results.

Before stating the main results, we give a short list of basic notations and definitions that appear in this paper.

Unless differently stated, throughout this paper Ω is a nonempty bounded open subset of $\mathbb{R}^N$, k is a non negative integer and p is a real number in $[1,\infty[$.

To begin with, we give a list of some symbols which will often occur: $\mathscr{A}(\Omega)$ is the collection of all open subsets of the open set Ω; $\mathscr{B}(\Omega)$ is the σ–field of all Borel subsets of Ω; $\mathscr{L}_N$ denotes Lebesgue measure in $\mathbb{R}^N$ and $|A| = \mathscr{L}_N(A)$ is the Lebesgue measure of the Borel set A.

We refer to usual functional analysis notation (cf. BREZIS and RUDIN [2]); to avoid any ambiguity, however, we recall that letters in boldface are integral multi–indices $\mathbf{a} = (a_1,\ldots,a_N)$ with norm $|\mathbf{a}| = a_1 + \ldots + a_N$ and $D^{\mathbf{a}}$ is the partial derivative

$$D^{\mathbf{a}} = \Big(\frac{\partial}{\partial x_1}\Big)^{a_1} \cdots \Big(\frac{\partial}{\partial x_N}\Big)^{a_N} .$$

We recall that a Borel function u belongs to Sobolev space $W^{k,p}(\Omega,\mathbb{R}^M)$ if $D^{\mathbf{a}}u$ is (represented by) a function in $L^p(\Omega,\mathbb{R}^M)$ for every multi–index $\mathbf{a}$ with $|\mathbf{a}| \le k$.

Let $I(k)$ be the set of all multi–indexes $\mathbf{a}$ with $|\mathbf{a}| = k$; we denote by $D^k u$ the k-th derivative of u, i.e. the function of Ω into $(\mathbb{R}^M)^{I(k)}$ defined by $D^k u(x) = \big(D^{\mathbf{a}}u(x)\big)_{\mathbf{a}\in I(k)}$ for all $x \in \Omega$.

Unless differently stated, every space is endowed with its norm topology. Every subspace of a space X is endowed with the topology inherited from X.

Given a Borel function $u : \Omega \to [-\infty,\infty]$, we say that a point $x \in \Omega$ is a Lebesgue point of u, and we write $x \in \mathscr{L}(u)$, when

$$\text{(0.2)} \qquad \lim_{r\to 0} \frac{\big|B(x,r) \cap u^{-1}(A)\big|}{|B(x,r)|} = 1 \qquad \text{for each open neighbourhood } A \text{ of } u(x).$$

We say that a point $x \in \Omega$ is a p–Lebesgue point of u, and we write $x \in \mathscr{L}(p,u)$, when $u(x) \in \mathbb{R}$ and

$$\text{(0.3)} \qquad \lim_{r\to 0} \frac{\|u - u(x)\|_{L^p(B(x,r))}}{|B(x,r)|^{1/p}} = 0 .$$

It is well–known that both $\mathscr{L}(u)$ and $\mathscr{L}(u,p)$ are Borel sets, moreover $\big|\Omega \setminus \mathscr{L}(u)\big| = 0$ for all u and $\big|\Omega \setminus \mathscr{L}(u,p)\big| = 0$ for all p–summable u with $1 \le p < \infty$ (cf. RUDIN [1], chapter 9). The same holds if we consider functions which takes value in a finite dimensional normed space.

By measure on Ω we mean any σ–additive set function defined on the σ–field $\mathscr{B}(\Omega)$ which takes values in $]-\infty,\infty]$. If λ is a measure on Ω, $|\lambda|$ denotes its total variation and λ^+, λ^- denote its positive and negative variations respectively. We say that λ is absolutely continuous with respect to Lebesgue measure ($\lambda \ll \mathscr{L}_N$) if $\lambda(E) = 0$ whenever $|E| = 0$ (cf. RUDIN [1], chapter 6).

If λ is a measure which is absolutely continuous with respect to Lebesgue measure, Radon–Nikodym theorem (cf. RAO, Proposition 1 of Section 5.4) states that there exists a Borel function f such that f^- is summable and $\lambda(E) = \int_E f\,dx$ for all Borel sets $E \subset \Omega$.

The supremum of a collection $\mathscr{F}$ of measures on Ω is defined by

$$\bigvee_{\lambda\in\mathscr{F}} \lambda\,(E) = \sup\Big\{\sum_n \lambda_n(E\cap B_n)\Big\} \qquad \text{for all } E \in \mathscr{B}(\Omega), \tag{0.4}$$

where the supremum is taken over all sequences $\{\lambda_n\} \subset \mathscr{F}$ and all Borel partitions $\{B_n\}$ of Ω. The infimum $\bigwedge_{\lambda\in\mathscr{F}} \lambda$ is defined in a similar way. It may be easily proved that both $\bigvee_{\lambda\in\mathscr{F}} \lambda$ and $\bigwedge_{\lambda\in\mathscr{F}} \lambda$ are measures.

We introduce a notion which may be very useful in dealing with infinite measures. Let λ be a measure on Ω and let $\mathscr{F}(\lambda)$ be the set of all finite measures μ such that $\lambda(E) \geq \mu(E)$ for all Borel sets E. Define the lower envelope of λ by

$$\lambda^* = \bigvee_{\mu\in\mathscr{F}(\lambda)} \mu\ . \tag{0.5}$$

We say that λ is *lower regular* when $\lambda = \lambda^*$, i.e. when $\lambda - \lambda^* = 0$.

REMARK 0.1. – Let λ be a measure on Ω, then

(i) for every Borel set E, $\lambda^*(E) = \sup\{\mu(E) :\ \mu$ is a finite measure and $\mu(B) \leq \lambda(B)$ for all $B \subset E\}$,

(ii) $\lambda^{**} = \lambda^*$,

(iii) $\lambda - \lambda^*$ is a positive measure and $(\lambda - \lambda^*)^* = 0$. In particular $\lambda - \lambda^*$ never takes finite values and for every positive finite measure μ such that $\lambda - \lambda^* \geq \mu$ we have $\mu = 0$.

(iv) λ is lower regular if and only if, for each Borel set $E \subset \Omega$ such that $\lambda(E) = \infty$ and $\lambda(B) = 0$ or $\lambda(B) = \infty$ for all $B \subset E$, there exists a positive finite measure μ such that $\mu(E) > 0$ and

$$\mu(B) = 0 \text{ for all Borel sets } B \subset E \text{ with } \lambda(B) = 0,$$

In particular, a measure λ is lower regular provided one of the following condition is fulfilled

(v) for every Borel set $E \subset \Omega$ with $\lambda(E) = \infty$ there exists a Borel set $B \subset E$ such that $0 < \lambda(B) < \infty$,

(vi) λ is of the form $\lambda(B) = \int_B f d\mu$ where μ is a lower regular measure and f is a positive Borel function,

(vii) λ is finite or σ–finite,

(viii) λ is an Hausdorff measure,

(ix) λ is outer regular or inner regular.

The proof of this statements is not difficult except for (iii) which is a corollary of Theorem 48 of ROGERS. Notice that the most measures we may usually deal with are lower regular but not all (Example 4.7).

Let X be a subspace of some $W^{k,p}(\Omega,\mathbb{R}^N)$. Any function $F : X\times\mathscr{B}(\Omega) \longrightarrow]-\infty,\infty]$ is called a *functional on* X .

DEFINITION 0.2. – Let X be a subspace of some $W^{k,p}(\Omega,\mathbb{R}^N)$. We say that X is a local subspace if it contains all polynomial functions $p : \mathbb{R}^N \to \mathbb{R}^M$ with $\deg p < k$ and is closed under multiplication by real functions of class C^∞ with compact support in Ω.

Our main extent is studying functionals the domains of which is $W^{k,p}$. Nevertheless it is important to point out that the most statements we prove hold even when X is a local subspace of $W^{k,p}$ and that the idea of local subspace plays an essential role in their proofs.

DEFINITION 0.3. – Let F be a functional defined on a subspace X of $W^{k,p}(\Omega,\mathbb{R}^M)$. According to BUTTAZZO and DAL MASO [1], [2] and [3], we say that F satisfies invariance condition (0.6) if

$$F(u,B) = F(u+p,B) \qquad \text{for all polynomials } p : \mathbb{R}^N \to \mathbb{R}^M \text{ with } \deg p < k. \tag{0.6}$$

Moreover we use the following definitions.

(i) F is a measure if $F(u,\cdot)$ is a measure on Ω for all $u \in X$. F is finite, absolutely continuous (a.c.) or lower regular when every measure $F(u,\cdot)$ is finite, absolutely continuous with respect to Lebesgue measure or lower regular.

(ii) F is local on $\mathscr{B}$ if $F(u,B) = F(v,B)$ whenever $u = v$ almost everywhere in B and is local on $\mathscr{A}$ if $F(u,B) = F(v,B)$ whenever there exists an open set $A \supset B$ such that $u = v$ almost everywhere in A. F is D^k–local on $\mathscr{B}$ if $F(u,B) = F(v,B)$ whenever $D^k u = D^k v$ almost everywhere in B and is D^k–local on $\mathscr{A}$ if $F(u,B) = F(v,B)$ whenever there exists an open set $A \supset B$ such that $D^k u = D^k v$ almost everywhere in A.

(iii) F is lower semicontinuous (l.s.c.) on $\mathscr{B}$ if $F(\cdot,B)$ is lower semicontinuous (on X) with respect to the $W^{k,p}$ norm topology for all Borel sets B. F is lower semicontinuous on $\mathscr{A}$ if $F(\cdot,A)$ is lower semicontinuous for all open sets A.

(iv) F is lower p–bounded when there exist a positive function $a \in L^1(\Omega)$ and a real number $b \geq 0$ such that

$$F(u,B) \geq -\int_B \big[a(x) + b|D^k u(x)|^p\big]dx \qquad \text{for all } u \in X,\ B \in \mathscr{B}.$$

It is obvious what upper p–bounded and p–bounded mean.

(v) F is weakly lower p–bounded when, for every $r>0$, there exist a positive constant M_r such that

$$F(u,B)\ge -M_r \qquad \text{for all } u\in X,\ \|u\|_{W^{k,p}}<r \text{ and all } B\in\mathscr{B}.$$

It is obvious what weakly upper p–bounded and weakly p–bounded mean.

In (ii) we have given many different definition of locality. It is not difficult to prove (see Proposition 1.1) that when F is a functional defined on a subspace X of $W^{k,p}$ which is a measure and verifies (0.6), D^k–locality on $\mathscr{B}$ $\Rightarrow$ locality on $\mathscr{B}$ $\Rightarrow$ locality on $\mathscr{A}$ $\Leftrightarrow$ D^k–locality on $\mathscr{A}$. In Theorem 0.4 we prove that all these definition are equivalent provided X is a local subspace of $W^{k,p}$, F is a lower regular measure and is lower semicontinuous on $\mathscr{B}$. Example 4.3 shows that in general lower semicontinuity on $\mathscr{B}$ can neither be dropped nor replaced by lower semicontinuity on $\mathscr{A}$.

A straightforward Corollary of this fact is that any functional of this kind is the sum of a fixed measure which does not depends on u and of a functional which is an absolutely continuous measure, D^k–local on $\mathscr{B}$ and lower semicontinuous on $\mathscr{B}$. Hence, when needed, functionals will be assumed to be absolutely continuous measures or D^k–local on $\mathscr{B}$.

Theorem 0.4 (Decomposition Result, see Section 1 for the proof). – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which is a lower regular measure, local on $\mathscr{A}$, satisfies* (0.6) *and is lower semicontinuous on $\mathscr{B}$. Then F is D^k–local on $\mathscr{B}$. Moreover there exists a measure λ and a functional F', which is an a.c. measure, D^k–local on $\mathscr{B}$ and lower semicontinuous on $\mathscr{B}$, such that*

$$F(u,B)=F'(u,B)+\lambda(B) \qquad \textit{for all } u\in X \textit{ and all } B\in\mathscr{B}(\Omega).$$

A very important class of functionals on (a local subspace of) $W^{k,p}$ is the class of p–bounded functionals (see Definition 0.3(iv)). We notice that a p–bounded functional is D^k–local on B and lower semicontinuous on $\mathscr{B}$ if and only if (0.6) holds, it is local on $\mathscr{A}$ and lower semicontinuous on $\mathscr{A}$ (see Corollary 0.6). p–boundedness also plays a very important role in many question related to Relaxation and Γ–convergence of integral functionals (see for instance Buttazzo and Dal Maso [2] and [4]).

In Theorem 0.5 we give a characterization of p–bounded functionals. In particular we show that when F is a functional on $W^{k,p}$ which satisfy (0.6), is an a.c. measure and local on $\mathscr{A}$, then F is p–bounded if and only if is finite. It is essential that F is defined on $W^{k,p}$, in example 4.3 we build a finite functional F which is defined on a local subspace of $W^{k,p}$ and is not p–bounded.

Notice that this theorem does not need any regularity hypothesis on F, in particular it holds even for functionals which cannot be represented as integral functionals in the form (0.1). Moreover it may be stated in the following (weaker) form which is a generalization of a well–known theorem about superposition operators on L^p (see for instance Theorem 2.7.2 of Buttazzo: let f be a real Borel function of $\Omega\times(\mathbb{R}^M)^{I(k)}$ and let T_f be the superposition operator defined by, for all $u\in W^{k,p}(\Omega,\mathbb{R}^M)$,

$$\big[T_f u\big](x)=f\big(x,D^k u(x)\big) \qquad \text{for all } x\in\Omega.$$

If $T_f u\in L^1(\Omega)$ for all $u\in W^{k,p}$, then there exists a function $a\in L^1(\Omega)$ and a positive real number b such that

$$|f(x,s)|\le a(x)+b|s|^p \qquad \text{for all } x\in\Omega \text{ and all } s\in(\mathbb{R}^M)^{I(k)}.$$

Theorem 0.5 (Characterization of p–bounded Functionals, see Section 2 for the proof). – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which is an a.c. measure, satisfies* (0.6) *and is local on $\mathscr{A}$. Then*

(i) *if $X=W^{k,p}$, F is lower p–bounded and it is upper p–bounded if and only if it is finite,*

(ii) *in general, F is lower p–bounded if and only if it is weakly lower p–bounded and it is upper p–bounded if and only if it is weakly upper p–bounded.*

Corollary 0.6 (see Section 2 for the proof). – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which is an a.c. measure, local on $\mathscr{A}$, satisfies* (0.6) *and is lower semicontinuous on $\mathscr{A}$. If either $X=W^{k,p}$ and F is finite or F is weakly p–bounded, then F is p–bounded, D^k–local on $\mathscr{B}$ and lower semicontinuous on $\mathscr{B}$.*

Section 3 is devoted to the proof of Theorem 0.7. In particular we show that every functional F which is defined on a local subspace $X\subset W^{k,p}$, is an a.c. measure, D^k–local on $\mathscr{B}$ and lower semicontinous on $\mathscr{B}$ (see Definition 0.3), is representable as an integral functional in the form (0.1) with a Borel function f which is lower semicontinuous with respect to second variable (statement (i) of Theorem 0.7). Examples 4.3 and 4.4 show that this statement is false in general if we weaken semicontinuity hypothesis.

Using a rather surprising result (Theorem 5.12) we prove the uniqueness of the integral representation (statement (ii) of Theorem 0.7) i.e. that if both f and f' represent F then f and f' agree everywhere but in a set $S\subset\Omega\times(\mathbb{R}^M)^{I(k)}$ the projection of which on Ω is Lebesgue negligible. Example 4.6 shows that this is false in general when X is a local subspace of $W^{k,p}$.

Theorem 0.7 (Representation Result). – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which is an a.c. measure, D^k–local on $\mathscr{B}$ and lower semicontinuous on $\mathscr{B}$. Then there exists a Borel function $f:\Omega\times(\mathbb{R}^M)^{I(k)}\longrightarrow]-\infty,\infty]$ which is lower semicontinuous with respect to second variable and*

(i) *f represents F in the form* (0.1), *i.e., for all $u\in X$ and $B\in\mathscr{B}(\Omega)$,*

$$F(u,B)=\int_B f\big(x,D^k u(x)\big)dx\ .$$

(ii) *When $X \supset C_0^k(\Omega,\mathbb{R}^N)$, f is uniquely determined in the following sense: if f' is a Borel function which represents F, then there exists a negligible Borel set $N \subset \Omega$ such that $f(x,s) = f'(x,s)$ for all $x \in \Omega \setminus N$ and all $s \in (\mathbb{R}^M)^{I(k)}$.*

1. – Local Functionals.

To begin with, we state a very simple proposition.

Proposition 1.1. – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which a measure and fulfills invariance condition* (0.6). *Consider the following statements:*

(*a*) *F is D^k–local on $\mathscr{B}$.*

(*b*) *F is local on $\mathscr{B}$.*

(*c*) *F is D^k–local on $\mathscr{A}$.*

(*d*) *F is local on $\mathscr{A}$.*

Then $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$. Notice that in general $(c) \not\Rightarrow (b) \not\Rightarrow (a)$ (cf. example 4.3).

Proof. – It is obvious that $(b) \Rightarrow (d)$.

$(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ follow from this well–known fact: when u and v are functions in $W^{1,1}$ and $u = v$ a.e. in the Borel set B, then $Du = Dv$ a.e. in B (this may be proved taking into account that a function u in $W^{1,1}$ is approximately differentiable in almost every points x with approximate gradient $Du(x)$, see also Brezis, chapter IX).

Let's prove that $(d) \Rightarrow (c)$. Let u, v be two functions in X such that $D^k u = D^k v$ a.e. in the open set A. It is well–known that in each connected component of A, $u - v$ agrees with a polynomial function $p : \mathbb{R}^N \to \mathbb{R}^M$ with $\deg p < k$. As F satisfies (0.6) and is local on $\mathscr{A}$, $F(u,B) = F(v,B)$ for every Borel set B which is included in some connected component of A and then $F(u,B) = F(v,B)$ for every Borel set $B \subset A$ because F is a measure. ■

We shall prove that the implication $(d) \Rightarrow (a)$ holds in general provided F satisfies some regularity conditions. If $X = L^p(\Omega,\mathbb{R}^M)$, we have the following optimal result.

Proposition 1.2. – *Let F be functional on $L^p(\Omega,\mathbb{R}^M)$ which is a measure, local on $\mathscr{A}$ and lower semicontinuous on $\mathscr{B}$. Then F is local on $\mathscr{B}$, i.e. $F(u,E) = F(u',E)$ for all functions $u, u' \in X$ and all Borel sets $E \subset \Omega$ such that $u = u'$ a.e. in E.*

Proof. – Of course it is enough to prove the inequality $F(u,E) \le F(u',E)$.

For $n = 1, 2, \ldots$, let $A_n \subset \Omega$ be open sets which include E and satisfy $\lim_{n\to\infty} |A_n \setminus E| = 0$ and let

$$u_n(x) = \begin{cases} u'(x) & \text{if } x \in A_n \\ u(x) & \text{if } x \notin A_n . \end{cases}$$

Hence, condition $u = u'$ a.e. in B implies that u_n converges to u in the L^p norm and then, as F is local on $\mathscr{A}$,

$$\begin{aligned} F(u,E) &\le \liminf_{n\to\infty} F(u_n,E) \\ &= \liminf_{n\to\infty} F(u',E) = F(u',E) \ . \ ■ \end{aligned}$$

A similar results holds when F is defined on a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ (Theorem 1.4) with the additional assumption that F is lower regular. To begin with, we need an approximation lemma.

Lemma 1.3. – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$. Suppose that u is a function in X, λ is a finite positive measure on Ω and $E \subset \Omega$ is a Borel set such that $D^k u = 0$ a.e. on E. Then, for every $\varepsilon > 0$ there exists a function $v \in X$ with compact support in Ω and an open set $A \subset \Omega$ such that*

(i) *$D^k v = D^k u$ a.e. in A and $\lambda(E \setminus A) \le \varepsilon$.*

(ii) *$\|v\|_{W^{k,p}} \le \varepsilon$.*

Proof. – By Theorem 5.8, there exists a Borel set $E' \subset E$ such that $\lambda(E \setminus E') = 0$ and, for every $x \in E'$,

$$\limsup_{r\to 0} \frac{\lambda(B(x,ar))}{\lambda(B(x,r))} \ge a^N \quad \text{for all } a \text{ with } 0 < a < 1, \tag{1.1}$$

Let δ be a real number with $0 < \delta < 1$ and let $P \subset \Omega$ be an open set which includes E.

Let $\mathscr{F}$ be the collection of all closed balls $\overline{B} = \overline{B(x,r)} \subset P$ with $x \in E'$ and

$$\frac{\lambda\big[B(x,(1-\delta)r)\big]}{\lambda\big[\overline{B(x,r)}\big]} \ge \frac{\lambda\big[B(x,(1-\delta)r)\big]}{\lambda\big[B\big(x,(1-\delta)^{-1}r\big)\big]} \ge (1-\delta)^{3N} \ . \tag{1.2}$$

(1.1) shows that for all $x \in E'$ there exist closed balls $\overline{B}$ with center x and arbitrary small radius which belong to $\mathscr{F}$ and then we may apply Theorem 5.4 to obtain disjoint closed balls $\overline{B}_i = \overline{B(x_i,r_i)}$ in $\mathscr{F}$ for $i = 1, \ldots, n$ such that

$$\lambda\Big[E' \setminus \bigcup_i \overline{B(x_i,r_i)}\Big] \le \delta \ .$$

Set $A = \bigcup B\big(x_i,(1-\delta)r_i\big)$. Taking into account (1.2), we obtain

$$\begin{aligned} \lambda(E \setminus A) = \lambda(E' \setminus A) &= \lambda\Big[E' \setminus \bigcup_i \overline{B(x_i,r_i)}\Big] + \sum_i \lambda\Big[\overline{B(x_i,r_i)} \setminus B\big(x_i,(1-\delta)r_i\big)\Big] \\ &\le \delta + \sum_i \big[1 - (1-\delta)^{3N}\big]\lambda(\overline{B}_i) \le \delta + \sum_i 3N\delta\,\lambda(\overline{B}_i) \\ &\le \big[1 + 3N\,\lambda(\Omega)\big]\,\delta \ . \end{aligned} \tag{1.3}$$

As $B_1, \dots, B_n$ are disjoint, we may apply the Glueing Lemma 5.10 to obtain a function $v \in X$ such that

(*a*) $v = 0$ out of the union of all B_i.

(*b*) $D^k v = D^k u$ a.e. in $B\big(x_i, (1-\delta)r_i\big)$ for all i.

(*c*) $\|D^k v\|_{L^p(B_i)} \le C\delta^{-k}\|D^k u\|_{L^p(B_i)}$ for all i.

Then (*a*), (*c*) and $B_i \subset P$ for all i yield

$$\|D^k v\|_p \le \left[\sum_i \Big(C\delta^{-k}\|D^k u\|_{L^p(B_i)}\Big)^p\right]^{1/p} \le C\delta^{-k}\,\|D^k u\|_{L^p(P)} \ .$$

As every ball in $\mathscr{F}$ was chosen relatively compact in P, (*a*) implies that v has compact support in Ω and we may apply Poincaré inequality (BREZIS, chapter IX) to obtain

$$\|v\|_{W^{k,p}} \le C_1\|D^k v\|_p \le C_1 C\delta^{-k}\|D^k u\|_{L^p(P)} \ . \tag{1.4}$$

where C_1 is a constant which depends on N, k and p only.

(*b*) yields $D^k v = D^k u$ a.e. in A and then, taking into account that $D^k u = 0$ a.e. in E and P is any open set which include E, we may choose δ and P small enough to have that $\big[1 + 3N\,\lambda(\Omega)\big]\,\delta < \varepsilon$ and $C_1 C\delta^{-k}\|D^k u\|_{L^p(P)} < \varepsilon$. Hence (1.3) and (1.4) yield (i) and (ii). ■

THEOREM 1.4. – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be functional on X which is a lower regular measure, D^k–local on $\mathscr{A}$ and lower semicontinuous on $\mathscr{B}$. Then F is D^k–local on $\mathscr{B}$.*

PROOF. – Notice that it is enough to prove that inequality $F(u,E) \le F(u',E)$ holds whenever $D^k u = D^k u'$ a.e. in E. As F is lower regular, $F(u,\cdot)$ is a lower regular measure and then (cf. remark 0.1)

$$F(u,E) = \sup\Big\{\mu(E) \ : \ \mu \text{ is finite and } \mu(B) \le F(u,B) \text{ for all Borel sets } B \subset E\ \Big\} \ .$$

Hence it is enough to prove that for every finite measure μ such that $\mu(B) \le F(u,B)$ for all Borel set $B \subset E$ we have

$$\mu(E) \le F(u',E) \ . \tag{1.5}$$

Let ϕ be the negative variation of $F(u,\cdot)$, notice that ϕ is finite because F never takes value $-\infty$ and set $\lambda = |\mu| + \phi$. λ is a finite positive measure.

As $D^k u = D^k u'$ a.e. in E, we may apply Lemma 1.3 to find functions $v_n \in X$ and open sets $A_n \subset \Omega$ for all integers n such that

(*a*) $D^k v_n = D^k(u'-u)$ a.e. in A_n and $\lambda(E \setminus A_n) \le 2^{-n}$,

(*b*) $\|v_n\|_{W^{k,p}} \le 2^{-n}$.

As F is D^k–local on $\mathscr{A}$, (*a*) yields $F(u+v_n,B) = F(u',B)$ for all n and all Borel sets $B \subset A_n$. As $u + v_n$ converge to u by (*b*), and F is lower semicontinous on $\mathscr{B}$, we have that

$$\begin{aligned} F(u,B) &\le \liminf_{n\to\infty} F(u+v_n,B) \\ &= \liminf_{n\to\infty} F(u',B) = F(u',B) \end{aligned} \tag{1.6}$$

for all Borel sets $B \subset \bigcap_{n>m} A_n$ with $m = 1, 2, \dots$. Set

$$A = \bigcup_m \Big(\bigcap_{n>m} A_n\Big)$$

and notice that inequality (1.6) holds for all Borel sets $B \subset A$ because F is a measure. In particular

$$F(u, E\cap A) \le F(u', E\cap A) \ . \tag{1.7}$$

By (*a*) we have

$$\begin{aligned} \lambda(E\setminus A) &= \inf_m \Big[\lambda\big(E \setminus \bigcap_{n>m} A_n\big)\Big] \le \inf_m \Big[\sum_{n>m} \lambda(E\setminus A_n)\Big] \\ &\le \inf_m \Big[\sum_{n>m} 2^{-n}\Big] = \inf_m \big[2^{-m}\big] = 0 \end{aligned}$$

and then, recalling that $\lambda = |\mu| + \phi$ and ϕ is the negative variation of the measure $F(u',\cdot)$, (1.7) and the fact that $\mu(B) \le F(u,B)$ for all Borel sets $B \subset \Omega$ yield

$$\begin{aligned} \mu(E) = \mu(E\cap A) + \mu(E\setminus A) &= \mu(E\cap A) \le F(u,E\cap A) \le F(u',E\cap A) \\ &= F(u',E) - F(u',E\setminus A) \le F(u',E) + \phi(E\setminus A) = F(u',E) \ , \end{aligned}$$

and (1.5) is proved. ■

Decomposition of Local Functionals and Proof of Theorem 0.4

In order to complete the proof of Theorem 0.4, we need the following decomposition result.

PROPOSITION 1.5 (Decomposition of positive local functionals). – *Let X be a subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which is a positive measure, (D^k–) local on $\mathscr{B}$. Then there exist a positive measure λ on Ω and a functional F' on X, which is a positive a.c. measure, (D^k–) local on $\mathscr{B}$, such that*

$$F(u,B) = F'(u,B) + \lambda(B) \qquad \textit{for all } u \in X,\ B \in \mathscr{B}(\Omega). \tag{1.8}$$

Notice that λ is finite if $F(\bar u,\Omega)<\infty$ for some $\bar u\in X$.

PROOF. – Let λ be the lower envelope of all measures $F(u,\cdot)$ with $u\in X$, i.e.

$$\lambda(\cdot)=\bigwedge_{u\in X}F(u,\cdot)\ .$$

It is obvious that λ is a positive measure which satisfies $\lambda(B)\le F(u,B)$ for all $u\in X$, $B\in\mathscr{B}(\Omega)$. Moreover, if B is a Borel set such that $|B|=0$ and u, u' are functions in X, then $u=u'$ and $D^ku=D^ku'$ a.e. in B and $F(u,B')=F(u',B')$ for all Borel sets $B'\subset B$ because F is $(D^k\text{–})$ local on $\mathscr{B}$. Hence

$$\lambda(B)=F(u,B)\qquad\text{for all } u\in X \text{ and all } B\in\mathscr{B}(\Omega)\text{ with } |B|=0. \tag{1.9}$$

If λ is a finite measure, it is enough to take $F'(u,B)=F(u,B)-\lambda(B)$ for all $u\in X$ and all $B\in\mathscr{B}(\Omega)$.

In general, when λ is not finite, this definition does not make sense.

Let $\mathscr{F}$ be the collection of all Borel sets B which are of σ–finite measure with respect to $\mathscr{L}_N+\lambda$; as $\mathscr{F}$ is closed under countable union, there exists a Borel set $E\in\mathscr{F}$ such that $\big[\mathscr{L}_N+\lambda\big](B\setminus E)=0$ for all $B\in\mathscr{F}$. Hence

$$|B|+\lambda(B)=0 \text{ or } |B|+\lambda(B)=\infty \text{ for all Borel sets } B\subset\Omega\setminus E \tag{1.10}$$

and moreover, as E belongs to $\mathscr{F}$, we may find a sequence of pairwise disjoint Borel sets E_n which have finite $\mathscr{L}_N+\lambda$ measure and cover E. Set

$$F'(u,B)=\sum_n F(u,B\cap E_n)-\lambda(B\cap E_n)\qquad\text{for all } B\in\mathscr{B}(\Omega). \tag{1.11}$$

Notice that all terms of the series make sense because $\lambda(B\cap E_n)<\infty$ and are positive because λ is everywhere less than each measure $F(u,\cdot)$ by definition. Hence F' is well–defined, is a positive measure, and (1.9), (1.11) yield $F'(u,B)=0$ for all Borel sets B with $|B|=0$, i.e. F' is an a.c. measure.

The proof will be complete if we show that (1.8) holds, in fact it is a straightforward corollary of (1.8) that F' is $(D^k\text{–})$ local on $\mathscr{B}$ when F is $(D^k\text{–})$ local on $\mathscr{B}$. As F, F' and λ are measures, by (1.10) it is enough to show that, for all $u\in X$, (1.8) holds in the following three cases:

(*a*) $B\subset E$,

(*b*) $B\subset\Omega\setminus E$ and $|B|+\lambda(B)=0$,

(*c*) $B\subset\Omega\setminus E$ and $|B|+\lambda(B)=\infty$.

In the case (*a*), (1.8) follows from

$$\begin{aligned}\lambda(B)+F'(u,B)&=\sum_n\lambda(B\cap E_n)+\sum_n F(u,B\cap E_n)-\lambda(B\cap E_n)\\&=\sum_n F(u,B\cap E_n)=F(u,B)\end{aligned}$$

In the case (*b*), $|B|=\lambda(B)=0$ and (1.9) yields $F(u,B)=0$, then $F'(u,B)=0$ by (1.11) and (1.8) holds.

In the case (*c*), $\lambda(B)=\infty$ yields $F(u,B)=\infty$ because $\lambda(B)\le F(u,B)$, and then (1.8) holds. ■

COROLLARY 1.6 (Decomposition of local functionals). – *Let X be a subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which is a measure, $(D^k\text{–})$ local on $\mathscr{B}$. Then there exists a measure λ on Ω and a functional F' on X which is an a.c. measure, $(D^k\text{–})$ local on $\mathscr{B}$, such that*

$$F(u,B)=F'(u,B)+\lambda(B)\qquad\text{for all } u\in X,\ B\in\mathscr{B}(\Omega). \tag{1.8}$$

Notice that λ is finite if $F(\bar u,\Omega)<\infty$ for some $\bar u\in X$.

PROOF. – Define F^+ and F^- (resp. the positive and negative variations of F) by $F^+(u,\cdot)=\big(F(u,\cdot)\big)^+$ and $F^-(u,\cdot)=\big(F(u,\cdot)\big)^-$ for all $u\in X$. $F=F^+-F^-$ and then it is enough to verify that both F^+ and F^- are positive measures $(D^k\text{–})$ local on $\mathscr{B}$, that F^- is finite and then apply Proposition 1.5. ■

PROOF OF THEOREM 0.4. – It is enough to apply Proposition 1.1, Theorem 1.4 and Corollary 1.6. ■

2. – Characterization of p–bounded Functionals.

Unless differently stated, throughout this section F will be a fixed functional defined on a local subspace X of $W^{k,p}(\Omega,\mathbb{R}^M)$ which is a positive a.c. measure and D^k–local on $\mathscr{A}$.

Since for each $u\in X$, $F(u,\cdot)$ is a positive measure which is absolutely continuous with respect to Lebesgue measure, then it is represented by a positive Borel function that we denote by f_u. In other words we have

$$F(u,B)=\int_B f_u\,dx\qquad\text{for all } B\in\mathscr{B}(\Omega). \tag{2.1}$$

DEFINITION 2.1. – Let $g:\Omega\to[0,\infty]$ be a positive Borel function and let $\mathscr{F}(g)$ be the collection of all functions f'_u given by

$$f'_u(x)=\begin{cases}f_u(x)&\text{if } |D^ku(x)|\le g(x)\\0&\text{otherwise.}\end{cases} \tag{2.2}$$

We denote by Sg the Borel essential supremum of the family $\mathscr{F}(g)$, that is, the positive Borel function h which satisfies $h\ge f'_u$ a.e. for every $u\in X$ and $h'\ge h$ a.e. for every positive Borel function h' such that $h\ge f'_u$ a.e. for every $u\in X$ (see for instance CASTAING and VALADIER).

Finally, for all $x \in \Omega$ and all $n \in \mathbb{N}$, we set

$$H_F(x,n) = Sn(x) \tag{2.3}$$

The following two statements are straightforward corollaries of the definition above.

PROPOSITION 2.2. – *If g and g' are two positive Borel functions of Ω and $B \subset \Omega$ is a Borel set such that $g = g'$ a.e. in B, then $Sg = Sg'$.*

PROPOSITION 2.3. – *If g is a Borel function which takes only integer values, then*

(i) $H_F\big(x, g(x)\big) = Sg(x)$ *a.e.,*

(ii) $\int_B H_F\big(x,g(x)\big)dx \ge F(u,B)$ *for all $u \in X$ such that $|D^k u| \le g$ almost everywhere in B.*

We shall prove that when either $X = W^{k,p}$ and F is finite or F is weakly (upper) p–bounded, $\int Sg\,dx < \infty$ for all positive functions $g \in L^p$ (Theorem 2.5). Hence Proposition 2.3(i) yields

$$\int_\Omega H_F\big(x,g(x)\big)dx < \infty \qquad \text{for all } g \in L^p(\Omega,\mathbb{N}).$$

Then we prove that when $H : \Omega \times \mathbb{N} \to [0,\infty]$ is a Borel function which satisfies this inequality, then

$$H(x,n) \le k(x) + mn^p \qquad \text{for all } x \in \Omega,\ n \in \mathbb{N},$$

for suitable $k \in L^1$ and $m \in \mathbb{N}$ (Theorem 2.7), and then, by assertion (ii) of Proposition of 2.3, we obtain that F is (upper) p–bounded (Theorem 2.8).

To begin with, we prove that the integral of Sg is finite for all positive functions $g \in L^p$ when either $X = W^{k,p}$ and F is finite or F is weakly (upper) p–bounded.

LEMMA 2.4. – *Let g be a positive function in $L^p(\Omega)$ and let r be a real number such that $r < \int Sg\,dx$. Then there exist a function $u \in X$ with compact support in Ω such that*

(i) $\|D^k u\|_p \le C2^{k+1}\|g\|_p$ *and C is the same constant of Lemma 5.10.*

(ii) $F(u,\Omega) \ge 2^{-(N+1)}r$.

PROOF. – Since $r < \int Sg\,dx$, we may find a positive function $h \in L^1$ such that $r < \int h\,dx$ and $h \le Sg$ a.e.. Set $\lambda(B) = \int_B h\,dt$ for all Borel sets $B \subset \Omega$ and let ε be a positive real number.

Since $h \le Sg$ a.e., we may find a countable collection $\mathscr{F} \subset X$ Borel sets $\{B_u\}_{u\in\mathscr{F}}$ which cover almost all of Ω such that $f'_u \ge h$ a.e. in B_u for every $u \in \mathscr{F}$.

Let E be the set of those points $x \in \Omega$ which are p–Lebesgue points of g, p–Lebesgue points of $D^k u$ for each $u \in \mathscr{F}$, 1–Lebesgue point of h, Lebesgue point of f_u for every $u \in \mathscr{F}$ and $h(x) > 0$. E is a Borel set λ–equivalent to Ω and for all $x \in E$ there exists a function $u_x \in \mathscr{F}$ such that

$$g(x) \ge |D^k u(x)| \text{ and } f_{u_x}(x) \ge h(x). \tag{2.4}$$

and then we may find a positive real number r_x such that, for all r with $0 < r < r_x$, $B(x,r)$ is relatively compact in Ω and (cf. (0.2) and (0.3))

$$2\|g\|_{L^p(x,r))} > \|D^k u\|_{L^p(B(x,r))} \tag{2.5}$$

$$\int_{B(x,r)} f_{u_x} dt \ge |B(x,r)| \cdot \big(f_{u_x}(x) - \varepsilon\big) \tag{2.6}$$

$$\int_{B(x,r)} h\,dt \le |B(x,r)| \cdot \big(h(x) + \varepsilon\big) \tag{2.7}$$

Set $\mathscr{F} = \big\{B(x,r) : x \in E,\ r < r_x\big\}$ and apply Theorem 5.4 to obtain disjoint balls $B(x_1,r_1),\dots,B(x_n,r_n)$ in $\mathscr{F}$ such that

$$\lambda\Big[E \setminus \bigcup_i B(x_i,r_i)\Big] \le \varepsilon\ .$$

Let $B_i = B(x_i,r_i)$ and $u_i = u_{x_i}$ for all i and apply Glueing Lemma 5.10 to obtain a function $u \in X$ such that

(a) $u = 0$ a.e. out of the union of all B_i.

(b) $D^k u = D^k u_i$ a.e. in $B(x_i, r_i/2)$ for all i.

(c) $\|D^k u\|_{L^p(B_i)} \le C2^k\|D^k u_i\|_{L^p(B_i)}$ for all i.

As every ball in $\mathscr{F}$ is relatively compact in Ω, (a) implies that u has compact support in Ω. By (c) and (2.5) we get

$$\|D^k u\|_p \le \bigg[\sum_i \Big(C2^k\|D^k u_i\|_{L^p(B_i)}\Big)^p\bigg]^{1/p} \le C2^{k+1}\|g\|_p\ ,$$

so that u satisfies (i). Taking into account (b), (2.4), (2.6), (2.7) and the choice of $B_1,\dots,B_n$, and recalling that F is D^k–local on $\mathscr{A}$ and that $\lambda(\Omega) = \int h\,dx > r$,

$$\begin{aligned}
F(u,\Omega) &\ge \sum_i F\big(u_i, B(x_i,r_i/2)\big) = \sum_i \int_{B(x_i,r_i/2)} f_{u_i} dt \\
&\ge \sum_i \big|B(x_i,r_i/2)\big| \cdot \big(f_{u_i}(x_i) - \varepsilon\big) \ge \sum_i 2^{-N}|B_i|\,\big(h(x_i) - \varepsilon\big) \\
&\ge 2^{-N}\sum_i \int_{B_i} (h - 2\varepsilon)dt = 2^{-N}\lambda(\cup B_i) - 2^{1-N}\big|\cup B_i\big|\varepsilon \\
&\ge 2^{-N}\big(\lambda(\Omega) - \varepsilon\big) - 2^{1-N}|\Omega|\varepsilon \ge 2^{-N}\,\big[r - \big(2|\Omega| + 1\big)\varepsilon\big]\ .
\end{aligned}$$

Hence (ii) holds if we have chosen ε small enough to have $\big(2|\Omega|+1\big)\varepsilon \le r/2$. ∎

THEOREM 2.5. – *Let g be a positive function in $L^p(\Omega)$ and suppose that either $X = W^{k,p}$ and F is finite or F is weakly (upper) p–bounded. Then $\int Sg\,dx < \infty$.*

PROOF. – Assume by contradiction that $\int Sg\,dx = \infty$. By Lemma 5.2 we may find a sequence of pairwise disjoint open sets Ω_n such that

$$\int_{\Omega_n} Sg\,dx > 1 \qquad \text{for } n = 1, 2, \ldots.$$

By Lemma 2.4, for all n there exist functions $u_n \in W^{k,p}$ with compact support in Ω_n such that

(*a*) $\|D^k u_n\|_{L^p(\Omega_n)} \le C 2^{k+1} \|g\|_{L^p(\Omega_n)}$ where C is the same constant of Lemma 5.10.

(*b*) $F(u_n, \Omega_n) \ge 2^{-(N+1)}$.

If $X = W^{k,p}$ and F is finite, set $u = \sum_1^\infty u_n$. As the functions u_n have pairwise disjoint compact supports included in Ω_n and g belongs to L^p, Poincaré inequality (BREZIS, chapter IX) and (*a*) imply that the series $\sum u_n$ converges in the $W^{k,p}$ norm and then u belongs to $W_0^{k,p}$. As the sets Ω_n are pairwise disjoint for all n, $u = u_n$ a.e. in Ω_n for $n = 1, 2, \ldots$. Hence, if take into account D^k–locality of F on $\mathscr{A}$ and (*b*), we obtain

$$F(u, \Omega) \ge \sum_n F(u_n, \Omega_n) \ge \sum_n 2^{-(N+1)} = \infty\ ,$$

and this contradicts the fact that F is finite.

If F is weakly (upper) p–bounded, set $v_m = \sum_1^m u_n$ for $m = 1, 2, \ldots$. Applying (*a*) and Poincaré inequality it may be proved that the functions v_m are functions in X with compact support in Ω and are uniformly bounded in the norm of $W^{k,p}$ and then there exists a real number M such that $M > F(v_m, \Omega)$ for all m because F is weakly p–bounded. But D^k–locality of F on $\mathscr{A}$ and (*b*) lead to the contradiction

$$F(v_m, \Omega) \ge \sum_1^m F(u_n, \Omega_n) \ge m 2^{-(N+1)} \qquad \text{for } m = 1, 2, \ldots. \ ∎$$

COROLLARY 2.6. – *If either $X = W^{k,p}$ and F is finite or F is weakly (upper) p-bounded, then H_F (see Definition 2.1) satisfies*

$$\int_\Omega H_F\big(x, g(x)\big)dx < \infty \qquad \textit{for all } g \in L^p(\Omega, \mathbb{N}). \tag{2.8}$$

PROOF. – It is enough to apply Theorem 2.5 and Proposition 2.3(i). ∎

Now we want to characterize those Borel functions $H : \Omega \times \mathbb{N} \to [0, \infty]$ which satisfy (2.8).

THEOREM 2.7. – *Let H be a Borel function of $\Omega \times \mathbb{N}$ into $[0, \infty]$. For all integers m set*

$$g_m(x) = \sup\big\{n : H(x, n) > m\, n^p\big\} \qquad \textit{for all } x \in \Omega. \tag{2.9}$$

Each g_m is a Borel function of Ω into $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and the following three statements are equivalent:

(i) *H satisfies* (2.8),

(ii) *H satisfies* (2.8) *and there exists an integer m such that g_m belongs to L^p,*

(iii) *there exist $m \in \mathbb{N}$ and $k \in L^1(\Omega)$ such that $H(x, n) \le k(x) + m\, n^p$ for all $x \in \Omega$, $n \in \mathbb{N}$.*

PROOF. – We omit to verify that each g_m is a Borel function.

(i) ⇒ (ii). By contradiction, suppose that $\int (g_m)^p dx = \infty$ for all integers m. By Lemma 5.3 there exist pairwise disjoint Borel sets A_m such that $\int_{A_m} (g_m)^p dx = \infty$ for $m = 1, 2, \ldots$.

Hence, (2.9) provides, for all integers m, Borel functions $f_m : \Omega \to \mathbb{N}$ such that

$$H\big(x, f_m(x)\big) > m\big(f_m(x)\big)^p \qquad \text{for all } x \in \Omega$$

and $\int_{A_m} (f_m)^p dx \ge 1/m^2$. As Lebesgue measure is non–atomic, there exist Borel sets $B_m \subset A_m$ such that

$$\int_{B_m} (f_m)^p dx = \frac{1}{m^2}\ . \tag{2.11}$$

Since the sets A_m are pairwise disjoint, the sets B_m are pairwise disjoint too and then we may find a Borel function f such that $f = f_m$ a.e. in each B_m and $f = 0$ a.e. outside the union of all B_m. Then (2.11) yields

$$\int_\Omega f^p dx = \sum_m \int_{B_m} (f_m)^p dx = \sum_m \frac{1}{m^2} < \infty\ ,$$

i.e. f belongs to L^p, (2.10) yields

$$\int_\Omega H\big(x, f(x)\big)dx > \sum_m \int_{B_m} m\big(f_m(x)\big)^p dx = \sum_m \frac{1}{m} = \infty$$

and this contradicts (i).

(ii) $\Rightarrow$ (iii). Let m be taken so that g_m belongs to L^p: hence, by (2.9), $H(x,n) \le m\, n^p$ for all $x \in \Omega$ and all integers $n > g_m(x)$. As $g_m < \infty$ a.e., we may find a Borel function $g : \Omega \to \mathbb{N}$ which is everywhere less than g_m and

$$H\bigl(x, g(x)\bigr) = \max\{H(x,n) : n \le g_m(x)\} \qquad \text{for a.a. } x \in \Omega.$$

Hence (iii) holds if we set $k(x) = H\bigl(x, g(x)\bigr)$. Infact, as g_m is p–summable and $g \le g_m$ everywhere, g belongs to L^p. This fact and (2.8) yield $k \in L^1$.

(iii) $\Rightarrow$ (i). Trivial. ∎

With Corollary 2.6 and Theorem 2.7 we can state and prove the main result of this section.

THEOREM 2.8. – *Let X be a local subspace of $W^{k,p}$ and let F be a functional on X which is an a.c. positive measure and D^k–local on $\mathscr{A}$. Then*

(i) *if $X = W^{k,p}$, F is (upper) p–bounded if and only if is finite,*

(ii) *in general, F is (upper) p–bounded if and only if is weakly (upper) p–bounded.*

PROOF. – Being obvious that F is finite and weakly p–bounded when is p–bounded, let's prove the opposite implication.

If $X = W^{k,p}$ and F is finite, or F is (upper) p–bounded, then H_F (cf. Definition 2.1) satisfies (2.8) by Corollary 2.6 and then Theorem 2.7 implies that there exist a positive integer m and a positive function $k \in L^1(\Omega)$ such that

$$H_F(x,n) \le k(x) + m\, n^p \qquad \text{for all } x \in \Omega \text{ and all } n \in \mathbb{N}. \tag{2.12}$$

Suppose that u is a function in X, choose a function $g \in L^p(\Omega, \mathbb{N})$ such that $|D^k u| \le g \le |D^k u| + 1$ everywhere and notice that, recalling Proposition 2.3(ii) and taking into account (2.12), for all Borel sets $B \subset \Omega$,

$$\begin{aligned} F(u,B) &\le \int_B H_F\bigl(x, g(x)\bigr) dx \\ &\le \int_B \bigl[k(x) + m\bigl(g(x)\bigr)^p\bigr]\, dx \le \int_B \bigl[k(x) + m\bigl(|D^k u(x)| + 1\bigr)^p\bigr]\, dx \\ &\le \int_B \bigl[k(x) + m2^{p-1} + m2^{p-1}|D^k u(x)|^p\bigr]\, dx \end{aligned}$$

(we have used inequality $(a+1)^p \le 2^{p-1}(a^p+1)$) and this prove that F is (upper) p–bounded. ∎

PROOF OF THEOREM 0.5. – Suppose that F is a functional on a local subspace X of $W^{k,p}(\Omega, \mathbb{R}^M)$ which is an a.c. measure and D^k–local on $\mathscr{A}$ (cf. Proposition 1.1) and define F^+ and F^- (resp. the positive and negative variations of F) by $F^+(u,\cdot) = \bigl(F(u,\cdot)\bigr)^+$ and $F^-(u,\cdot) = \bigl(F(u,\cdot)\bigr)^-$ for all $u \in X$. Verify that both F^+ and F^- are a.c. positive measures, D^k–local on $\mathscr{A}$, and notice that F^- is finite because F never takes values $-\infty$. Finally, apply Theorem 2.8. ∎

PROPOSITION 2.9. – *Let X be a subspace of $W^{k,p}(\Omega, \mathbb{R}^M)$ and let F be a functional on X which is p–bounded and lower semicontinuous on $\mathscr{A}$. Then F is lower semicontinuous on $\mathscr{B}$.*

PROOF. – Suppose that $\{u_n\}$ is a sequence of functions in X converging to a function $u \in X$ in the norm of $W^{k,p}$, E is a Borel set and ε is a positive real number. As F is p–bounded, there exists a positive function $a \in L^1(\Omega)$ and a positive real number b such that

$$|F(u,B)| \le \bigl[a(x) + b|D^k u(x)|^p\bigr] \qquad \text{for all } u \in X,\ B \in \mathscr{B}(\Omega). \tag{2.13}$$

Notice that the functional $G(u,B) = \int_B \bigl[a(x) + b|D^k u(x)|^p\bigr] dx$ is a positive finite measure and continuous on $\mathscr{B}$. Hence there exists an open set A such that $G(u, A \backslash E) \le \varepsilon$ and then, taking into account (2.13) and the lower semicontinuity of F on $\mathscr{A}$,

$$\begin{aligned} \liminf_{n\to\infty} F(u_n, E) &\ge \liminf_{n\to\infty} \bigl[F(u_n, A) - G(u_n, A \setminus E)\bigr] \\ &\ge \liminf_{n\to\infty} F(u_n, A) - \lim_{n\to\infty} G(u_n, A \setminus E) \\ &\ge F(u,A) - G(u, A \setminus E) \ge F(u,E) - 2G(u, A\setminus E) \ge F(u,E) - 2\varepsilon \end{aligned}$$

and the proof is complete because ε may be chosen arbitrary small. ∎

PROOF OF COROLLARY 0.6. – Let F be a functional on a local subspace X of $W^{k,p}(\Omega, \mathbb{R}^M)$ which is an a.c. measure, D^k–local on $\mathscr{A}$ (cf. Proposition 1.1) and is lower semicontinuous on $\mathscr{A}$. Suppose that either $X = W^{k,p}$ and F is finite or F is weakly p–bounded.

By Theorem 0.5 we have that F is p–bounded and then we may apply Proposition 2.9 to obtain that F is lower semicontinuous on $\mathscr{B}$ and Theorem 0.4 to obtain that F D^k–local on $\mathscr{B}$. ∎

3. – Representation Theorem.

Unless differently stated, throughout this section F is a fixed functional defined on a local subspace X of $W^{k,p}(\Omega, \mathbb{R}^M)$ which is an a.c. measure, D^k–local on $\mathscr{B}$ and lower semicontinuous on $\mathscr{B}$.

Since for all $u \in X$, $F(u,\cdot)$ is a measure on $\mathscr{B}(\Omega)$ which is absolutely continuous with respect to Lebesgue measure, it may be represented by a Borel function that we denote by f_v. In other words we have

$$F(u,B) = \int_B f_u\, dx \qquad \text{for all } B \in \mathscr{B}(\Omega).$$

DEFINITION 3.1. – Let $\mathcal{S}$ be a countable subset of X and let A be the set of all $x \in \Omega$ such that $f_v(x) \neq f_{v'}(x)$ and $D^k v(x) = D^k v'(x)$ for some v, v' in $\mathcal{S}$. As F is D^k–local on $\mathscr{B}$, $f_v = f_{v'}$ a.e. in B for all $v, v' \in \mathcal{S}$ and all $B \in \mathscr{B}(\Omega)$ such that $D^k v = D^k v'$ a.e. in B and then we have that $|A| = 0$.

For all (x,s) in $\Omega \times (\mathbb{R}^N)^{I(k)}$ set

$$f'_{\mathcal{S}}(x,s) = \begin{cases} f_v(x) & \text{if } x \notin A \text{ and } v \in \mathcal{S} \text{ exists such that } D^k v(x) = s \\ \infty & \text{otherwise .} \end{cases} \tag{3.1}$$

By definition of A, $f'_{\mathcal{S}}$ is well–defined. Thus, for all (x,s) in $\Omega \times (\mathbb{R}^N)^{I(k)}$, we may set

$$f_{\mathcal{S}}(x,s) = \liminf_{t \to s} f'_{\mathcal{S}}(x,t) \ , \tag{3.2}$$

i.e. $f_{\mathcal{S}}$ is the relaxation of $f'_{\mathcal{S}}$ with respect to second variable.

REMARK 3.2. – An easy computation shows that both $f'_{\mathcal{S}}$ and $f_{\mathcal{S}}$ are Borel functions of $\Omega \times (\mathbb{R}^M)^{I(k)}$, $f_{\mathcal{S}}$ is l.s.c. in the second variable and $f_v(x) \geq f_{\mathcal{S}}\big(x, v(x)\big)$ a.e. for all $v \in \mathcal{S}$.

We shall prove that $f_{\mathcal{S}}$ represents F, i.e. $f_{\mathcal{S}}\big(x, D^k u(x)\big) = f_u(x)$ a.e. for all $u \in X$, when $\mathcal{S}$ is a suitable countable subset of X: in Theorem 3.4 we prove that for every countable $\mathcal{S} \subset X$ we have

$$f_{\mathcal{S}}\big(x, D^k u(x)\big) \geq f_u(x) \quad \text{a.e. for all } u \in X$$

and in Theorem 3.8 we prove that the opposite inequality holds for a suitable $\mathcal{S}$.

To begin with, we prove the following approximation lemma.

LEMMA 3.3. – *Let $\mathcal{S}$ a countable subset of X and let u be a function in X. Define $g(x) = f_{\mathcal{S}}\big(x, D^k u(x)\big)$ and let A be a Borel set in which g takes only finite values. Then, for each $\varepsilon > 0$, there exist a function $w \in X$ and a Borel set B such that*

(i) *$g \geq f_{u+w} - \varepsilon$ a.e. on B and $|A \setminus B| \leq \varepsilon$.*

(ii) *$\|w\|_{W^{1,p}} \leq \varepsilon$.*

PROOF. – Let δ be a positive real number.

Let E be the set of all $x \in A$ which are p–Lebesgue points of $D^k u$, $D^k v$ and Lebesgue points of g, f_v for all $v \in \mathcal{S}$. E is a Borel set which is Lebesgue equivalent to A (cf. (0.2) and (0.3)).

For every x in E we have $g(x) = f_{\mathcal{S}}\big(x, D^k u(x)\big) < \infty$ and by definition of $f_{\mathcal{S}}$ and $f'_{\mathcal{S}}$ there exists $v_x \in \mathcal{S}$ such that

$$\big|D^k v_x(x) - D^k u(x)\big| < \delta^{k+1} \quad \text{and} \quad g(x) \geq f_{v_x}(x) - \delta$$

and then, taking into account that x is a p–Lebesgue point of $D^k u$ and $D^k v_x$ and a Lebesgue point of g and f_{v_x}, there exists a positive real number r_x such that, for all r with $0 < r < r_x$, $B(x,r)$ is relatively compact in Ω and (cf. (0.2) and (0.3))

$$\|D^k v_x - D^k u\|_{L^p(B(x,r))} < 2|B(x,r)|^{1/p}\delta^{k+1} \ , \tag{3.3}$$

$$\Big|B(x,r) \cap \big\{t : g(t) \geq f_{v_x}(t) - 2\delta\big\}\Big| \geq (1-\delta)|B(x,r)| \ . \tag{3.4}$$

As usual, set $\mathscr{F} = \big\{B(x,r) : x \in E,\ r < r(x)\big\}$ and apply Theorem 5.4 to obtain disjoint balls $B_i = B(x_i, r_i)$ in $\mathscr{F}$ for $i = 1, \dots, n$ such that

$$\big|E \setminus \cup B_i\big| \leq \delta \ . \tag{3.5}$$

Let $v_i = v_{x_i}$ for all i and apply Glueing Lemma 5.10 to obtain a function $w \in X$ such that

(a) $w = 0$ a.e. out of the union of all B_i.

(b) $D^k w = D^k(v_i - u)$ a.e. in $B\big(x_i, (1-\delta)r_i\big)$ for all i.

(c) $\|D^k w\|_{L^p(B_i)} \leq C\delta^{-k}\|D^k(v_i - u)\|_{L^p(B_i)}$ for all i.

Taking into account (a), (c) and (3.3), and recalling that $B_1, \dots, B_n$ are disjoint balls in Ω, we obtain

$$\|D^k w\|_p \leq \bigg[\sum_i \Big(C\delta^{-k}\|D^k(v_i - u)\|_{L^p(B_i)}\Big)^p\bigg]^{1/p} \leq 2C\delta\Big[\sum_i |B_i|\Big]^{1/p} \leq 2C|\Omega|^{1/p}\delta \ .$$

Since all balls in $\mathscr{F}$ are relatively compact in Ω, (a) implies that w has compact support in Ω and by Poincaré inequality we obtain

$$\|w\|_{W^{k,p}} \leq C_1\|D^k w\|_p \leq 2C_1 C|\Omega|^{1/p}\delta \tag{3.6}$$

where C_1 is a constant which depends on N, k and p only.

Let B be the set of all t such that

$$g(t) \geq f_{u+w}(t) - 2\delta \ . \tag{3.7}$$

For all i, (b) yields $D^k(u+w) = D^k v_i$ a.e. in $B\big(x_i, (1-\delta)r_i\big)$ and, taking into account that F is D^k–local on $\mathscr{B}$, $f_{u+w} = f_{v_i}$ a.e. in $B\big(x_i, (1-\delta)r_i\big)$. Hence (3.4) yields

$$\Big|B\big(x_i, (1-\delta)r_i\big) \setminus B\Big| \leq \delta\,\Big|B\big(x_i, (1-\delta)r_i\big)\Big| \quad \text{for } i = 1, \dots, n$$

and then, recalling (3.5),

$$\begin{aligned} |A \setminus B| = |E \setminus B| &\leq \Big|E \setminus \bigcup_i B(x_i, r_i)\Big| + \sum_i \big|B(x_i, r_i) \setminus B\big| \\ &\leq \delta + \sum_i \bigg[\ \Big|B(x_i, r_i) \setminus B\big(x_i, (1-\delta)r_i\big)\Big| + \Big|B\big(x_i, (1-\delta)r_i\big) \setminus B\Big|\ \bigg] \\ &\leq \delta + \sum_i \Big[\ \big(1 - (1-\delta)^N\big)|B_i| + \delta\,\Big|B\big(x_i, (1-\delta)r_i\big)\Big|\ \Big] \\ &\leq \delta + \sum_i (N\delta + \delta)\,|B_i| = \big[1 + (N+1)|\Omega|\big]\,\delta \ . \end{aligned} \tag{3.8}$$

(3.6), (3.7) and (3.8) yields (i) and (ii) if we have chosen δ small enough to have that $2C_1C|\Omega|^{1/p}\delta \le \varepsilon$, $2\delta \le \varepsilon$ and $\big[1+(N+1)|\Omega|\big]\,\delta \le \varepsilon$. ■

THEOREM 3.4. – *Let $\mathcal{S}$ be a countable subset of X, then $f_{\mathcal{S}}\big(x, D^k u(x)\big) \ge f_u(x)$ a.e. for all $u \in X$.*

PROOF. – Let u be a fixed function in X. Define $g(x) = f_{\mathcal{S}}\big(x, D^k u(x)\big)$ and let A be the set of all $x \in \Omega$ such that $g(x) < \infty$. It is enough to prove that

$$g(x) = f_{\mathcal{S}}\big(x, D^k u(x)\big) \ge f_u(x) \qquad \text{a.e. in } A. \tag{3.9}$$

By Lemma 3.3, for every integer n we may find a function $w_n \in X$ and a Borel set B_n such that

(a) $g \ge f_{u+w_n} - 2^{-n}$ a.e. on B_n and $|A \setminus B_n| \le 2^{-n}$.

(b) $\|w_n\|_{W^{1,p}} \le 2^{-n}$.

For each integer m set $C_m = \Big(\bigcap_{n>m} B_n\Big)$. By ($a$)

$$g \ge f_{u+w_n} - 2^{-m} \qquad \text{a.e in } C_m \text{ for all } n > m.$$

By (b), $w_n + u$ converge to u in the $W^{k,p}$ norm and then, taking into account the definition of g and the lower semicontinuity of F on $\mathscr{B}$, for all integers m and Borel sets $B \subset C_m$ we have

$$\begin{aligned}\int_B f_{\mathcal{S}}\big(x, D^k u(x)\big)dx &= \int_B g(x)dx \\ &\ge \liminf_{n\to\infty} \int_B \big[f_{u+w_n}(x) - 2^{-m}\big]dx \\ &= \liminf_{n\to\infty} F(u+w_n, B) - |B|\,2^{-m} \ge F(v,B) - |\Omega|\,2^{-m}\ .\end{aligned}$$

Inequality (3.9) immediately follows since previous inequality holds for every integer m, F is an a.c. measure and (a) yields

$$|A \setminus C_m| \le \sum_{n>m} |A \setminus B_m| \le \sum_{n>m} 2^{-n} = 2^{-m}\!\downarrow\! 0 \qquad \text{when } m \uparrow \infty. \ ■$$

COROLLARY 3.5. – *Let $\mathcal{S}$ be a countable subset of X, then $f_{\mathcal{S}}\big(x, D^k v(x)\big) = f_v(x)$ a.e. for all $v \in \mathcal{S}$.*

PROOF. – It is enough to apply Theorem 3.4 and recall Remark 3.2. ■

LEMMA 3.6. – *Let $f : \Omega \times \mathbb{R}^M \longrightarrow]-\infty, \infty]$ be a Borel function which is lower semicontinuous with respect to second variable. Suppose that there exist a positive function $a \in L^1(\Omega)$ and a positive real number b such that*

$$f(x,s) \ge -\big[a(x) + b|s|^p\big] \qquad \textit{for all } x \in \Omega \textit{ and all } s \in \mathbb{R}^M$$

and let F_f be the integral functional on $L^p(\Omega, \mathbb{R}^M)$ which is associated to f by usual formula

$$F_f(u,B) = \int_B f\big(x, u(x)\big)dx\ .$$

Then F_f is well defined and lower semicontinuous on $\mathscr{B}$

PROOF. – notice that the functional $G(u,B) = \int_B \big[a(x) + b|u(x)|^p\big]dx$ is continuous on $\mathscr{B}$ and then F_f is l.s.c. on $\mathscr{B}$ if and only if $F_f + G$ is. Hence we may suppose with no loss in generality that f is a positive function.

Let $\{u_n\}$ be a sequence of functions which converges to u almost everywhere and in the L^p norm. As f is lower semicontinuous with respect to second variable we have that

$$\liminf_{n\to\infty} f\big(x, u_n(x)\big) \ge f\big(x, u(x)\big) \qquad \text{a.e.}$$

and then Fatou's lemma yields, for all Borel sets B,

$$\liminf_{n\to\infty} F_f(u_n, B) = \liminf_{n\to\infty} \int_B f\big(x, u_n(x)\big)dx \ge \int_B f\big(x, u(x)\big)dx = F_f(u,B)\ .$$

As for every sequence which converges in L^1_{loc} we may find a subsequence which converges almost everywhere, we have just proved that F_f is lower semicontinuous on $\mathscr{B}$. ■

COROLLARY 3.7. – *Let k be a positive integer and set*

$$F^k(u,\cdot) = \big(F(u,\cdot) \vee -k\mathscr{L}_N(\cdot)\big) \wedge k\mathscr{L}_N(\cdot) \qquad \textit{for all } u \in X$$

($\vee$ and $\wedge$ must be intended in the sense of measures). Then F^k is lower semicontinuous on $\mathscr{B}$.

PROOF. – Let $\{u_n\}$ be a sequence which converges to u_∞ in X and set $\mathcal{S} = \{u_n : n \in \overline{\mathbb{N}}\}$. Corollary 3.5 yields

$$F(u_n, B) = \int_B f_{\mathcal{S}}\big(x, D^k u_n(x)\big)dx$$

for all integers $n \in \overline{\mathbb{N}}$ and all Borel sets $B \subset \Omega$, and then

$$F^k(u_n, B) = \int_B \Big[f_{\mathcal{S}}\big(x, D^k u_n(x)\big) \vee -k\Big] \wedge k\, dx \tag{3.10}$$

for all integers $n \in \overline{\mathbb{N}}$ and all Borel sets $B \subset \Omega$. Notice that $f_{\mathcal{S}}$ is a Borel function of $\Omega \times (\mathbb{R}^M)^{I(k)}$ which is l.s.c. in the second variable and then $(f_{\mathcal{S}} \vee -k) \wedge k$ is a Borel function which is l.s.c. in the second variable, everywhere greater than $-k$ and less than k. Hence, Lemma 3.6 yields

$$\liminf_{n\to\infty} F^k(u_n, B) \ge F^k(u_\infty, B) \qquad \text{for all Borel sets } B \subset \Omega. \ ■$$

LEMMA 3.8. – *If (T,d) is a separable metric space and $f : T \to [-\infty,\infty]$ is a lower semicontinuous function, there exists a countable set $S \subset T$ such that f is the relaxation on X of its restriction within S, i.e.*

$$f(x) = \liminf_{y\in S,\ y\to x} f(y) \qquad \textit{for all } x \in T,$$

PROOF. – By semicontinuity it's enough to find a countable set S such that

$$f(x) \geq \liminf_{y\in S,\ y\to x} f(y) \qquad \text{for all } x \in T. \tag{3.11}$$

Set $C_q = \{x : f(x) \leq q\}$; for all rational q choose a countable set $S_q \subset C_q$ which is dense in C_q and finally set $S = \bigcup\{S_q : q \in \mathbf{Q}\}$. Let $x \in T$; by definition of S_q, for all rational numbers $q > f(x)$ and for all $\varepsilon > 0$ there exists $y \in S_q \subset S$ such that $d(y,x) < \varepsilon$ and $f(y) \leq q$. Thus inequality (3.11) immediately follows. ■

THEOREM 3.9. – *There exists a countable set $\mathcal{S} \subset X$ such that*

$$f_u(x) \geq f_{\mathcal{S}}\big(x, D^k u(x)\big) \qquad \textit{a.e. for all } u \in X. \tag{3.12}$$

PROOF. – Consider $\mathscr{B}(\Omega)$ as a subset of $L^1(\Omega)$. As L^1 is a separable metric space, we may find a countable collection of Borel sets $\mathscr{H}$ which is dense in $\mathscr{B}(\Omega)$. For $k = 1, 2, \ldots$, define F^k as in Corollary 3.7; as each F^k is lower semicontinuous on $\mathscr{B}$ and X is a separable metric space, by Lemma 3.8 we may find a countable set $\mathcal{S} \subset X$ such that

$$F^k(u,B) = \liminf_{v\in\mathcal{S},\ v\to u} F^k(u,B)$$

for all $k \in \mathbb{N}$, $u \in X$ and $B \in \mathscr{H}$.

Taking into account Corollary 3.5 and applying Lemma 3.6, for all $k \in \mathbb{N}$ and all $B \in \mathscr{H}$ we get

$$\begin{aligned} F^k(u,B) = \liminf_{v\in\mathcal{S},\ v\to u} F^k(v,B) &= \liminf_{v\in\mathcal{S},\ v\to u} \int_B \Big[f_{\mathcal{S}}\big(x, D^k v(x)\big) \vee -k\Big] \wedge k \; dx \\ &\geq \int_B \Big[f_{\mathcal{S}}\big(x, D^k u(x)\big) \vee -k\Big] \wedge k \; dx \; . \end{aligned}$$

As $\mathscr{H}$ is dense in $\mathscr{B}(\Omega)$, this inequality holds for all Borel sets $B \subset \Omega$ and all positive integers k and yields

$$\big[f_u(x) \vee -k\big] \wedge k \geq \Big[f_{\mathcal{S}}\big(x, D^k u(x)\big) \vee -k\Big] \wedge k \qquad \text{a.e. for all } v \in X.$$

Now (3.12) follows immediately since k is arbitrary. ■

THEOREM 3.10 (Representation Theorem). – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$ and let F be a functional on X which is an a.c. measure, D^k–local on $\mathscr{B}$ and l.s.c. on $\mathscr{B}$. Then there exists a Borel function f of $\Omega \times (\mathbb{R}^M)^{I(k)}$ into $]-\infty,\infty]$ which is lower semicontinuous with respect to second variable and represents F in the form* (0.1), *i.e.*

$$F(u,B) = \int_B f\big(x, D^k u(x)\big) dx$$

for all functions $u \in X$ and all Borel sets $B \subset \Omega$.

PROOF. – By Theorem 3.9 there exists a countable $\mathcal{S} \subset X$ such that $f_u(x) \geq f_{\mathcal{S}}\big(x, D^k u(x)\big)$ a.e. for all $u \in X$. By Theorem 3.4 the opposite inequality holds and then $f_{\mathcal{S}}$ satisfies

$$f_u(x) = f_{\mathcal{S}}\big(x, D^k u(x)\big) \qquad \text{a.e. for all } u \in X,$$

and then $f_{\mathcal{S}}$ represents F and is a Borel function lower semicontinuous with respect to second variable (Remark 3.2). ■

In order to complete the proof of Theorem 0.7 we need another lemma.

THEOREM 3.11. –(Uniqueness of Integral Representation) *Let $f, f' : \Omega \times (\mathbb{R}^M)^{I(k)} \longrightarrow [-\infty,\infty]$ be two Borel functions such that, for all $u \in C_0^k(\Omega,\mathbb{R}^M)$,*

$$f\big(x, D^k u(x)\big) = f'\big(x, D^k u(x)\big) \qquad \textit{a.e. in } \Omega. \tag{3.13}$$

Then there exists a negligible Borel set $N \subset \Omega$ such that $f(x,s) = f'(x,s)$ for all $x \in \Omega \setminus N$ and all $s \in (R^M)^{I(k)}$.

In particular, if F is a functional on a space $X \subset W^{k,p}(\Omega,\mathbb{R}^M)$ which contains C_0^k and f, f' are two Borel functions which represents F as an integral functional in the form (0.1), then there exists a negligible Borel set N such that $f(x,s) = f'(x,s)$ for all $x \in \Omega \setminus N$ and all $s \in (R^M)^{I(k)}$.

PROOF. – Let S be the set of all (x,s) such that $f(x,s) \neq f'(x,s)$ and let π be the projection of $\Omega \times (\mathbb{R}^M)^{I(k)}$ into Ω. Aumann measurable selection theorem (cf. CASTAING and VALADIER, Theorems III.22 and III.23) states that

(*a*) $\pi(S)$ is a Lebesgue measurable set,

(*b*) there exists a Lebesgue measurable function $w : \pi(S) \longrightarrow (\mathbb{R}^M)^{I(k)}$ the graph of which is a subset of S.

Since $\pi(S)$ is Lebesgue measurable, it is enough to show that $|\pi(S)| = 0$. Assume by contradiction that $|\pi(S)| > 0$; (*b*) and Lusin theorem yields a continuous function $v : \Omega \longrightarrow (\mathbb{R}^M)^{I(k)}$ with compact support in Ω and a compact set $B \subset \pi(S)$ such that $|B| > 0$ and $v = w$ in B and then $\big(x, v(x)\big) = \big(x, w(x)\big) \in S$ for all $x \in B$.

If $k = 0$ this contradicts hypothesis (3.13) and the proof is complete. If $k > 0$, we may apply Theorem 5.12 to obtain a function $u \in C_0^k(\Omega, \mathbb{R}^M)$ and a compact set $K \subset \Omega$ such that $|\Omega \setminus K| \leq |B|/2$ and $D^k u = v$ in K. Hence

$$\big(x, D^k u(x)\big) = \big(x, v(x)\big) = \big(x, w(x)\big) \in E \qquad \text{for all } x \in K \cap B$$

and this contradicts hypothesis (3.13) because $|K \cap B| \geq |B| - |\Omega \setminus K| \geq |B|/2 > 0$. ■

PROOF OF THEOREM 0.7. – Let F be a functional on X which is an a.c. measure, D^k–local on $\mathscr{B}$ and l.s.c. on $\mathscr{B}$. By Theorem 3.10 there exists a Borel function $f : \Omega \times (\mathbb{R}^M)^{I(k)} \longrightarrow [-\infty, \infty]$ which is lower semicontinuous in the second variable and represents F in the form (0.1) and (i) is proved. (ii) is a straightforward corollary of Theorem 3.11. ■

4. – Some Examples.

Given a Borel function $f : \Omega \times (\mathbb{R}^M)^{I(k)} \longrightarrow [-\infty, \infty]$, in the following we denote by F_f the integral functional which is associated to f by usual formula (0.1) (of course, when it makes sense, cf. Lemma 3.6).

To begin with, we notice a simple property of integral functional (Proposition 4.1) and then we apply it to obtain a useful criterion of non–representability (Corollary 4.2).

PROPOSITION 4.1. – *Let $f, f' : \Omega \times (\mathbb{R}^M)^{I(k)} \longrightarrow [-\infty, \infty]$ be two Borel functions and suppose that a function $v \in C^{k+1}$ exists so that, for every homogeneous polynomial function $p : \mathbb{R}^N \to \mathbb{R}^M$ with $\deg p = k$,*

$$f\big(x, D^k v(x) + D^k p(x)\big) = f'\big(x, D^k v(x) + D^k p(x)\big) \qquad \textit{for almost all } x \in \Omega.$$

Then $f = f'$ for almost all (x, s) in $\Omega \times (\mathbb{R}^M)^{I(k)}$.

PROOF. – Let $\mathbf{P}$ be the (finite dimensional) vector space of all homogeneous polynomial functions $p : \mathbb{R}^N \to \mathbb{R}^M$ with $\deg p = k$ and consider the following map of $\Omega \times \mathbf{P}$ into $\Omega \times (\mathbb{R}^M)^{I(k)}$:

$$\Psi_v : (x, p) \longmapsto \big(x, D^k v(x) + D^k p(x)\big) \ .$$

An easy computation shows that Ψ_v is a diffeomorphism of class C^1 and by hypothesis , for all $p \in \mathbf{P}$,

$$\big[f \circ \Psi_v\big](x, p) = \big[f' \circ \Psi_v\big](x, p) \qquad \text{for almost all } x \in \Omega.$$

Hence $f \circ \Psi_v = f' \circ \Psi_v$ a.e. in $\Omega \times \mathbf{P}$ by Fubini's theorem and $f = f'$ a.e. in $\Omega \times (\mathbb{R}^M)^{I(k)}$ because Ψ_v is a diffeomorphism. ■

COROLLARY 4.2. – *Let F be a functional on a subspace X of $W^{k,p}(\Omega, \mathbb{R}^N)$. Suppose that functions $v, v' \in X \cap C^{k+1}$ and Borel functions $f, f' : \Omega \times (\mathbb{R}^N)^{I(k)} \longrightarrow [-\infty, \infty]$ exist so that, for every homogeneous polynomial function $p : \mathbb{R}^N \to \mathbb{R}^M$ with $\deg p = k$,*

$$F(v{+}p, B) = F_f(v{+}p, B) \quad , \quad F(v'{+}p, B) = F_{f'}(v'{+}p, B) \qquad \textit{for all Borel sets } B \subset \Omega$$

but f and f' do not agree almost everywhere in $\Omega \times (\mathbb{R}^N)^{I(k)}$. Then F cannot be represented as an integral functional in the form (0.1) by any Borel function.

PROOF. – If F can be represented as an integral functional in the form (0.1) by a Borel function g, then we should have that f, f' and g agree almost everywhere in $\Omega \times (\mathbb{R}^N)^{I(k)}$ by Proposition 4.1. ■

EXAMPLE 4.3. – Let $X \subset W^{k,p}(\Omega, \mathbb{R}^M)$ be the local subspace of all functions which belong to $W^{k+1,p}$ and set

$$F(u, B) = \int_B |D^{k+1} u|^p dx \qquad \text{for all } u \in X,\ B \in \mathscr{B}(\Omega).$$

It is obvious that F is a finite a.c. measure and is lower semicontinuous on $\mathscr{B}$.

It is well known that if u and u' are two functions in $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^M)$ and B is a Borel set such that $u = u'$ a.e. in B then $Du = Du'$ a.e. in B (cf. the proof of Proposition 1.1 or BREZIS, chapter IX). It follows immediately that F is D^k–local on $\mathscr{B}$.

Now we apply Corollary 4.2 to prove that F is not representable as an integral functional in the form (0.1) by any Borel function. Let $v = 0$ and notice that $F(v + p, B) = 0$ for all Borel sets $B \subset \Omega$ and all polynomial functions $p : \mathbb{R}^N \to \mathbb{R}^M$ with $\deg p \leq k$. Let $v' \in X$ be any function of class C^{k+1} such that $|D^{k+1} v| = 1$ everywhere in Ω and notice that $F(v' + p, B) = |B|$ for all Borel sets $B \subset \Omega$ and all polynomial functions $p : \mathbb{R}^N \to \mathbb{R}^M$ with $\deg p \leq k$. Apply Corollary 4.2 with $f = 0$ and $f' = 1$.

F is not lower semicontinuous on $\mathscr{B}$, otherwise we could apply Theorem 0.7 and obtain that F is representable as an integral functional by a Borel function f.

F is not p–bounded, otherwise it would be lower semicontinuous on $\mathscr{B}$ by Proposition 2.9.

We have considered a very particular functional defined on a local subspace of $W^{k,p}$. We may ask whether this functional may be extended to the whole of $W^{k,p}$ and what happens in this case.

Let u be a function in $W^{k,p}(\Omega, \mathbb{R}^M)$ and let $\mathscr{F}$ be the collection of all open sets $A \subset \Omega$ such that u belongs to $W^{k+1,p}_{\text{loc}}(A, \mathbb{R}^M)$. It may be proved that $\mathscr{F}$ is closed under countable union and then Lindeloff theorem yields a maximal element of $\mathscr{F}$ which we denote by A_u. For all $u \in W^{k,p}$ and all $B \in \mathscr{B}(\Omega)$, set

$$G(u, B) = \begin{cases} \infty & \text{if } |B \setminus A_u| > 0, \\[2ex] \displaystyle\int_{B \cap A_u} |D^{k+1} u|^p dx & \text{otherwise.} \end{cases}$$

One may easily verify that G is an a.c. measure, D^k–local on $\mathscr{A}$ and lower semicontinuous on $\mathscr{A}$. Moreover, for all functions $u \in W^{k+1,p}$ and all Borel sets $B \subset \Omega$, $G(u,B) = F(u,B)$ where F is the previously described functional.

G is not D^k–local on $\mathscr{B}$, otherwise F would be D^k–local on $\mathscr{B}$ and we have just proved this impossible.

G is not lower semicontinuous on $\mathscr{B}$, otherwise it would be D^k–local on $\mathscr{B}$ by Theorem 0.4.

EXAMPLE 4.4 (cf. BUTTAZZO and DAL MASO, [1] and [3]). – Let u be a function in $W^{k,p}(\Omega,\mathbb{R}^M)$ and let $\mathscr{F}$ be the collection of all Borel sets E such that $D^k u$ agrees with a constant a.e. in E. It may be proved that there exists a Borel set E_u which is covered by countably many elements of $\mathscr{F}$ and $|E_u \setminus E| = 0$ for all $E \in \mathscr{F}$. Set

$$F(u,B) = |B \cap E_u| \quad \text{for all } u \in W^{k,p},\ B \in \mathscr{B}(\Omega).$$

It is obvious that F is a finite a.c. measure, D^k–local on $\mathscr{B}$.

Let $v = 0$ and v' be any function of class C^{k+1} such that $D^{k+1}v' \neq 0$ for all $x \in \Omega$. Notice that, for all Borel sets B and all polynomial functions p with $\deg p \leq k$, $F(v+p,B) = |B|$ and $F(v'+p,B) = 0$. Apply Corollary 4.2 with $f = 1$ and $f' = 0$ and obtain that F is not representable as an integral functional in the form (0.1) by any Borel function.

We want to point out that that the functionals F described in examples 4.3 and 4.4 may be represented as integral functionals by functions which are not Borel measurable, infact the following representation theorem holds (see for instance APPELL).

THEOREM 4.5. – *Let X be a subset of $W^{k,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^M)$. Suppose that F is a functional on X which is an a.c. measure and D^k–local on $\mathscr{B}$, and that Continuum Hypothesis holds. Then there exists a function $f : \Omega \times (\mathbb{R}^M)^{I(k)} \longrightarrow]-\infty,\infty]$ such that $f\big(x,u(x)\big)$ is a Borel function for all $u \in X$ and*

$$F(u,B) = \int_B f\big(x, D^k u(x)\big)dx \quad \text{for all } u \in X \text{ and } B \in \mathscr{B}.$$

PROOF. – We prove the case $k = 0$ only (the proof of the general case is slight generalization of this).

As Continuum Hypothesis holds and the set of all Borel functions of Ω into $\mathbb{R}^M$ has the same cardinality of $2^{\mathbb{N}}$, $\mathrm{card}(X) \leq \aleph_1$ and then we may find a well order $(\prec)$ of X such that each element of X is preceded by a countable lot of elements only (cf. HALMOS).

For all $u \in X$, let f_u be a Borel function which represents the measure $F(u,\cdot)$ and set

$$f(x,s) = \begin{cases} 0 & \text{if } u(x) \neq s \text{ for every } u, \\ f_v(x) & \text{otherwise, with } v = \min\{u : u(x) = s\}. \end{cases}$$

Let u be a fixed function in X and, for all $v \preceq u$, set

$$A_v = \Big\{ x \in \Omega :\ v(x) = u(x) \text{ and } w(x) \neq u(x) \text{ for all } w \prec v \Big\}.$$

An easy computation shows that $\big\{A_v : v \preceq u\big\}$ is a *countable* Borel partition of Ω and $f\big(x,u(x)\big) = f_v(x)$ for all $v \preceq u$ and all $x \in A_v$: hence $f\big(x,u(x)\big)$ is a Borel function and $f\big(x,u(x)\big) = f_u(x)$ a.e. because $f_v = f_u$ a.e. in A_v for all $v \preceq u$ by the locality of F on $\mathscr{B}$. ■

EXAMPLE 4.6. – Let v be a Borel real function of $\mathbb{R}$ whose graph Γv is purely $\mathscr{H}_1$–unrectifiable, i.e. $\mathscr{H}_1(M \cap \Gamma v) = 0$ for all 1–dimensional manifold M of class C^1 (see, for instance, FEDERER and MORGAN). For example take

$$v(t) = \sum_0^\infty 4^{-n} w(4^n t) \quad \text{for all } t \in \mathbb{R}$$

where w is the periodic function

$$w(t) = \begin{cases} 0 & \text{if } 2h-1 \leq t < 2h \text{ for some } h \in \mathbf{Z}, \\ 4 & \text{if } 2h \leq t < 2h+1 \text{ for some } h \in \mathbf{Z}. \end{cases}$$

Let $N = 1$, $M = 1$. Suppose that Ω is an nonempty open subset of $\mathbb{R}$ and set $S = \Gamma v \cap \Omega \times \mathbb{R}$.

S is a Borel set of $\Omega \times \mathbb{R}$ the projection of which on Ω is Ω. Notice that for all functions u of class C^{k+1} on Ω, $\big|\{t : D^k u(t) = v(t)\}\big| = 0$ and then, if f and f' are two Borel functions of $\Omega \times \mathbb{R}$ into $[-\infty,\infty]$ which agree everywhere but in S,

$$f\big(t, D^k u(t)\big) = f'\big(t, D^k u(t)\big) \quad \text{a.e. in } \Omega$$

for all functions u of class C^{k+1}. This shows that Theorem 3.11 does not hold in general when X does not contain C^k_0 and in particular we may find functions f and f' which do not agree in a set S with non negligible projection but represent the same functional on X.

EXAMPLE 4.8. – For all Borel sets $E \subset \Omega$, define

$$\lambda(E) = \begin{cases} 0 & \text{if } E \text{ is of first category}, \\ \infty & \text{otherwise}. \end{cases}$$

(we recall that a set $E \subset \Omega$ is of first category if it is covered by countably many closed sets with empty interiors).

It is obvious that λ is a measure.

$\lambda(\Omega) = \infty$ because Ω is not of first category (Baire's theorem).

We want to prove that λ is not lower regular. Notice that, for every positive finite Borel measure μ on Ω and every Borel set $E \subset \Omega$,

$$\mu(E) = \sup \Big\{ \mu(K): \; K \text{ is compact with empty interior and } K \subset E \Big\} ,$$

infact, for every $\varepsilon > 0$ we may find a dense open set A with $\mu(A) \le \varepsilon$ and, for every Borel set E, a compact set $C \subset E$ such that $\mu(E \setminus C) \le \varepsilon$. Then $K = C \setminus A$ is a compact set with empty interior and $\mu(E \setminus K) \le 2\varepsilon$.

Suppose that μ is a positive finite measure on Ω such that $\mu(E) = 0$ whenever $\lambda(E) = 0$, then $\mu(E) = 0$ for all Borel sets E of first category and in particular $\mu(K) = 0$ for all compact sets K with empty interior. Hence previous remark yields $\mu(E) = 0$ for all Borel sets E. Then λ is not lower regular (cf. Remark 0.1).

5. – Appendix on Measure Theory.

Some Lemmas in Integration Theory

In the following, we say that a sequence of sets $\{A_n\}$ is strictly increasing if $\overline{A}_n \subset A_{n+1}$ for all n.

LEMMA 5.1. – *Let λ be a positive non–atomic infinite measure on Ω. Then there exists a strictly increasing sequence of open sets $\{B_n\}$ such that $\lambda\big(\Omega \setminus \cup B_n\big) = 0$ and $\lambda(\Omega \setminus B_n) = \infty$ for all integers n.*

PROOF. – Let $\{A_n\}$ be a strictly increasing sequence of open sets which cover Ω. Set

$$A = \Big\{ x \; : \lambda\big(B(x,r)\big) = \infty \text{ for all } r > 0 \Big\} .$$

First case: $A = \emptyset$. Each compact sets $K \subset \Omega$ can be covered by a finite collection of open balls with finite measure, and then K has finite measure too. Hence it is enough to take $B_n = A_n$ for all integers n.

Second case: $A \ne \emptyset$. There exists $x \in \Omega$ such that $\lambda\big(B(x,r)\big) = \infty$ for all $r > 0$. Hence it is enough to take $B_n = A_n \setminus \overline{B(x, 1/n)}$ for all integers n. ■

LEMMA 5.2. – *Let λ be a positive non–atomic infinite measure on Ω. Then there exists a countable disjoint collection $\mathscr{F}$ of open sets such that $\lambda(A) > 1$ for all $A \in \mathscr{F}$.*

PROOF. – By Lemma 5.1 there exists a strictly increasing sequence of open sets $\{B_n\}$ such that $\lambda\big(\Omega \setminus \cup B_n\big) = 0$ and $\lambda(\Omega \setminus B_n) = \infty$ for all n. We may suppose that $B_0 = \emptyset$.

By induction on n, we choose integers m_n so that $m_0 = 0$ and $\lambda\big(B_{m_{n+1}} \setminus B_{m_n+1}\big) > 1$ for all $n \ge 0$. Let n be a fixed integer and let m_n be chosen. By the choice of the sets B_n we have that

$$\lim_{m \to \infty} \lambda\big(B_m \setminus B_{m_n+1}\big) = \lambda\big(\Omega \setminus B_{m_n+1}\big) = \infty$$

and then there exists an integer m_{n+1} such that $\lambda\big(B_{m_{n+1}} \setminus B_{m_n+1}\big) > 1$.

Set $A_n = B_{m_{n+1}} \setminus \overline{B}_{m_n}$ for all $n \in \mathbb{N}$. Recalling that B_{m-1} is relatively compact in B_m for all m, it may easily be proved that the collection $\mathscr{F} = \{A_n : n \in \mathbb{N}\}$ satisfies our thesis. ■

LEMMA 5.3. – *For all integers $n > 0$, let Borel functions $f_n : \Omega \to [0, \infty]$ be given so that $\int_\Omega f_n dt = \infty$. Then there exist pairwise disjoint Borel sets A_n such that $\int_{A_n} f_n dt = \infty$ for $n = 1, 2, \ldots$.*

PROOF. – In this proof we omit to check that any considered set is a Borel set. n and m are always positive integers while k and h always belong to $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

For every couple n, k, set

$$D_{n,k} = \{x : k \le f_n(x) < k+1\} \quad \text{if } k < \infty \text{, and} \quad D_{n,\infty} = \{x : f_n(x) = \infty\} . \tag{5.1}$$

First step. For all n, k, we find a set $C_{n,k} \subset D_{n,k}$ such that

$$|C_{n,k}| = 2^{-n} |D_{n,k}| \tag{5.2}$$

$$\Big| C_{n,k} \bigcap \Big(\bigcup_{m>n;\, h \in \overline{\mathbb{N}}} C_{m,h} \Big) \Big| \le 2^{-n} |C_{n,k}| . \tag{5.3}$$

We act by induction on n: let n be fixed and let $C_{m,k} \subset D_{m,k}$ be chosen for all $m < n$ and $k \in \overline{\mathbb{N}}$. For each $m < n$ set

$$\mathscr{F}_m = \Big\{ C_{m,k} : k \in \overline{\mathbb{N}} \Big\} \bigcup \Big\{ \Omega \setminus \bigcup_{k \in \overline{\mathbb{N}}} C_{m,k} \Big\} .$$

$\mathscr{F}_m$ is a countable partition of Ω because the sets $\big\{ C_{m,k} : k \in \overline{\mathbb{N}} \big\}$ are pairwise disjoint. Hence, also

$$\mathcal{G} = \Big\{ \bigcap_{m<n} E_m \;\; : \quad E_m \in \mathscr{F}_m \text{ for } m = 1, \ldots, n-1 \Big\} .$$

is a countable partition of Ω and recalling that Lebesgue measure is non-atomic, for each k we may find $C_{n,k} \subset D_{n,k}$ such that for every $E \in \mathcal{G}$

$$|C_{n,k} \cap E| = 2^{-n} |D_{n,k} \cap E| . \tag{5.4}$$

Let n and k be fixed, (5.4) implies that $C_{n,k}$ satisfies (5.2) and that for all m with $m > n$ and all $h \in \overline{\mathbb{N}}$,

$$|C_{m,h} \cap C_{n,k}| = 2^{-m} |D_{m,h} \cap C_{n,k}| .$$

Hence, taking into account that for every m the collection $\big\{ D_{m,h} : h \in \overline{\mathbb{N}} \big\}$ is a partition of Ω,

$$\begin{aligned} \Big| C_{n,k} \bigcap \Big(\bigcup_{m>n,\, h \in \overline{\mathbb{N}}} C_{m,h} \Big) \Big| &\le \sum_{m>n,\, h \in \overline{\mathbb{N}}} |C_{m,h} \cap C_{n,k}| \\ &\le \sum_{m>n} 2^{-m} \Big(\sum_{h \in \overline{\mathbb{N}}} |D_{m,h} \cap C_{n,k}| \Big) \\ &= \sum_{m>n} 2^{-m} |C_{n,k}| \; = 2^{-n} |C_{n,k}| \end{aligned}$$

and $C_{n,k}$ satisfies (5.3).

Second step. For all n and k, set

$$B_{n,k} = C_{n,k} \setminus \Big(\bigcup_{m>n,\, h\in\overline{\mathbb{N}}} C_{m,h} \Big) \,.$$

As $B_{n,k} \subset C_{n,k} \subset D_{n,k}$ for all n, k, and taking into account that for every n the collection $\{D_{n,k} : k \in \overline{\mathbb{N}}\}$ is a partition of Ω, we obtain that $B_{n,k} \cap B_{n',k'} = \emptyset$ whenever $(n,k) \neq (n',k')$. (5.2), (5.3) yield

$$|B_{n,k}| \geq (1-2^{-n})|C_{n,k}| \geq \frac{1}{2}|C_{n,k}| = 2^{-(n+1)}|D_{n,k}| \,,$$

and then, recalling (5.1) and $\int_\Omega f_n dx = \infty$, we obtain that for all n

$$\sum_k \int_{B_{n,k}} f_n dt \geq \sum_{0<k\leq\infty} (k-1)\,|B_{n,k}| \geq 2^{-(n+1)} \sum_{0<k\leq\infty} (k-1)\,|D_{n,k}| \geq 2^{-(n+1)} \int_\Omega (f_n - 1) dt = \infty \,.$$

Hence it is enough to set $A_n = \bigcup_{0<k\leq\infty} B_{n,k}$ for all n. ■

Covering Lemma and Applications

To begin with, we recall two corollaries of the the well–known Besicovitch covering lemma (see for instance SIMON, FEDERER or MORGAN).

THEOREM 5.4 (cf. MORGAN, Theorem 2.7). – *Suppose λ is a finite positive measure on $\mathbb{R}^N$, E is a Borel subset of Ω and $\mathscr{F}$ is a collection of non trivial closed balls such that* $\inf\{r : \overline{B(x,r)} \in \mathscr{F}\} = 0$ *for all $x \in E$. Then for every $\varepsilon > 0$ there exists a finite disjoint collection $\mathscr{F}' \subset \mathscr{F}$ such that*

$$\lambda\Big(E \setminus \bigcup_{\overline{B}\in\mathscr{F}'} \overline{B}\Big) < \varepsilon \,.$$

REMARK 5.5. – Notice that Theorem 5.4 holds even if $\mathscr{F}$ is a collection of non trivial open balls B such that $\lambda(\partial B) = 0$. In particular this happens when λ is (absolutely continuous with respect to) Lebesgue measure and in this case we often apply Theorem 5.4 to collections of open balls.

THEOREM 5.6 (cf. SIMON, Theorem 4.7). – *Suppose that λ_1, λ_2 are (locally) finite positive measures on $\mathbb{R}^N$. Then*

$$\frac{d\lambda_2}{d\lambda_1}(x) = \lim_{r\to 0} \frac{\lambda_2(B(x,r))}{\lambda_1(B(x,r))}$$

exists except for a λ_1-negligible Borel set of points and is a Borel function of x which represents the component of λ_2 which is absolutely continuous with respect to λ_1 in the Lebesgue decomposition of λ_2 relative to λ_1.

This theorem has a straightforward corollary.

COROLLARY 5.7. – *If λ is a finite measure on $\mathbb{R}^N$, then we have that the limit*

$$\lim_{r\to 0} \frac{r^N}{\lambda(B(x,r))}$$

exists and is finite except a λ–negligible Borel set of points x, in particular it is λ–a.e. 0 when λ and $\mathscr{L}_N$ are mutually singular.

THEOREM 5.8. – *Let λ be a positive finite measure on $\mathbb{R}^N$. Then there exists a Borel set E such that $\lambda(\mathbb{R}^N \setminus E) = 0$ and, for all $x \in E$,*

$$\limsup_{r\to 0} \frac{\lambda(B(x,ar))}{\lambda(B(x,r))} \geq a^N \quad \textit{for all } a \textit{ with } 0 < a < 1. \tag{5.5}$$

PROOF. – Let x be a point such that (5.5) does not hold for some a with $0 < a < 1$, then there exist positive real numbers $b < a^N$ and $\bar r > 0$ such that

$$\frac{\lambda(B(x,ar))}{\lambda(B(x,r))} \leq b < a^N \quad \text{for all } r \text{ with } 0 < r \leq \bar r.$$

Hence, for all integers $m > 0$,

$$\frac{\lambda\big(B(x,a^m\bar r)\big)}{\lambda\big(B(x,\bar r)\big)} = \prod_{i=1}^m \frac{\lambda\big(B(x,a^i\bar r)\big)}{\lambda\big(B(x,a^{i-1}\bar r)\big)} \leq b^m$$

and then, if we set $r = a^m \bar r$, we have

$$\lambda(B(x,r)) \leq \lambda(B(x,\bar r))b^m \leq \lambda(B(x,\bar r))b^{\frac{\log(r/\bar r)}{\log a}} = M\, r^{\frac{\log b}{\log a}} \tag{5.6}$$

where $M = \lambda(B(x,\bar r))\bar r^{-\frac{\log b}{\log a}}$. Notice that $b < a^N$ and $a < 1$ yield $\frac{\log b}{\log a} > N$ and then, by (5.6),

$$\limsup_{r\to 0} \frac{r^N}{\lambda(B(x,r))} \geq \lim_{r\to 0} \frac{r^N}{M\, r^{\frac{\log b}{\log a}}} = \infty$$

and Corollary 5.7 shows that this may happen for a λ–negligible (Borel) set of points x only. ■

Glueing Lemma and Applications

PROPOSITION 5.9 (Poincaré–Wirtinger Inequality). – *Let B be an open ball in $\mathbb{R}^N$ with radius r and let u be a function in $W^{k,p}(B,\mathbb{R}^M)$ such that the mean value of $D^h u$ on B is 0 for $h=0,\dots,N-1$. Then*

$$\|D^h u\|_{L^p(B)} \le (Cr)^{k-h}\|Du^k\|_{L^p(B)} \quad \text{for } h=0,\dots,k-1 \tag{5.7}$$

where C is a constant which depends on N and p only.

PROOF. – The case $k=1$, $h=0$ is well–known and the general case follows by iteration. ∎

LEMMA 5.10 (Glueing Lemma). – *Let X be a local subspace of $W^{k,p}(\Omega,\mathbb{R}^M)$, $1\le p\le\infty$. For $i=1,\dots,n$, let be given pairwise disjoint open balls $B_i=B(x_i,r_i)$ and functions $u_i\in X$. Then, for every $\varepsilon>0$ there exists $u\in X$ such that*

(i) *$u=0$ a.e. out of the union of all B_i.*

(ii) *$D^k u = D^k u_i$ a.e. in $B\big(x_i,(1-\varepsilon)r_i\big)$ for all i.*

(iii) *$\|D^k u\|_{L^p(B_i)} \le C\varepsilon^{-k}\|D^k u_i\|_{L^p(B_i)}$ for all i, where C is a constant which depends on N, k and p only.*

PROOF. – We may suppose that $\varepsilon<1$. It is well–known that for all i we may find a polynomial function $p_i:\mathbb{R}^N\to\mathbb{R}^M$ with $\deg p_i<k$, such that $D^h(u_i-p_i)$ has mean value 0 on B_i for $h=0,\dots,k-1$. Hence, possibly replacing each u_i with u_i-p_i (cf. Definition 0.2), we may assume that $D^h u_i$ has mean value 0 on B_i for $h=0,\dots,k-1$.

An easy computation shows that for all i there exist functions $\phi_i:\mathbb{R}^N\to\mathbb{R}$ of class C^∞ such that $\phi_i=1$ in $B\big(x_i,(1-\varepsilon)r_i\big)$, $\phi_i=0$ out of $B(x_i,r_i)$ and

$$\|D^h\phi_i\|_\infty \le C'(r_i\varepsilon)^{-h} \quad \text{for } h=0,\dots,k \tag{5.8}$$

where C' is a constant which depends on N and k only. Set

$$u=\sum_i \phi_i u_i\ . \tag{5.9}$$

u belong to X because X is a local subspace (cf. Definition 0.2); the choice of ϕ_i and the fact that the balls B_i are mutually disjoint yield (i) and (ii). By definition (5.9), for all $\mathbf{a}$ with $|\mathbf{a}|=k$,

$$D^{\mathbf{a}}u=\sum_{\mathbf{b}+\mathbf{c}=\mathbf{a}} D^{\mathbf{b}}\phi_i\, D^{\mathbf{c}}u_i \quad \text{in each } B_i$$

and then, taking into account (5.7), (5.8) and $\varepsilon<1$,

$$\begin{aligned}\|D^{\mathbf{a}}u\|_{L^p(B_i)} &\le \sum_{\mathbf{b}+\mathbf{c}=\mathbf{a}} \|D^{\mathbf{b}}\phi_i\|_\infty \|D^{\mathbf{c}}v_i\|_{L^p(B_i)}\\ &\le \Big[\sum_{\mathbf{b}+\mathbf{c}=\mathbf{a}} C'(r_i\varepsilon)^{-|\mathbf{b}|}(Cr_i)^{|\mathbf{b}|}\Big]\|D^k u_i\|_{L^p(B_i)}\\ &\le \Big[\sum_{\mathbf{b}+\mathbf{c}=\mathbf{a}} C'C^{|\mathbf{b}|}\Big]\,\varepsilon^{-k}\|D^k u_i\|_{L^p(B_i)}\ .\end{aligned}$$

and this proves (iii) with a suitably chosen constant. ∎

We apply Glueing Lemma in Theorem 5.12 to prove a generalization of a Lusin type theorem which may be found in ALBERTI. The proof is a slight modification of the argument used in that paper.

LEMMA 5.11. – *Let k be a positive integer and λ be a positive finite measure on the open set $\Omega\subset\mathbb{R}^N$. Suppose that $v:\Omega\to(R^M)^{I(k)}$ is a continuous function and η and ε are positive real numbers. Then there exists a compact set $K\subset\Omega$ and a function $u\in C_0^k(\Omega)$ such that*

(i) *$|D^k u - v|\le\eta$ in K and $\lambda(\Omega\setminus K)\le\lambda(\Omega)\,\varepsilon$*

(ii) *$\|D^k u\|_\infty \le C\varepsilon^{-k}\|v\|_\infty$ where C is a constant which depends on N and k only.*

PROOF. – For all $x\in\Omega$, there exists a a positive real number r_x such that for all r, $0<r<r_x$, $B(x,r)$ is relatively compact in Ω and

$$|v(t)-v(x)|\le\eta \quad \text{for all } t\in B(x,r). \tag{5.10}$$

By Theorem 5.8, for λ–almost all x,

$$\limsup_{r\to0}\frac{\lambda(B(x,ar))}{\lambda(B(x,r))}\ge a^N \quad \text{for all } a \text{ with } 0<a<1. \tag{5.11}$$

Let $\mathscr{F}$ be the collection of all closed balls $\overline{B}$ with center $x\in\Omega$ and radius r, $0<r<r_x$, such that

$$\frac{\lambda\big[B\big(x,(1-\delta)r\big)\big]}{\lambda\big[\overline{B(x,r)}\big]}\ge\frac{\lambda\big[B\big(x,(1-\delta)r\big)\big]}{\lambda\big[B\big(x,(1-\delta)^{-1}r\big)\big]}\ge(1-\delta)^{3N}\ . \tag{5.12}$$

Let $\delta=\varepsilon/(1+3N)$, notice that, for almost all x, (5.11) yields closed balls $\overline{B}\in\mathscr{F}$ with center x and arbitrary small radius. Hence we may apply Theorem 5.4 to obtain closed balls $\overline{B(x_i,r_i)}\in\mathscr{F}$ for $i=1,\dots,n$ such that

$$\lambda\big(\Omega\setminus\cup\overline{B}_i\big)\le\lambda(\Omega)\,\delta\ .$$

For all i let $v_i\in(R^M)^{I(k)}$ be the mean value of v in B_i and let $u_i:\mathbb{R}^N\to\mathbb{R}^M$ be the polynomial function

$$u_i(t)=\sum_{\mathbf{a}\in I(k)}(v_i)_{\mathbf{a}}(t-x_i)^{\mathbf{a}}\ .$$

Let $X=W^{k,\infty}(\Omega,\mathbb{R}^M)\bigcap C^k(\Omega,\mathbb{R}^M)$. As X is a local subspace of $W^{k,\infty}$, we may apply Glueing Lemma 5.10 to obtain a function $u\in X$ such that

(a) $u=0$ out of the union of all B_i.

(b) $D^k u = D^k u_i = v_i$ in $B\big(x_i,(1-\delta)r_i\big)$ for all i.

(*c*) $\|D^k u\|_{L^\infty(B_i)} \le C\delta^{-k}\|D^k u_i\|_{L^\infty(B_i)}$ for all i.

(*a*) and the fact that every ball B_i is relatively compact in Ω implies that u has compact support in Ω and then $u \in C_0^k(\Omega, \mathbb{R}^M)$. Set

$$K = \bigcup_i \overline{B\big(x_i, (1-\delta)r_i\big)} \ .$$

Taking into account definition of v_i, (*b*) and (5.10) yield

$$|D^k u(t) - v(t)| \le |v_i - v(t)| \le \eta \quad \text{for all } i \text{ and all } t \in \overline{B\big(x_i, (1-\delta)r_i\big)},$$

moreover, taking into account (5.12) and the choice of B_i and recalling that $\varepsilon = (1+3N)\delta$,

$$\begin{aligned}\lambda(\Omega \setminus K) &\le \lambda\Big[\Omega\setminus\bigcup_i \overline{B(x_i,r_i)}\Big] + \sum_i \lambda\Big[\overline{B(x_i,r_i)}\setminus B\big(x_i,(1-\delta)r_i\big)\Big] \\ &\le \lambda(\Omega)\,\delta + \big[1-(1-\delta)^{3N}\big]\sum_i \lambda(B_i) \le (1+3N)\delta\,\lambda(\Omega) = \varepsilon\,\lambda(\Omega)\end{aligned}$$

and then (i) holds. Applying (*c*) and (*a*) and taking into account definition of v_i, we obtain

$$\begin{aligned}\|D^k u\|_\infty &= \sup_i \|D^k u\|_{L^\infty(B_i)} \le \sup_i C\delta^{-k}\|D^k u_i\|_\infty \\ &\le C(1+3N)^k\varepsilon^{-k}\sup_i |v_i| \le C(1+3N)^k\varepsilon^{-k}\|v\|_\infty\end{aligned}$$

and (ii) holds with a suitably chosen constant. ■

THEOREM 5.12. – *Let k be a positive integer and let λ be a positive finite measure on Ω. Suppose that $v : \Omega \to (R^M)^{I(k)}$ is a continuous function and let ε be a positive real number. Then there exists a compact set $K \subset \Omega$ and a function $u \in C_0^k(\Omega, \mathbb{R}^M)$ such that*

(i) $D^k u = v$ *everywhere in K and* $\lambda(\Omega \setminus K) \le \lambda(\Omega)\varepsilon$

(ii) $\|D^k u\|_\infty \le C\varepsilon^{-k}\|v\|_\infty$ *where C is a constant which depends on N and k only.*

PROOF. – by induction on n we build a sequence $\{K_n, u_n, v_n\}$ as follows: set $K_0 = \Omega$, $u_0 = 0$ and $v_0 = v$.

Let $n > 0$ and let K_{n-1}, u_{n-1} and v_{n-1} be given. Apply Lemma 5.11 to obtain a compact set $K_n \subset \Omega$ and a function $u_n \in C_0^k(\Omega)$ such that

(*a*) $|v_{n-1} - D^k u_n| \le \|v\|_\infty 4^{-nk}$ in K_n and $\lambda(\Omega \setminus K_n) \le \lambda(\Omega)2^{-n}\varepsilon$,

(*b*) $\|D^k u_n\|_\infty \le C2^{nk}\varepsilon^{-k}\|v_{n-1}\|_\infty$.

Define $v_n(x) = v_{n-1}(x) - D^k u_n(x)$ for all $x \in K_n$ and apply Titze's lemma to extend v_n to the whole Ω so that

$$\sup_{x\in\Omega} |v_n(x)| = \sup_{x\in K_n} |v_n(x)| \le \|v\|_\infty 4^{-nk} \ . \tag{5.13}$$

We set $K = \cap K_n$, $u = \sum u_n$ and then we show that these definitions make sense and satisfy (i) and (ii).

Taking into account (*a*), (*b*) and (5.13),

$$\begin{aligned}\sum_1^\infty \|D^k u_n\|_\infty &\le \sum_1^\infty C2^{nk}\varepsilon^{-k}\|v_{n-1}\|_\infty \\ &\le C2^k\varepsilon^{-k}\Big[\|v\|_\infty + \sum_1^\infty 2^{nk}\|v\|_\infty 4^{-nk}\Big] \le C2^{k+1}\varepsilon^{-k}\|v\|_\infty\end{aligned}$$

Poincaré inequality (cf. BREZIS, ch.IX) shows that the series $\sum_n u_n$ converges in the norm of $C_0^k(\Omega, \mathbb{R}^M)$ to a function u which satisfies (ii) with a suitably chosen constant.

(*a*) yields

$$\lambda(\Omega \setminus K) \le \sum_1^\infty \lambda(\Omega \setminus K_n) \le \sum_1^\infty \lambda(\Omega)2^{-n}\varepsilon = |\Omega|\,\varepsilon \ ,$$

By the definition of v_n we have that, for all $x \in K$ and all integer $m > 1$, $v(x) - \sum_1^{m-1} D^k u_n(x) = v_m(x)$ and then (*b*) yields

$$|v(x) - D^k u(x)| \le |v_m(x)| \ + \ \sum_m^\infty |D^k u_n(x)| \le \|v\|_\infty 4^{-mk} + \sum_m^\infty \|D^k u_n\|_\infty \ .$$

Hence (i) immediately follows because the sequence $m \mapsto \sum_m^\infty \|D^k u_n\|_\infty$ converge to 0 . ■

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