

Many-valued logic and its interactions with ordered algebraic structures, games, probability and substructural logics

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Abstract.

While the most convenient logic for mathematics is probably classical logic, or perhaps intuitionistic logic, many-valued logic is more appropriate in order to describe concrete situations. In particular, many-valued logic is the best framework for the treatment of imprecise or uncertain data. It offers approximate solutions of problems whose precise solution would be extremely complicate. However, many-valued logic has also a very nice mathematics: for instance, the algebraic semantics of many-valued logic is strictly related to the theory of lattice ordered (abelian) groups, and the proof theory for Lukasiewicz logic involves techniques of desingularization which arise from toric geometry.

In the first part of this talk we briefly touch many aspects of many-valued logic, namely:

- (a) Many-valued logic and algebra (MV-algebras and lattice-order abelian groups, free MV-algebras and Mc-Naughton functions, MV-algebras with product, MV-algebras with composition, etc.)
- (b) Many-valued logic and probability (probability theory as a modal many-valued logic, probability over MV-algebras, the coherence problem in subjective probability theory).
- (c) Many-valued logic and Ulam game (the treatment of this topic will be very short, as Daniele Mundici will explain better some recent developments of this subject).

In the second part of this talk, we will discuss in more details the work of the group of Siena (excluding Panti's work, which will be presented by himself), with reference to the following topics:

- (d) BL-algebras. According to Hjek's book 'Metamathematics of Fuzzy Logic', many-valued logics are presented as logics of continuous t-norms (a continuous t-norm, the semantic interpretation of conjunction, is a continuous

binary operation on $[0,1]$ which makes it a commutative and integral ordered monoid; a continuous t-norm \star has a residual \rightarrow (semantic interpretation of implication) which satisfies $a \leq b \rightarrow c$ iff $a \star b \leq c$. BL-algebras form a variety which is generated by all structures of continuous t-norms and their residuals. The most recent results deal with an algebraic investigation of interpolation and Beth's property in BL.

- (e) Many-valued logics and substructural logics. The weak systems of many-valued logic arise both as generalizations of the most common many-valued logics as Lukasiewicz, Goedel-Dummett (Skolem?) and product logics, as well as specializations of substructural logics like the Full Lambek Calculus. It turns out that the weakest reasonable commutative many-valued logic is roughly the multiplicative fragment of intuitionistic multiplicative linear logic (without exponentials) with an extra structural rule called 'communication rule'. We are investigating the following problems: (i) which weak many-valued logics are complete with respect to residuated lattices over the reals? and (ii) investigate many-valued logics with a modality (the goal is to obtain interpretation of a many-valued logic in another one with modality, like e.g. the interpretation of intuitionism in S4 or in Linear Logic).