CHARACTERIZATION OF HOLDER AND Zygmund CLASSES AS INTERPOLATION SPACES

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0. INTRODUCTION

In this paper we are concerned with the characterization of certain interpolation spaces between the domain of a second-order strongly elliptic operator, with a regular oblique derivative boundary condition, and the space of continuous functions in which the domain is embedded.

The interpolation spaces considered here can be abstractly defined as follows: let $E$ be a Banach space, and let $A:D(A)\subseteq E$ be a closed linear operator which generates a bounded semi-group $e^{tA}$ in $E$; for each $\varepsilon \geq 0$, $\eta$ set

$$D_\varepsilon(\eta):=\{x \in D(A): \sup_{t>0} t^{-\varepsilon} \|e^{tA}x\|_E < \eta\},$$

$$D_\varepsilon(\eta):=\{x \in D(A): i \lim_{t \to 0^+} t^{-\varepsilon} i e^{tA}x = 0\}.$$

Although this definition seems to depend on the semi-group $e^{tA}$, it can be shown that in fact the spaces $D_\varepsilon(\eta)$ and $D_\varepsilon(\eta)$ depend only on $E$ and the domain $D(A)$; actually $D_\varepsilon(\eta)$ coincides with Lions' interpolation space $(D(A),E),_{-\varepsilon,\eta}$ (see Lions [10], Lions-Peetre [11]), whereas $D_\varepsilon(\eta)$ is the so-called "continuous interpolation space" $(D(A),E),_{-\varepsilon,\eta}$ introduced by Da Prato-Grisvard [12].

The spaces $D_\varepsilon(\eta)$ and $D_\varepsilon(\eta)$ have been recently studied by several authors in connection with the theory of abstract evolution equations: see Da Prato-Grisvard [12], Arlotto-Riccucci [5], Da Prato-Sinestrari [9], Sinestrari [13, 17], Lunardi [12, 13, 14, 15], Acquistapace-Terreni [1, 2].

An important feature of these spaces is their "maximal regularity" property. Maximal regularity means the following: if $f$ is continuous with values in a Banach space $Y$, then
the evolution problem
\[
\begin{cases}
u' - Au = f \\
u(0) = 0
\end{cases}
\]
has a unique $C^1$ solution $u$ such that $u'$ and $Au$ are continuous with values in $Y$. This property is not true for every Banach space $Y$ (see Bailon [6], Trava [22]), but it holds when $Y = D_{A}^{(8)}$, where $A$ is the infinitesimal generator of an analytic semi-group in some other Banach space $E$; note that we cannot replace here $D_{A}^{(8)}$ by $D_{A}^{(8)=}$ (see Da Prato-Grisvard [8]).

However, a similar property is true for $D_{A}^{(8),=}$ (with $A$ as before); namely, if $f$ is continuous with values in $E$ and bounded with values in $D_{A}^{(8),=}$, then the same is true for $u'$ and $Au$. For a proof of these facts see Sinestrari [19].

Our characterization of the spaces $D_{A}^{(8)}$ and $D_{A}^{(8),=}$ concerns the case in which $E = C^{0}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded connected open set with smooth boundary, and $A$ is a second-order strongly elliptic operator with a regular first-order differential condition at the boundary. In this situation, $A$ is the infinitesimal generator of an analytic semi-group in $E$ (see Stewart [20]). Under Stewart's assumptions we prove that $D_{A}^{(8),=}$ (resp. $D_{A}^{(8)}$) coincides with the space of $2\alpha$-Hölder (resp. $2\alpha$-little Hölder) continuous functions if $0 \leq \alpha \leq 1/2$, and with the space of differentiable functions which satisfy the boundary condition and whose gradient is $2(\alpha-1)$-Hölder (resp. $2(\alpha-1)$-little Hölder) continuous if $0 \leq \alpha \leq 1/2$. We recall that a function $f$ is $\alpha$-Hölder (resp. $\alpha$-little Hölder) continuous, $0 \leq \alpha \leq 1$, if
\[
|f(x) - f(y)| = o(|x-y|^\alpha) \quad \text{(resp. } |f(x) - f(y)| = O(|x-y|^\alpha)\text{)}
\]
as $|x-y| \to 0$.

When $\alpha = 1/2$, we obtain as $D_{A}^{(8),=}$ (resp. $D_{A}^{(8)}$) the so-called Zygmund classes, i.e. the spaces of continuous functions verifying
\[
|f(x) - f(y)| = o(|x-y|^\alpha) \quad \text{and } |x-y| = o(|x-y|^\alpha)
\]
as $|x-y| \to 0$.

and satisfying an additional property along the boundary, which is in some sense a weak form of the boundary differential condition.

We prove in addition that the interpolation spaces do not change if we replace $D_{A}$ by the (smaller) space $C_{B}^{0}(\Omega)$ of twice continuously differentiable functions which satisfy the boundary condition; in other words, we show that
\[
D_{A}^{(8),=}(\Omega) = C_{B}^{0}(\Omega)
\]
and
\[
D_{A}^{(8)}(\Omega) = C_{B}^{0}(\Omega)
\]

The results which we prove here are already known (except for the case $\alpha = 1/2$) in dimension $n = 1$, with $\Omega = (a, b)$ (see Rosati-Sangare-Terrani [3], Da Prato-Grisvard [8]); in this case (0.1) is obvious since $D_{A} = C_{B}^{0}(\Omega)$.

The characterization of $D_{A}^{(8),=}$ and $D_{A}^{(8)}$ in the case of boundary conditions of Dirichlet type has been given by Lunardi [13]; she obtains the Hölder and little Hölder spaces if $\alpha = 1/2$, and the Zygmund classes if $\alpha = 1/2$, with the additional requirement to the functions, in either case, to vanish along $\partial \Omega$. Lunardi's result can be found again by our method, which also allows slightly weaker assumptions about the smoothness of $\Omega$ and of the coefficients of the differential operator $A$.

Let us describe now the subject of the next sections. In section 1 we list our definitions and assumptions, and state some preliminary results to be used later on; among these ones, we mention some properties of the Zygmund spaces which
do not seem to be completely straightforward. Section 2, 3, 4 are concerned with the proof of the inclusions
\[ X_0^0 \subset (C^0(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta, \alpha}, \quad Y_0^0 \subset (C^0(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta, \alpha}, \]
where \( X_0^0 \) and \( Y_0^0 \) symbolize the concrete function spaces which we will characterize as \( D_A(\theta, \alpha), D_A(\theta) \) respectively. More precisely, in Section 2 we consider the case of the half-space \( \mathbb{R}^n_+ \), with a boundary condition \( \nu u = 0 \) whose principal part is the derivative with respect to the normal to \( \partial \Omega \); Section 3 still concerns the case \( \mathbb{R}^n_+ \), with a boundary condition \( \nu u = 0 \) with general principal part (i.e. a derivative along a non-tangential direction with respect to \( \partial \Omega \)); finally in Section 4 we treat the general case in a bounded connected open set with \( C^2 \) boundary. In Section 5 we consider the reverse inclusions, i.e. the inclusions
\[ D_A(\theta, \alpha) \subset X_0^0, \quad D_A(\theta) \subset Y_0^0, \]
where \( X_0^0 \) and \( Y_0^0 \) have the same meaning as before. Finally, in Section 6 we draw the conclusions, stating our main theorems.

1. PRELIMINARIES

Let us list some notations and definitions. Let \( \Omega \) be a (possibly unbounded) connected open set of \( \mathbb{R}^n \), \( n \geq 1 \); further assumptions on \( \Omega \) will be specified when necessary. We denote by \( \overline{\Omega} \) the closure of \( \Omega \) and by \( \partial \Omega \) its boundary.

**Definition 1.1.** For each \( \Omega \in \mathbb{R}^n \) set:

(i) \( C^0(\overline{\Omega}) = \{ f : \overline{\Omega} \to \mathbb{C} \text{ uniformly continuous and bounded in } \overline{\Omega} \} \),

(ii) \( C^{0,0}(\overline{\Omega}) = \{ f : \overline{\Omega} \to \mathbb{C} \text{ continuous}\} \),

(iii) \( C^{1,0}(\overline{\Omega}) = \{ f : \overline{\Omega} \to \mathbb{C} \text{ continuous and } f_1, f_2 \text{ continuous}\} \),

(iv) \( C^{2,0}(\overline{\Omega}) = \{ f : \overline{\Omega} \to \mathbb{C} \text{ continuous and } f_1, f_2, f_3 \text{ continuous}\} \),

(v) \( C^{\infty}(\overline{\Omega}) = \{ f : \overline{\Omega} \to \mathbb{C} \text{ continuous and } f_1, f_2, f_3, \ldots \text{ continuous}\} \).

Again, let \( \mathbb{R}^n_{k} \) be any set; define for each \( \theta \in (0, 2] \)

\[ C^0(\overline{\mathbb{R}^n_{k}}) = \sup \{|f(x)|, x \in \mathbb{R}^n_{k}\}. \]

We will also denote the norm \( \| f \|_{C^0(\overline{\Omega})} \) by \( \| f \|_{C^0(\overline{\Omega})} \) or simply \( C^0(\overline{\Omega}) \) when no confusion can arise.

For each \( x_0 \in \mathbb{R}^n_{k} \) and \( r > 0 \) define \( \Omega_0(x, \mathbb{R}^n_{k}) = \{ x \in \mathbb{R}^n_{k} : |x - x_0| < r \} \) if \( x \in \mathbb{R}^n_{k} \) is any set, define also for each \( \theta \in (0, 1) \):

\[ [f]_{\theta} = \sup_{x, y \in \mathbb{R}^n_{k}} \frac{|f(x) - f(y)|}{|x - y|^{\theta}}. \]

When \( \mathbb{R}^n_{k} \) we simply write \( [f] \) instead of \( [f]_{0} \).

**Definition 1.2.** For each \( \theta \in (0, 1) \) and \( \mathbb{R}^n_{k} \setminus \Omega \) set:

(i) \( C^{\theta,0}(\overline{\Omega}) = \{ f : \mathbb{R}^n_{k} \to \mathbb{C} \text{ continuous}\} \),

(ii) \( C^{\theta,0}(\overline{\mathbb{R}^n_{k}}) = \{ f : \mathbb{R}^n_{k} \to \mathbb{C} \text{ continuous}\} \),

(iii) \( C^{\theta,0}(\overline{\mathbb{R}^n_{k}}) = \{ f : \mathbb{R}^n_{k} \to \mathbb{C} \text{ continuous}\} \),

(iv) \( C^{\infty}(\overline{\mathbb{R}^n_{k}}) = \{ f : \mathbb{R}^n_{k} \to \mathbb{C} \text{ continuous}\} \).

Again, let \( \mathbb{R}^n_{k} \) be any set; define for each \( \theta \in (0, 2] \)
\[ f \star_{\theta, \mu} = \sup \left( \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|}{|x-y|^\theta} \right)_{x \neq y} \leq \mu, x \neq y \]  

(1.2)

when \( \mu = 0 \) we write \( f \star_\theta \) for \( f \star_{\theta, \mu} \).

**Definition 1.3.** For each \( \varepsilon \in (0, 1) \) set:

1. \( C^{\varepsilon, 0}(\tilde{\Omega}) = \{ f \in C^0(\tilde{\Omega}) : f \star_{\varepsilon, 0} = 0 \} \).

(1.3)

2. \( h^{\varepsilon, 0}(\tilde{\Omega}) = \{ f \in C^{\varepsilon, 0}(\tilde{\Omega}) : \lim_{r \to 0^+} \sup_{x \in \tilde{\Omega}} [f(x) - f(\tilde{x}, \frac{x+y}{2}) = 0] \} \).

Remark 1.4 (i) It is clear that for \( \varepsilon \in (0, 1) \) and \( x = 0, 1, 2, \ldots \)

(\ldots)\)

\( h^{\varepsilon, 0}(\tilde{\Omega}) \) is a closed subspace of \( C^{\varepsilon, 0}(\tilde{\Omega}) \). Similarly \( h^{\varepsilon, 0}(\tilde{\Omega}) \) is a closed subspace of \( C^{\varepsilon, 0}(\tilde{\Omega}) \) for each \( \varepsilon \in (0, 1) \).

(ii) The spaces \( C^{\varepsilon, 0}(\tilde{\Omega}), h^{\varepsilon, 0}(\tilde{\Omega}) \) have been studied by Szegö [24], [25], in the one-dimensional case (see also Butzer-Sorensen [27]). A wide description of the case \( \varepsilon = 1 \) can be found in Stein [19]; see also Taibleson [21]. It is well-known that

\[
\begin{align*}
C^{\varepsilon, 0}(\tilde{\Omega}) &= C^{\varepsilon, 1}(\tilde{\Omega}), \quad h^{\varepsilon, 0}(\tilde{\Omega}) = h^{\varepsilon, 1}(\tilde{\Omega}) \quad \text{whenever } \varepsilon \in (0, 1), \\
h^{\varepsilon, 0}(\tilde{\Omega}) &= h^{\varepsilon, 1}(\tilde{\Omega}) \quad \text{whenever } \varepsilon \in (0, 1),
\end{align*}
\]

with equivalence of norms; on the other hand we have the proper (continuous) inclusions

\[
\begin{align*}
C^{\varepsilon, 1}(\tilde{\Omega}) &\subset C^{\varepsilon, 1}(\tilde{\Omega}) \subset h^{\varepsilon, 0}(\tilde{\Omega}), \\
h^{\varepsilon, 1}(\tilde{\Omega}) &\subset h^{\varepsilon, 1}(\tilde{\Omega}), \\
h^{\varepsilon, 1}(\tilde{\Omega}) &\subset h^{\varepsilon, 2}(\tilde{\Omega}) \subset h^{\varepsilon, 1}(\tilde{\Omega}), \quad \varepsilon \in (0, 1),
\end{align*}
\]

whereas \( h^{\varepsilon, 0}(\tilde{\Omega}) \) = (in constants), \( h^{\varepsilon, 2}(\tilde{\Omega}) \) = (affine functions) for bounded \( \tilde{\Omega} \). For all these results see Szegö [24], [25] in the

**Remark 1.5.** Let \( Y \) be a Banach space and let \( X \) be any of the symbols \( C^k, C^{k,0}, h^k, h^{k,0} \). Definitions 1.1, 1.2 and 1.3 can be obviously adapted to the case of functions \( f \in \mathbb{R} \); in this case we denote the corresponding space by \( X(\tilde{\Omega}, Y) \). However if \( \mathbb{R} = X(\tilde{\Omega}, Y) \) instead of \( X(\tilde{\Omega}, Y) \), provided no confusion can arise.

In particular, suppose \( \mathbb{R} C^{k,0}(\tilde{\Omega}), h^{k,0} \), and denote by \( d^s f(x) \) the \( s \)-th order gradient of \( f \) at \( x \), i.e., the \( (\varepsilon\alpha^s) \)-vector whose components are \( (d^s f(x)) \); then we have

\[
\begin{align*}
d^s f(x) &= |\alpha| = s
\end{align*}
\]

with simplicity of notation or simply \( d^s f(x) \).

**Definition 1.6.** Let \( Y \) be a Banach space. For each \( \varepsilon \in (0, 1) \) set

\[
\begin{align*}
C^\varepsilon(1, 0, 1) &= C(1, 0, 1) = \{ f : f(t) = t^\varepsilon \} \text{ is continuous and bounded in } [0, 1] \\
\mathbb{R}C^\varepsilon(1, 0, 1) &= \sup_{t \in [0, 1]} t^\varepsilon \text{ in } [0, 1],
\end{align*}
\]

(1.4)

(11) \( C^\varepsilon((0, 1), Y) = \mathbb{R}C^\varepsilon((0, 1), Y), Y = t^\varepsilon \) \( \mathbb{R}C^\varepsilon(0, 1, Y) \).

Clearly \( C^\varepsilon((0, 1), Y) \) is a closed subspace of \( C^\varepsilon((0, 1), Y) \); note that \( C^\varepsilon(0, 1, Y) = C^0(0, 1, Y) \).
Now we recall the definition of Lions' interpolation spaces.

**Definition 1.7.** Let \( Y, Z \) be Banach spaces with \( Y \in \mathbb{K} \) (continuous inclusion), and let \( 0 \leq \theta \leq 1 \).

(i) We say that \( x \in (Y, Z)_{\theta, \infty} \) if there exists \( u \in C^0([0, 1], Z) \) such that \( u(0) = x \) and moreover,

\[
\|u\|_{C^0([0, 1], Z)} \leq M
\]

where \( x \in (Y, Z)_{\theta, \infty} \).

(ii) We say that \( x \in (Y, Z)_{\theta} \) if there exists \( u \in C^0([0, 1], Z) \) such that \( u(0) = x \) and moreover,

\[
\|u\|_{C^0([0, 1], Z)} < \infty
\]

Remark 1.8. Clearly \( (Y, Z)_{\theta, \infty} \) is a closed subspace of \( (Y, Z)_{\theta} \) and in fact it coincides with the closure of \( Y \) in the norm of \( (Y, Z)_{\theta} \); a proof is in Sinestrari [16]. More details about the spaces \( (Y, Z)_{\theta, \infty} \) can be found in Lions-Peetre [11], Serrano-Beranes [7], Triebel [23]; for the spaces \( (Y, Z)_{\theta} \) see Da Prato-Grisvard [8].

**Remark 1.9.** Let \( A \) be the infinitesimal generator of an analytic semi-group \( e^{tA} \) (possibly not strongly continuous at \( t = 0 \)) on the Banach space \( Z \). Thus, in particular, the resolvent set of \( A \) contains a sector \( \{\omega \in \mathbb{C} : \arg \omega < \eta \} \) with \( \eta = \pi/2, \pi \}, \) and there exists \( M > 0 \) such that

\[
\|e^{tA}x\|_Z \leq M \|x\|_Z \quad \text{for all } x \in Z, t > 0.
\]
All these properties are proved in Sinestrari [17].

Now we list a series of auxiliary results which will be needed in the following sections.

**Lemma 3.9.** Suppose \( \Omega \) is bounded with \( \partial \Omega \) of class \( C^1 \) (or, alternatively, suppose \( \Omega = \mathbb{R}^n \) or \( \Omega = \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : x < 0\} \)). There exists \( \Lambda > 0 \) such that

\[
\|\text{Du}(\cdot)\|_{L^2(\Omega)} \leq \Lambda \|\text{u}\|_{L^2(\Omega)} + \frac{1}{2} \|\text{D}^2\text{u}(\cdot)\|_{L^2(\Omega)}.
\]

**Proof.** By considering separately \( \text{Re} u \) and \( \text{Im} u \), and replacing \( \Omega \) by \( 2\Omega \), we can assume that \( u \) is real-valued. Suppose first \( \Omega = \mathbb{R}^n \) or \( \Omega = \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : x < 0\} \) and fix \( \Lambda \) such that \( \text{Du}(x_0) \neq 0 \) (if \( \text{Du}(x_0) = 0 \) the result is obvious). By Taylor's formula we have, denoting by \( \langle \cdot, \cdot \rangle \) the scalar product in \( \mathbb{R}^n \) or \( \mathbb{C}^n \):

\[
u(x) = u(x_0 + h\text{Du}(x_0)) + \frac{1}{2} \text{Re} \left( \langle h, \text{D}^2\text{u}(x_0)h \rangle \right) \nu_h \text{Re} \left( \langle h, \text{D}^2\text{u}(x_0)h \rangle \right)
\]

where \( \nu \) is a suitable point in the segment joining \( x \) and \( x_0 \).

We can assume (possibly replacing \( u \) by \( -u \)) that

\[
\frac{\text{Du}(x_0)}{\|\text{Du}(x_0)\|} \in \Omega \quad \forall \nu \geq 0.
\]

Hence if we choose \( \nu = \lambda x_0 + \frac{1}{2} \text{Du}(x_0) \),

we easily get

\[
\|\text{Du}(x_0)\| \leq \frac{2}{\lambda} \|\text{u}\|_{L^2(\Omega)} + \frac{2}{\lambda} \|\text{D}^2\text{u}(\cdot)\|_{L^2(\Omega)} \quad \forall \nu > 0.
\]

The expression on the right-hand side attains its minimum value when \( \nu = 2\lambda \|\text{u}\|_{L^2(\Omega)} \). Hence

\[
\|\text{Du}(x_0)\| \leq 4 \|\text{u}\|_{L^2(\Omega)} + \frac{1}{4} \|\text{D}^2\text{u}(\cdot)\|_{L^2(\Omega)} \quad \forall \nu \in \Omega
\]

and the result follows with \( \Lambda = 2 \).

Suppose now \( \Omega \) bounded with \( \partial \Omega \) of class \( C^1 \) and fix \( x_0 \in \partial \Omega \) with \( \text{Du}(x_0) \neq 0 \). Then we can select a finite number of open balls \( \Omega_j \), \( j = 1, \ldots, k \), centered in points of \( \partial \Omega \), with the following properties:

\[
(a) \quad \partial \Omega \subseteq \bigcup_{j=1}^k \Omega_j, \quad \text{where} \quad \nu_j = \Omega_j \cap \partial \Omega,
\]

(b) for \( j = 1, \ldots, k \), there exists a diffeomorphism

\[
\nu_j : \nu_j \setminus B(0,1) = \nu(\partial \Omega) \cup \nu(x_0) : \nu(x) \in \Omega, \quad \nu_j : \nu_j \setminus B(0,1) = \nu(\partial \Omega) \cup \nu(x_0) : \nu(x) \in \Omega
\]

Define now, for \( c > 0 \),

\[
\nu = [x \in \nu : \text{dist}(x, \partial \Omega) > c], \quad A_c = \{x \in \nu : \text{dist}(x, \partial \Omega) = c\}
\]

where \( v(x) \) is the unit exterior normal vector at \( x \). Obviously in general \( A_c \) is not contained in \( \Omega \), but this is true for small \( c \); in this case \( A_c = \nu \cap B(0,c) \). Hence we can choose \( c > 0 \) such that

\[
\begin{align*}
A_{2c} = \nu & - \Omega_{2c} \subseteq \bigcup_{j=1}^k \Omega_j, \quad \nu_j = \Omega_j \cap \partial \Omega, \quad \text{and we can assume (possibly replacing \( u \) by \( -u \)) that} \\
& \left( \text{Du}(x_0) \right) \in \mathbb{R}^n \quad \forall \nu \in \nu_{2c}.
\end{align*}
\]

Define

\[
\begin{align*}
\nu = [x \in \nu : \text{dist}(x, \partial \Omega) > c], \quad A_c = \{x \in \nu : \text{dist}(x, \partial \Omega) = c\}
\end{align*}
\]

Now, if \( x \in \nu_{2c} \),

\[
\begin{align*}
[\text{Du}(x_0)] & = \left[2c^{-1} \frac{\text{Du}(x_0)}{1} + \frac{1}{2} \text{D}^2\text{u}(\cdot)\right]_v \quad \forall \nu \in \nu_{2c}.
\end{align*}
\]

On the other hand, suppose \( x \in \nu_{2c} \). Take a point \( \nu \in \nu_{2c} \), such that \( |x - \nu_{2c}| = \text{dist}(x, \partial \Omega) \), then \( v(x) = \frac{x - \nu}{|x - \nu|} \) and we can assume (possibly replacing \( u \) by \( -u \)) that \( \left( \text{Du}(x_0) \right) \in \mathbb{R}^n \) and \( \nu(x) \in \Omega_{2c} \) for all \( \nu \in \nu_{2c} \). We have:
\[ -q(x,v(x),u) - (q - 1) \frac{Du(x)}{|Du(x)|^n} n \geq \frac{1}{\sqrt{s}}. \]  

(1.8)

Set \( x = x_0 + \epsilon q \); then the segment joining \( x_0 \) and \( x \) lies in \( U \) (indeed, for each \( \epsilon \geq 0 \), \( x \) is the point \( x_0 + \epsilon q \) belongs to the ball with center \( x_0 \) and radius \( \epsilon \)). Hence by Taylor's formula and (1.8) we easily get

\[ \frac{1}{\sqrt{s}} \frac{|Du(x_0)|}{2u(x)} \leq 21u(x) + x^2 \frac{u(x^2)}{2u(x)} \quad \forall x \in \overline{U}, \]

which, together with (1.7), yields the result. \( \Box \)

**Lemma 1.11.** Let \( X \) be a Banach space and let \( \epsilon \in (0,1) \). For each \( \psi \in C^0([0,1],Y) \) such that \( \psi \in C^1([0,1],Y) \) we have:

\[ L_{\psi} = \inf_{Y \in C^0([0,1],Y)} \left\{ \int_0^1 \psi(t) \, dt \right\} \]

**Proof:** We have for each \( t \in [0,1] \)

\[ \frac{1}{\sqrt{s}} \int_0^1 \psi(t) \, dt \leq \int_0^1 \psi(t) \, dt \quad \forall \psi \in C^0([0,1],Y). \]

Let us define now some suitable subspaces of the spaces \( C^0([0,1],Y) \), \( C^1([0,1],Y) \), which we introduced in Definitions 1.1, 1.2 and 1.3.

**Definition 1.12.** Denote by \( X \) any of the symbols \( C^k, C^{k,0}, C^{k,0}_c, h^k, h^{k,1}, h^{k,1}_c \), \( k \geq 0 \), \( k = 0,1,2 \), and set

\[ X_{0,1}(\overline{U}) = \{ f \in X(U) : f(x) = 0 \quad \forall x \in \partial U \}. \]

**Definition 1.13.** Suppose \( 0 \in U \) is of class \( C^1 \) and consider the boundary differential operator

\[ B_0 = n(x) + (q(x) |Du(x)|^n) \quad \forall x \in \partial U, \]

where \( \alpha \in C^0(\partial U) \). If \( X \) is any of the symbols \( C^k, C^{k,0}, C^{k,0}_c, h^k, h^{k,1}, h^{k,1}_c \), \( k = 0,1,2 \), set

\[ X_{0,1}(\overline{U}) = \{ f \in X(U) : f(x) = 0 \quad \forall x \in \partial U \}. \]

**Definition 1.14.** Suppose \( 0 \in U \) is of class \( C^1 \), and denote by \( v(x) \) the unit exterior normal vector at \( x \). Then \( \psi \in C^0(\partial U) \).

Let \( B \) be as in Definition 1.13, and suppose in addition that \( \psi \in C^0(\partial U) \), \( \beta \in C^0(\partial U) \), \( \gamma \in C^0(\partial U) \), \( (\beta(x)) \psi(x) > 0 \forall x \in \partial U \). Define

\[ \langle f \rangle_{1,\beta} = \sup_{x \in \partial U} \frac{1}{\beta(x)} \bigg| (x) \bigg| f(x) \bigg| (x) \bigg| \quad \forall x \in \partial U, \gamma \in C^0(\partial U). \]

**Lemma 1.15.** Let \( 0 \in U \) be bounded, with \( 0 \in C^1 \), let \( \beta \in C^0(\partial U) \), \( \gamma \in C^0(\partial U) \). Then there exists \( \delta > 0 \) such that

\[ x - \delta \in \partial U \quad \forall x \in \partial U, \gamma \in C^0(\partial U). \]

Consequently in the semi-norm (1.9) we can take \( \alpha \in C^0(\partial U) \).
where $\sigma_0$ is any sufficiently small fixed positive number (independent of $x_0$).

**Proof.** As $0$ is of class $C^1$ and compact, we have $(B(x)|v(x)) < \infty$ and $\forall x \in \Omega$; in addition, $\Omega$ can be covered by a finite number of open balls $W_j$, $1 \leq j \leq k$, with the following property: there exist functions $g_j \in C^1(\overline{W_j}, \mathbb{R})$ with $|Dg_j(x)| \geq \delta_j > 0$

$\forall x \in \overline{W_j}$, such that

$$W_j \cap \Omega = \{x \in \Omega : v_j(x) > 0\}, \quad \Omega \cap \overline{W_j} = \{x \in \overline{W_j} : g_j(x) = 0\}.$$  

We can also suppose that the covering is minimal, i.e., $\bigcup_{j=1}^k \overline{W_j}$ does not contain $\Omega$ for each $s=1,...,k$. Choose $\sigma$ so small that:

(a) $x \in \Omega \cap W_j$ and $x \in W_s$, for some $j=1,..,k$; we have $x \in \overline{W_s}$ and also $x \in \partial(\Omega \cap \overline{W_j})$.

(b) $x \in \Omega \cap W_j$ and $x \in \partial(\Omega \cap \overline{W_j})$.

This is clearly possible. Now take $x \in \Omega$; then, by (a) and (b), for some $j=1,..,k$ we have $x \in \overline{W_j}$ and also $x \in \partial(\Omega \cap \overline{W_j})$. In addition, by Taylor's formula,

$$g_j(x \circ \sigma(x)) = g_j(x) - \frac{Dg_j(x)}{|Dg_j(x)|} \sigma \circ o(x) + o(\sigma) \circ o(x),$$

and, since $v(x) = -\frac{Dg_j(x)}{|Dg_j(x)|}$ and $g_j(x) = 0$, we get

$$g_j(x \circ \sigma(x)) \geq |Dg_j(x)| \{v(x) | B(x)\} \sigma \circ o(x) \circ o(x) \circ o(\sigma) \circ o(x),$$

Hence, possibly replacing $\sigma$ by a smaller number, we get

$$g_j(x \circ \sigma(x)) > 0 \quad \forall x \in \overline{W_j},$$

which implies $x \in \overline{W_j}$.

We will need later another geometric property of $\Omega$, which we express in the following lemma.

**Lemma 1.16.** Suppose $\Omega$ is bounded with $\partial \Omega$ of class $C^1$. There exist $\sigma > 0$, $N \geq 1$ satisfying the following property: if $x \in \overline{\Omega}$ and $|x-y| \leq \sigma$, there exists a continuously differentiable path $\Gamma_0(0,1) = \Omega$ such that

$$\Gamma(0) = x, \quad \Gamma(1) = y, \quad L(\Gamma) \leq N |x-y|,$$

where $L(\Gamma) = \int_0^1 |\dot{\Gamma}(t)| \, dt$ is the length of $\Gamma$.

**Proof.** As in the proofs of Lemmas 1.10 and 1.15 we have $\overline{\Omega} \supset \bigcup_{j=1}^k \overline{W_j}$, where $\overline{W_j} \supset \overline{\Omega}$ and $\overline{W_j}$ is an open ball, centered in a point of $\partial \Omega$, having the following property: there exists a diffeomorphism $\psi_j : \overline{W_j} \rightarrow \overline{B(0,1)} = \{x \in \mathbb{R}^n : |x| \leq 1\}$ of class $C^1$, such that $|D\psi_j(y) \circ \sigma_0 \circ o(x) = \overline{B(0,1)}$, and $\psi_j(W_j) = \overline{B(0,1)}$.

We can also suppose that $\bigcup_{j=1}^k \overline{W_j}$ does not contain $\partial \Omega$ for each $j=1,..,k$. Define again, for $0, \tau \in \mathbb{R}$, $\tau = \text{dist}(x, \partial \Omega) > 0$, $\tau = \text{dist}(x, \partial \Omega) > 0$, and take $\sigma > 0$ such that (1.6) holds.

Next, choose $\sigma_0 \in (0,\sigma]$ small enough, in order that the following properties are satisfied:

(a) $x \in \overline{\Omega} \supset \bigcup_{j=1}^k \overline{W_j}$ and $x \in \overline{W_s}$, for some $j=1,..,k$; we have $x \in \overline{W_s}$ and also $x \in \partial(\overline{\Omega} \cap \overline{W_j})$.

(b) $x \in \overline{\Omega} \supset \bigcup_{j=1}^k \overline{W_j}$ and $x \in \partial(\overline{\Omega} \cap \overline{W_j})$.

where $\sigma_0 \in (0,\sigma]$. Clearly this choice is possible. Take now $x, y \in \overline{\Omega}$ with $|x-y| \leq \sigma_0$; three cases can occur:

Case 1: $x, y \in \overline{\Omega}$. Then the ball $B(x, \sigma_0)$ is contained in $\overline{\Omega}$, and $y \in B(x, \sigma_0)$. Thus we can take as $\Gamma$ the segment joining $x$ and $y$. 

Lemma 1.19. Suppose $\Omega$ is bounded with $\partial \Omega$ of class $C^1$ (or, alternatively, suppose $\Omega$ is bounded and convex). Let $f, g \in C^1(\bar{\Omega})$ (resp. $h^{1,1}(\bar{\Omega})$) and $g \in C^1(\bar{\Omega})$ (resp. $h^{1,1}(\bar{\Omega})$) where $h^{1,1}(\bar{\Omega}) = C^{1,0}(\bar{\Omega}) \cap C^0(\bar{\Omega})$, and suppose that $g(\bar{\Omega}) \subseteq \mathbb{R}$. If $a, b \in [0, 2]$, then $f \circ g^{a,1}(\bar{\Omega})$ and $h^{1,1}(\bar{\Omega})$ and there exists $C > 0$ such that

$$\int_{\partial \Omega} [f(\gamma) - f(\gamma')] d\gamma \leq C \int_{\partial \Omega} [g(\gamma) - g(\gamma')] d\gamma,$$

where $[f(\gamma) - f(\gamma')] = f(\gamma) + 2(\gamma - \gamma') \cdot f'(\gamma') - f(\gamma')$.

Proof. Suppose that $\Omega$ is convex. If $x, y \in \overline{\Omega}$ then $\frac{1}{2} \|g(y) - g(y')\|_{\overline{\Omega}}$; hence we can write

$$\int_{\partial \Omega} [f(\gamma) - f(\gamma')] d\gamma = \int_{\partial \Omega} \left[ f(\gamma) - f(\gamma') \right] d\gamma \leq \int_{\partial \Omega} \left[ \frac{1}{2} (g(y) - g(\gamma'))^2 \right] d\gamma$$

and by Remark 1.4 (ii)

$$\int_{\partial \Omega} [f(\gamma) - f(\gamma')] d\gamma \leq \int_{\partial \Omega} \left[ \frac{1}{2} |g(y) - g(\gamma')|^2 \right] d\gamma \leq \frac{1}{2} \|g(y) - g(\gamma')\|_{\overline{\Omega}}^2.$$

Now we need to study some properties of the spaces $C^{1,1}(\bar{\Omega})$, $C^{1,1}(\bar{\Omega})$ in more detail.

Lemma 1.18. If $f \in C^{1,1}(\bar{\Omega})$ (resp. $h^{1,1}(\bar{\Omega})$) then $f \in C^{1,1}(\bar{\Omega})$ (resp. $h^{1,1}(\bar{\Omega})$) and

$$\left[ \int_{\partial \Omega} \left[ f(\gamma) - f(\gamma') \right] d\gamma \right]^2 \leq C \int_{\partial \Omega} \left[ \int_{\partial \Omega} \left[ g(\gamma) - g(\gamma') \right] d\gamma \right]^2.$$

Proof. Let $x, y \in \overline{\Omega}$ be such that $x, y \in \overline{\Omega}$. Then

$$f(x) - f(y) = f(y) - f(x) + \frac{1}{2} f(x) + \frac{1}{2} f(y) - f(x) + f(x) =$$

$$= [f(x) - f(y)] \cdot [g(x) - g(y)] + f(x) - f(x) + f(y) - f(y) +$$

and taking into account Remark 1.4 (ii), the result follows.
certainly the case if either \( g(x) \) or \( g(y) \) belongs to \( \mathcal{U}/2 \). Hence we can reduce ourselves to the following situation:

\[
|g(x)-g(y)| < \varepsilon/2; \quad (g(x), g(y), g(x+y))/2 \in \mathcal{U}; \quad \frac{1}{2} |g(x+y) - g(y)| \in \mathcal{U}. \tag{1.12}
\]

Set \( \sigma = \max\{|g(x)-g(y)|, |g(x+y)-g(y)|/2, |g(y)-g(x+y)/2|\}; \) obviously

\[
\sigma \leq |g|_1 \cdot |x-y| \leq \varepsilon/2 \tag{1.13}
\]

We want now to find a point \( w \in \mathcal{U} \) satisfying

\[
\frac{1}{2} w + \frac{1}{2} w + g(x+y)/2, \quad \frac{1}{2} w + g(x+y)/2, \quad \frac{1}{2} w + \frac{1}{2} (g(x)+g(y)) \in \mathcal{U} \tag{1.14}
\]

and

\[
|g(x)-w| \leq 3\sigma, \quad |g(y)-w| \leq 3\sigma, \quad |g(x+y)/2-w| \leq \frac{7}{2} \sigma \tag{1.15}
\]

Suppose that this has been done; then by (1.14) we can write

\[
\frac{1}{2} (g(x)+g(y)) - 2\alpha \leq g(x+y)/2 \leq \frac{1}{2} (g(x)+g(y)) + 2\alpha \left( \frac{1}{2} |g(x+y) - g(x)| \right) + \frac{1}{2} \left( |g(x) + g(y)| - |g(x+y)| \right) - 2\alpha \left( \frac{1}{2} |g(x)+g(y)| \right) - 2\alpha \left( \frac{1}{2} |g(x+y)| \right) - \frac{1}{2} \left( |g(x)+g(y)| \right) - \frac{7}{2} \alpha \left( \frac{1}{2} |g(x+y)| \right) \leq \frac{1}{2} \left( |g(x)+g(y)| \right) + \alpha \left( \frac{1}{2} |g(x+y)| \right) - \frac{7}{2} \alpha \left( \frac{1}{2} |g(x+y)| \right)
\]

and hence, by (1.13) and (1.15)

\[
|f(x)| + |f(y)| - 2\alpha |g(x+y)|/2 \leq f(x)x, y (|g(x)-w| + |g(y)-w| + |g(x)-g(y)| + |g(x+y)/2-w|) + \frac{1}{2} \left( |g(x)+g(y)| - |g(x+y)| \right) + \frac{1}{2} \left( |g(x)+g(y)| \right) + \frac{1}{2} \left( |g(x+y)| \right) + \frac{1}{2} \left( |g(x+y)| \right) \leq \frac{1}{2} \left( |g(x)+g(y)| \right) + \alpha \left( \frac{1}{2} |g(x+y)| \right) + \frac{7}{2} \alpha \left( \frac{1}{2} |g(x+y)| \right)
\]

This implies the result, provided (1.14) and (1.15) hold.

In order to find the point \( w \in \mathcal{U} \) satisfying (1.14) and (1.15), we start with observing that, by the definition of \( \sigma \), we have

\[
\text{dist} \left( \frac{1}{2} (g(x)+g(y)), \mathcal{U} \right) \leq \sigma/2. \tag{1.16}
\]

and the unit exterior normal vector at \( z \) is

\[
v(z) = \frac{\frac{1}{2} (g(x)+g(y)) - z}{|\frac{1}{2} (g(x)+g(y)) - z|}.
\]

Define now \( w = z - 2\sigma z \): by (1.16) and (1.13) \( w \in \mathcal{U} \) and

\[
\text{dist}(w, \mathcal{U}) = 2\sigma \leq \frac{1}{2} \varepsilon \tag{1.17}
\]

in particular \( \frac{1}{2} (g(x)+g(y)) \in \mathcal{U} \) because it lies in the segment joining \( w \) and \( z \).

By (1.17), we see that (1.14) will follow if we show that the points \( \frac{1}{2} w + g(x)/2 \), \( \frac{1}{2} w + g(y)/2 \), \( \frac{1}{2} w + g(x+y)/2 \) lie in a ball centered in \( w \) with radius less than \( 2\sigma \); as we will see, this will also imply (1.15). Indeed, we have by (1.16)

\[
\frac{1}{2} |w + g(x)|/2 - |g(x)|/2 + |g(y)|/2 \leq \frac{1}{2} |w + g(x)|/2 - |g(x)|/2 + |g(y)|/2 \leq \frac{1}{2} \sigma \leq \sigma
\]

and in particular \( |g(x)-w| \leq 3\sigma \); similarly

\[
|\frac{1}{2} w + g(y)|/2 - |g(y)|/2 \leq \frac{1}{2} |w + g(y)|/2 - |g(y)|/2 \leq \frac{1}{2} \sigma \leq \sigma
\]

and finally

\[
\frac{1}{2} |w + g(x+y)|/2 - |g(x+y)|/2 \leq \frac{1}{2} |w + g(x+y)|/2 - |g(x+y)|/2 \leq \frac{1}{2} |w + g(x+y)|/2 - |g(x+y)|/2 \leq \sigma
\]

\[
+ \frac{1}{2} |g(x+y)| - |g(x+y)| \leq \frac{1}{2} |g(x+y)| - |g(x+y)| \leq \frac{1}{2} \sigma \leq \frac{1}{2} \sigma
\]

\[
+ \frac{1}{2} |g(x+y)| - |g(x+y)| \leq \frac{1}{2} |g(x+y)| - |g(x+y)| \leq \frac{1}{2} \sigma \leq \frac{1}{2} \sigma
\]
which also implies $\left| y^{\frac{1}{2}} \right| < \frac{1}{2}$. Thus, (1.14) and (1.15) are proved and this concludes the proof. \(\square\)

We finish this section with a version of the well-known Sobolev's Theorem. Set:

$$I^p(\Omega) = \{ f: \Omega \to \mathbb{C}, f \text{ is Lebesgue measurable and} \}$$

$$\mathbb{P} \in L^p(\Omega) \cap \mathbb{H}^1(\Omega) = \{ \mathbb{P} : \int_{\Omega} \mathbb{P} dx \} \text{ such that}$$

where $\mathbb{P} \in L^p(\Omega)$, $i=1, \ldots, n$, and similarly one defines the spaces $L^p(\Omega, \mathbb{R}^n)$, $H^1(\Omega, \mathbb{R}^n)$ and $L^p(\Omega, \mathbb{R}^n)$, $H^1(\Omega, \mathbb{R}^n)$. The result is the following:

**Lemma 1.20.** Suppose $\Omega$ is of class $C^1$ and set $\Omega(x_0, r) = \{(x, y) : |x - x_0| < r\}$, where $x_0 \in \Omega$ and $r > 0$. Suppose $\Omega \cap \Omega$ and $\Omega = 1 - \frac{1}{p}$; then $H^1(\Omega) = H^0(\Omega)$, moreover there exist $C_1, C_2 > 0$ such that:

$$|| u ||_{H^1(\Omega)} \leq C_1 \| u \|_{L^2(\Omega)} \leq C_2 \| u \|_{H^1(\Omega)}$$

**Proof.** The first inequality is Sobolev-Morrey's inequality (see, e.g., Adams [4], Lemma 5.17), the second one follows by Hölder's inequality. \(\square\)

Finally we remark that in the forthcoming sections any number $C$ appearing in the estimates will denote a constant which is independent of the estimated quantities.

2. THE CASE OF A HALF-SPACE WITH A NORMAL BOUNDARY CONDITION

Let us consider the closed half-space

$$\mathbb{R}^n = \mathbb{R}^{n-1} \times \{ 0, \ldots, = (x, y) : x \in \mathbb{R}^{n-1}, y \geq 0 \}$$

obviously the unit exterior normal vector at each point of the boundary $\mathbb{R}^{n-1} \times \{ 0 \}$ is $\mathbf{n} = -\mathbf{e}_n$. Let $\mathfrak{C}^1(\mathbb{R}^{n-1}, \mathbb{R})$ be a real non-negative function; define the boundary operator

$$\mathfrak{N}(x) = a(x)u(x, 0) + (v(x)u(x, 0)) = a(x)u(x, 0) - \mathbf{n} \cdot \mathbf{u}(x, 0), \quad (2.1)$$

and consider the spaces (see Definitions 1.13 and 1.14)

$$\begin{align*}
\mathfrak{C}_k^{\mathfrak{N}}(\mathbb{R}^n) &= \{ f : \mathbb{C}^{1, \mu, a}(\mathbb{R}^n) : \mathfrak{N}f = 0 \}, & k = 1, 2; \\
\mathfrak{C}^{1, \alpha, a}(\mathbb{R}^n) &= \{ f : \mathbb{C}^{1, \alpha, a}(\mathbb{R}^n) : \mathfrak{N}f = 0 \}, & \alpha \in (0, 1], \\
\mathfrak{C}^{1, \alpha, a}(\mathbb{R}^n) &= \{ f : \mathbb{C}^{1, \alpha, a}(\mathbb{R}^n) : \mathfrak{N}f = 0 \}, & \alpha \in (0, 1], \\
\mathfrak{C}_\nu^{1, \alpha, a}(\mathbb{R}^n) &= \{ f : \mathbb{C}^{1, \alpha, a}(\mathbb{R}^n) : \mathfrak{N}f = 0 \}, & \nu \in (0, 1].
\end{align*}$$

These spaces are complete with respect to the norm

$$\begin{align*}
\mathfrak{C}_k^{\mathfrak{N}}(\mathbb{R}^n) &\quad \mathfrak{C}_k^{1, \alpha, a}(\mathbb{R}^n) \quad \mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n) \quad \mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n)
\end{align*}$$

Our goal is the following theorem:

**Theorem 2.1.** Let $\mathfrak{C}^2(\mathbb{R}^{n-1}, \mathbb{R})$ with $\alpha = 0$ and let $N$ be the operator defined in (2.1). The following continuous inclusions hold:

$$\begin{align*}
\mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n) &\hookrightarrow \mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n), \\
\mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n) &\hookrightarrow \mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n), \\
\mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n) &\hookrightarrow \mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n), \\
\mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n) &\hookrightarrow \mathfrak{C}_{\nu}^{1, \alpha, a}(\mathbb{R}^n).
\end{align*}$$
which also implies \(|y|^{1-\alpha} = n \frac{\sqrt{\alpha}}{2} \). Thus, (1.14) and (1.15) are proved and this concludes the proof. \(\square\)

We finish this section with a version of the well-known Sobolev's Theorem. Set

\[ L^p(\Omega) = \{ f: \Omega \to \mathbb{C} : f \text{ is Lebesgue measurable and} \] \[ \| f \|_{L^p(\Omega)} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p} \] \[ \text{if} \quad p \neq 1 \quad \text{and} \quad \sup_{x \in \Omega} |f(x)| = \| f \|_{L^\infty(\Omega)} \quad \text{if} \quad p = 1 \] \[ \text{where} \ \mathcal{P} \{ f : \Omega \to \mathbb{C} \} \] \[ \text{Similarly one defines the spaces} \ L^p(\Omega, \mathbb{R}^m), \ H^1(\Omega, \mathbb{R}^m), \text{and} \ H^1(\Omega, \mathbb{C}^m) \]. The result is the following:

**Lemma 1.20.** Suppose \( u \) is of class \( C^1 \) and set \( u(x_0, r) = u(x_0) - \frac{1}{r} \frac{d}{dr} \frac{d}{dt} u(x(t, r)) \) where \( x_0 \in \Omega \) and \( r > 0 \). Suppose \( u \in C^2(\Omega, \mathbb{C}^m) \) and \( \alpha = 1/n/q \) then \( H^1(\Omega, \mathbb{C}^m) \supset H^1(\Omega, \mathbb{C}^m) \). Moreover there exist \( C_1, C_2 > 0 \) such that for each \( x_0 \in \Omega \), \( r > 0 \), and \( \omega \in H^1(\Omega, \mathbb{C}^m) \)

\[ \| u \|_{L^2(\Omega, \mathbb{C}^m)} \leq C_1 \| \mathbf{Du} \|_{L^2(\Omega)} + C_2 \| \mathbf{Du} \|_{L^2(\Omega)} \]

**Proof.** The first inequality is Sobolev-Morrey's inequality (see e.g. Adams [4], Lemma 5.17), the second one follows by Hölder's inequality. \(\square\)

Finally we remark that in the forthcoming sections any number \( C \) appearing in the estimates will denote a constant which is independent of the estimated quantities.

2. THE CASE OF A HALF-SPACE WITH A NORMAL BOUNDARY CONDITION

Let us consider the closed half-space

\[ \mathbb{R}^n_+ = \{ (x, y) : x \in \mathbb{R}^{n-1}, y \geq 0 \} \]

which is obviously the unit exterior normal vector at each point of the boundary \( \partial = \mathbb{R}^{n-1} \times \{ 0 \} \) is \( \nu = -n \). Let \( u \in C^2(\mathbb{R}^{n-1}, \mathbb{R}) \) be a real non-negative function; define the boundary operator

\[ \nu u(x) = u(x)u(x_0) + \frac{1}{2} D u(x_0), \quad u(x_0) = u(x_0(x, 0)) - \frac{1}{2} y \frac{1}{y} u(x_0(x, 0)), \quad x \in \mathbb{R}^{n-1} \]

and consider the spaces (see Definitions 1.13 and 1.14)

(a) \( C^k_\infty(\mathbb{R}^n_+) = \{ f \in \mathcal{C}^k(\mathbb{R}^n_+) : \text{supp} f \subset (0, \infty) \}, \)

(b) \( C^k_\infty(\mathbb{R}^n_+) = \{ f \in \mathcal{C}^k(\mathbb{R}^n_+) : \text{supp} f \subset (0, \infty) \}, \)

(c) \( C^k_\infty(\mathbb{R}^n_+) = \{ f \in \mathcal{C}^k(\mathbb{R}^n_+) : \text{supp} f \subset (0, \infty) \}, \)

These spaces are complete with respect to the norms

(a) \[ \| f \|_{C^k(\mathbb{R}^n_+)} = \sum_{|\alpha| \leq k} \| D^\alpha f \|_{L^2(\mathbb{R}^n_+)} \]

(b) \[ \| f \|_{C^k_\infty(\mathbb{R}^n_+)} = \sum_{|\alpha| \leq k} \| D^\alpha f \|_{L^2(\mathbb{R}^n_+)} \]

(c) \[ \| f \|_{C^k_\infty(\mathbb{R}^n_+)} = \sum_{|\alpha| \leq k} \| D^\alpha f \|_{L^2(\mathbb{R}^n_+)} \]

Our goal is the following theorem:

**Theorem 2.1.** Let \( \varphi \in C^2(\mathbb{R}^{n-1}, \mathbb{R}) \) with \( \alpha > 0 \) and let \( N \) be the operator defined in (2.1). The following continuous inclusions hold:

\[ (C^2(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-\alpha, \alpha} \quad \subset \quad (C^{1+\alpha}(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-2\alpha, \alpha} \]

\[ (C^{1+\alpha}(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-2\alpha, \alpha} \quad \subset \quad (C^{2}(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-\alpha, \alpha} \]

\[ (C^{2}(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-\alpha, \alpha} \quad \subset \quad (C^{2+\alpha}(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-2\alpha, \alpha} \]

\[ (C^{2+\alpha}(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-2\alpha, \alpha} \quad \subset \quad (C^{3+\alpha}(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+))_{1-3\alpha, \alpha} \]
hence it will be sufficient to prove that

$$
\begin{align*}
\omega \in C^0([0,1] \times \mathbb{R}^n), & \quad u(0,x,y) = f(x,y) \quad \forall (x,y) \in \mathbb{R}^n, \\
\partial^2 u, & \quad u_{\epsilon}^\prime C_{1-0}([0,1],C^0(\mathbb{R}^n)) \quad \text{(resp. } C_{1-0}([0,1],C^0(\mathbb{R}^n)) \text{)} \quad (2.1) \\
|\nu(t,\cdot,\cdot)|(x) = 0 & \quad \forall x \in \mathbb{R}^{n-1}, \forall t \in [0,1].
\end{align*}
$$

Here and in the following the symbol $\partial$ denotes the gradient with respect to the only coordinates $(x,y)=(x_1,x_2, \ldots, x_{n-1},y)$.

We will proceed as follows. First we will construct an extension $\tilde{f}$ of $f$ to the whole $\mathbb{R}^n$, in such a way that $\tilde{f}$ is as smooth as $f$ and satisfies the additional condition along $\Sigma$ whenever $f$ does.

Next, we will construct a function $w(t,x,y)$, defined in $[0,1] \times \mathbb{R}^n$, satisfying (2.1) in $[0,1] \times \mathbb{R}^n$; this is done by taking the convolution $\tilde{f} * \rho$, where $\rho$ is a mollifier of parameter $t^{1/2}$, and adding to it a suitable term in order to satisfy the condition $\nu w(t,\cdot,\cdot)=0$. Finally, the restriction of $w$ to $[0,1] \times \mathbb{R}^n$ will be the desired function $u$.

**Step 1.** The extension of $f$ is the following:

$$
F(x,y) = \begin{cases} 
\mathcal{L}(x,y) & x \in \mathbb{R}^{n-1}, y \geq 0 \\
0 & x \in \mathbb{R}^{n-1}, y < 0.
\end{cases}
$$

**Remark 2.2** - (i) If $f$ vanishes somewhere, this definition does not assure the uniform continuity of $F$; in this case we replace $F(x,y)$ by $F(x,y) - a(y)$, where $a \in C^\infty(\mathbb{R})$ is such that $a(1) = 0$ and $a(0) = 1$.

(ii) If the boundary operator (2.1) is of Neumann type, i.e., $a(x) \equiv 0$, then the extension $F$ defined in (2.3) reduces to the even extension of $f$: hence it has the property that, for smooth $\tilde{f}$, the function $y \frac{\partial \tilde{F}}{\partial y}(x,y)$ (whose evaluation at $y=0$ yields the boundary operator) is odd.

The same property holds in the general case, namely the extension $F$ given by (2.3) is constructed in such a way that, for smooth $\tilde{f}$,
the function $y = \alpha(x, y) P(y) - \frac{\partial F}{\partial y}(x, y)$ turns out to be odd. This
obviously guarantees that $NF=0$ whenever $NF=0$; it is a remarkable
fact that, in addition, $F$ has exactly the same degree of smoothness
of $f$, even in the case of non-differentiable $f$, i.e. when
no boundary condition is required for $f$. This will be shown in
the next proposition.

PROPOSITION 2.3 - Let $F$ belong to any of the following spaces:

(i) $C^0(R^n)$; (ii) $C^{0,28}(R^n)$ (resp. $C^{0,28}(R^n)$) with $6\leq 0,1/2$;

(iii) $C^{1,-1}(R^n)$ (resp. $C^{1,-1}(R^n)$); (iv) $C^5(R^n)$;

(v) $C^{5,28-1}(R^n)$ (resp. $C^{5,28-1}(R^n)$) with $6\leq 1,2$.

Then the function $F$ defined in (2.3) satisfies accordingly:

(i) $F \in C^0(R^n)$; (ii) $F \in C^{0,28}(R^n)$ (resp. $F \in C^{0,28}(R^n)$);

(iii) $F \in C^{1,-1}(R^n)$ and $\sup_{y \neq 0} \frac{|F(x,y) - F(x,0)|}{|y|} = 0$

(resp. $F \in C^{1,-1}(R^n)$); (iv) $F \in C^{5}(R^n)$ and $NF=0$; (v) $F \in C^{5,28-1}(R^n)$ (resp. $F \in C^{5,28-1}(R^n)$)

and $NF=0$.

Moreover we have in any case

$$|F| \leq a(|x|)$$

in the corresponding norm; in particular, in case (iii) we set

$$|F| = \sup_{y \neq 0} \frac{|F(x,y) - F(x,0)|}{|y|} = 0$$

Proof. (i)-(ii) The results follow by straightforward computations.

(iii) This proof is more delicate. Suppose $F \in C^{1,-1}(R^n)$ and let

$$F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})$$. This is easy if $y, y'$ have the
same sign, for we can suppose $y \geq 0$ and two cases can occur:

(a) $y \geq \frac{y+y'}{2} \geq y'$; (b) $y > 0$.

In case (a) we can write

$$2F(\frac{x+x'}{2}, \frac{y+y'}{2}) = F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})$$

$$-2a(x') f \exp[a(x')(y'-s)] f(x', s) ds - 2\left[\frac{x+x'}{2}, \frac{y+y'}{2}\right]$$

$$= f(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})+2\left[\frac{x+x'}{2}, \frac{y+y'}{2}\right]$$

$$-2a(x') f \exp[a(x')(y'-s)] f(x', s) ds$$

and consequently, since $|y+y'| |y'\leq |y-y'|$, we get

$$|F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})| \leq 2\left[\frac{x+x'}{2}, \frac{y+y'}{2}\right]$$

$$a(x') f \exp[a(x')(y'-s)] f(x', s) ds$$

$$|C_{x'}$$

In case (b) we have

$$F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})$$

$$-2a(x') f \exp[a(x')(y'-s)] f(x', s) ds - 2\left[\frac{x+x'}{2}, \frac{y+y'}{2}\right]$$

$$+4\left[\frac{x+x'}{2}, \frac{y+y'}{2}\right] \exp[a(x')(y'-s)] f(x', s) ds$$

$$= f(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})+2\left[\frac{x+x'}{2}, \frac{y+y'}{2}\right]$$

$$-2a(x') f \exp[a(x')(y'-s)] f(x', s) ds$$

$$+4\left[\frac{x+x'}{2}, \frac{y+y'}{2}\right] \exp[a(x')(y'-s)] f(x', s) ds$$

$$Y'$$
which implies, since \(|y_1y'| \leq |y'_1| \leq |y'y''|\),

\[
|F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})| \leq |y_1|, 1 \text{ for } |x-x'|+|y-y'| \geq 2, +\frac{|x+x'|}{2} + \left( \frac{|y+y'|}{2} \right) + 2 \left( \frac{|y+y'|}{2} \right) \leq (2.7)
\]

\[
\leq \alpha \int \chi_1^*(x',y') |x-x'| + |y-y'|
\]

By (2.5) and (2.7) we deduce that \(F \in \mathcal{C}_0^1(\mathbb{R}^n)\) and \(F \in \mathcal{C}_0^1(\mathbb{R}^n) \leq \alpha \int \chi_1^*(x',y') \; \text{on the other hand it is easily seen that for } \mathcal{C}_0^1(\mathbb{R}^n) \),

each \(x \in \mathbb{R}^n + y \in \mathbb{R}^n\)

\[
|F(x,y)-F(x,0)| \leq \begin{cases} 
\frac{|y_1|}{1, y_1} & \text{if } y \geq 0 \\
\frac{|y_1|}{1, y_1} + \alpha \int \chi_1^*(x',y') & \text{if } y < 0
\end{cases}
\]

which implies

\[
\lim_{y \to 0} \frac{F(x,y)-F(x,0)}{y} \in \mathcal{C}_0^1(\mathbb{R}^n)
\]

Suppose now, in addition, that \(F \in \mathcal{C}_0^1(\mathbb{R}^n)\). We have to show that

\[
|F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})| = |(x-x')^2 + (y-y')^2| = \mathcal{O}(1) \quad \text{as } x \to 0^+ \quad \text{and} \quad y \to 0^-
\]

\[
\lim_{y \to 0} \frac{F(x,y)-F(x,0)}{y} = \alpha(x)f(x,0) \quad \forall x \in \mathbb{R}^n
\]

The proof of (2.8) is easy if \(y, y'\) have the same sign. Otherwise, we again reduce to case (a) or (b). In case (a) (b) we deduce as \(|x-x'| + |y-y'| = o^+\)

\[
|F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})| = o(|x-x'|^2 + |y-y'|^2) + (y-y')^2 |(x-x')f(\frac{x+x'}{2}, 0) + o(1)| + (y-y')^2 |(y-y')f(\frac{x+x'}{2}, 0) + o(1)|
\]

In case (b), by (2.6) similarly we get as \(|x-x'| + |y-y'| = o^+\)

\[
|F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})| = o(|x-x'| + |y-y'|)
\]

\[
\lim_{y \to 0} \frac{F(x,y)-F(x,0)}{y} = \alpha(x)f(x,0) \quad \forall x \in \mathbb{R}^n
\]

This proves (2.6). Finally it is easy to see that as \(y \to 0\)

\[
F(x,y)-F(x,0) = \begin{cases} 
\alpha(x)f(x,0) + o(1) & \text{if } y \to 0 \\
\alpha(x)f(x,0) + o(1) & \text{if } y \to 0
\end{cases}
\]

which implies (2.3). This concludes the proof of (i).
Step 2. We want to construct a function \( w(t,x,y) \), defined in \([0,1] \times \mathbb{R}^3 \), with the following properties:

1. \( w \in C^0([0,1] \times \mathbb{R}^3), w(0,\cdot,\cdot) = \Phi \)

2. \( D^2 w \in C_{1-\theta}([0,1] \times \mathbb{R}^3) \) (resp. \( D^2 w \in C_{1-\theta}([0,1] \times \mathbb{R}^3) \) and
   \[
   \lim_{t \to 0^+} D^2 w(t,\cdot,\cdot) \big|_{t=0^+} = 0
   \]
   \[
   \lim_{t \to 0^+} D^2 w(t',\cdot,\cdot) = 0 \quad \text{for all} \quad \varphi \in C^0(\mathbb{R}^3)
   \]

3. \( \{ w(t,\cdot,\cdot) \}_{t \in [0,1]} \) is \( C^0(\mathbb{R}^3) \) and \( w \in C_{0-\theta}([0,1] \times \mathbb{R}^3) \) (resp. \( w \in C_{0-\theta}([0,1] \times \mathbb{R}^3) \) and
   \[
   \lim_{t \to 0^+} w(t,\cdot,\cdot) \big|_{t=0^+} = 0
   \]

First of all let us note that for each \( t \in [0,1] \) and \( (x,y) \in \mathbb{R}^2 \)

\[
\Phi (t,x,y) = t^{-\frac{n}{2}} \phi \left( t^{-\frac{1}{2}} x, t^{-\frac{1}{2}} y \right)
\]

where \( \phi \in C^0(\mathbb{R}^2) \) is an even non-negative function with support contained in the ball \( B(0,1) = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 \leq 1\} \) and such that \( \int \Phi (x,y) \, dx \, dy = 1 \). Next, define

\[
v(t,x,y) = \begin{cases} 
  P(x,y) & \text{if } t = 0, (x,y) \in \mathbb{R}^2 \\
  \Phi (t,x,y) = t^{-n/2} \int_{\mathbb{R}^2} \phi \left( t^{-1/2} x - \xi, t^{-1/2} y - \eta \right) \, d\xi \, d\eta & \text{if } t \neq 0, (x,y) \in \mathbb{R}^2
\end{cases}
\]

and

\[
g(t,x) = \begin{cases} 
  0 & \text{if } t = 0, x \in \mathbb{R}^n \\
  \frac{a(x)v(t,x,0) - \frac{\partial}{\partial y} \frac{\partial}{\partial y} v(t,x,0)}{\sum_{i=1}^{n} a(x)[\phi^\delta \Phi] (x,0) - \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \Phi (x,0)} & \text{if } t \neq 0, x \in \mathbb{R}^n
\end{cases}
\]

Finally set

\[
w(t,x,y) = v(t,x,y) - \frac{1}{2} (1 - \tau) \frac{\partial}{\partial y} g(t,x) - \frac{1}{2} \tau (x,y) + \mathbf{1} (x) \]

where \( \mathbf{1} \in C^0(\mathbb{R}) \), \( \mathbf{1} \) in \([-1,1] \), \( \mathbf{1} \) outside \([-2,2] \) and \( 0 \leq \mathbf{1} \leq 1 \).

Remark 2.4 - It can be verified that the function \( v \) defined by (2.11) satisfies conditions (i), (ii) and (iv) of (2.10), whereas in general condition (2.10) (iii) needs not to be true. For this reason we have to introduce the function \( g \) defined in (2.12) which is suitably constructed in order that (2.10) (iii) holds automatically. We will see that, consequently, the function \( w \) given by (2.13) satisfies (2.10).

The auxiliary function \( g \) is unnecessary if in the boundary operator (2.1) we have \( a(x) = \text{constant} \), because in this case it can be shown that (2.10) (iii) holds, i.e.

\[
\left( N + 2 \Phi \right) (x) = \frac{1}{2} \phi \Phi (x,0) - \frac{\partial}{\partial y} \frac{\partial}{\partial y} \Phi (x,0) \quad \text{with } x \in \mathbb{R}^n.
\]

This is clear if \( \Phi \) is smooth, i.e. if \( \phi \) is smooth; indeed, as observed in Remark 2.2, for a differentiable \( \Phi \) the function

\[
G(y) = \phi \Phi (x,y) - \frac{\partial}{\partial y} \frac{\partial}{\partial y} \Phi (x,y)
\]

is odd, as the kernel \( \phi \) is an even function, the convolution

\[
[a(x) \phi \Phi (x,y) - \frac{\partial}{\partial y} \frac{\partial}{\partial y} \Phi (x,y)] (x,y)
\]

is also odd, and therefore it vanishes at \( y = 0 \). However, even if \( \phi \) is smooth enough to give sense to \( N \), a direct computation can show that \( N \phi \Phi = 0 \). We will prove this fact indirectly by formula (2.14) below, since that equality reduces to \( g(t,x) = 0 \) if \( a(x) = \text{constant} \) .

We have to verify (2.10) for the function \( w \) defined in (2.13).
We start with the following result:

**Proposition 2.5** (i) $w$ is twice differentiable in $[0,1] \times \mathbb{R}^n$.

(ii) $[w(t, \cdot, \cdot)](x)=\langle x, w(t, x, 0) \rangle - \frac{3}{2} \frac{\partial y}{\partial t}(t, x, 0) \equiv 0$ for all $t \in [0,1], \forall x \in \mathbb{R}^n$.

(iii) $\partial w(0,1,\mathbb{R}^n)$ and $w(0, \cdot, y) = \partial y_{\mathbb{R}^n}$.

**Proof.** (i) It is a straightforward consequence of the mollifying properties of the convolution and of the regularity of $a$.

(ii) For each $y \in \mathbb{R}^n$ we have

$$a(x)w(t, y, x) = \frac{3}{2} \frac{\partial y}{\partial t}(t, x, y) - \frac{3}{2} \frac{\partial y}{\partial x}(t, x, y) - \frac{1}{1+a(x)} \frac{\partial y}{\partial x}(t, x, y)$$

and choosing $y=0$ we get

$$[w(t, \cdot, \cdot)](x)=a(x)\langle x, 0 \rangle - \frac{3}{2} \frac{\partial y}{\partial x}(t, x, 0) - g(t, x);$$

by (2.11) and (2.12) the result follows.

(iii) We need an alternative expression for $g$, namely

$$g(t, x) = a(x)\langle x, \int \frac{e^{-|y|^2}}{4\pi|y|^2} \xi \partial_x e^{iy(x, \xi)} \rangle (x, 0). \tag{2.14}$$

To prove (2.14) it suffices to show that

$$\frac{3}{2} \frac{\partial y}{\partial x}(t, x, 0) = \langle x, \partial_x e^{iy(x, \xi)} \rangle (x, 0). \tag{2.15}$$

In fact, we have

$$\frac{3}{2} \frac{\partial y}{\partial x}(t, x, 0) = \frac{1}{8} \int \frac{3}{2} \frac{\partial y}{\partial x} \left( t^{-1/2}(x, \xi), t^{-1/2} \right) \xi e^{iy(x, \xi)} d\xi$$

$$+ \frac{1}{8} \int \frac{3}{2} \frac{\partial y}{\partial x} \left( t^{-1/2}(x, \xi), t^{-1/2} \right) \xi e^{iy(x, \xi)} d\xi$$

$$- \frac{1}{8} \int \frac{3}{2} \frac{\partial y}{\partial x} \left( t^{-1/2}(x, \xi), t^{-1/2} \right) \xi e^{iy(x, \xi)} d\xi$$

$$- \frac{1}{8} \int \frac{3}{2} \frac{\partial y}{\partial x} \left( t^{-1/2}(x, \xi), t^{-1/2} \right) \xi e^{iy(x, \xi)} d\xi.$$

By (2.14) we get as $t \to 0^+$

$$g(t, x) = a(x)\langle x, 0 \rangle - a(x)\langle x, 0 \rangle = 0 \text{ uniformly in } x \in \mathbb{R}^{n-1};$$

As $y + \frac{3}{2} \frac{\partial y}{\partial x}(x, y)$ is an odd function, the first two terms cancel each other. Thus by an integration by parts in the variable $\eta$ we obtain

$$\frac{3}{2} \frac{\partial y}{\partial x}(t, x, 0) = \int_{\mathbb{R}^{n-1}} \int_0^\infty f(t^{-1/2}(x, \xi), t^{-1/2} \eta) \xi e^{iy(x, \xi)} \eta \exp \left( \frac{e^{-|y|^2}}{4\pi|y|^2} \xi \partial_x e^{iy(x, \xi)} \right) d\xi d\eta - \frac{3}{2} \frac{\partial y}{\partial x}(t, x, 0).$$

On the other hand, as $y \cdot e^{iy(x, \xi)}$ is an even function, we have

$$\int_{\mathbb{R}^{n-1}} \int_0^\infty f(t^{-1/2}(x, \xi), t^{-1/2} \eta) \xi e^{iy(x, \xi)} \eta \exp \left( \frac{e^{-|y|^2}}{4\pi|y|^2} \xi \partial_x e^{iy(x, \xi)} \right) d\xi d\eta = 0.$$
on the other hand we have \( v(0, t, \cdot', \cdot') = F \) and
\[
\lim_{t \to 0} \frac{v(0, t, \cdot', \cdot') - v(t, \cdot', \cdot')}{C(0, t)} = 0.
\]
(These facts follow since \( F \) is uniformly continuous). Thus by (2.13) we get
\[
u(t, x, y) - F(x, y) = \frac{(1 - t) \theta(y)}{\alpha(x)} q(0, x) = F(x, y) \text{ uniformly in } (x, y) \in \mathbb{R}^n,
\]
i.e. (2.15) holds. The proof is complete. \( \Box \)

Conditions (ii) and (iii) of (2.10) are proved; we have now to verify (iv) and (iv) of (2.10).

**Lemma 2.6.** We have for each \( t \in [0, 1] \)
\[
\begin{align*}
1^2 \varphi(t, \cdot, \cdot') \in & C^0(\mathbb{R}^n) \quad \text{if} \quad 1^2 \varphi(t, \cdot, \cdot') \in C^2(\mathbb{R}^n) \\
& C^0(\mathbb{R}^n) \quad \text{if} \quad C^2(\mathbb{R}^n)
\end{align*}
\]

**Proof.** It is a straightforward consequence of (2.13) and (2.14). \( \Box \)

Thus to prove (iv) of (2.10) we have to estimate the \( C^2 \)-norms of the convolutions \( \varphi * F, \varphi * (F - F) \), and, in view of Lemma 2.10, it will be sufficient to estimate their \( C^0 \)-norms and the \( C^0 \)-norms of their second-order gradients. This is the goal of the next two lemmas.

**Lemma 2.7.** For each \( \varphi \in C^0(\mathbb{R}^n) \), let \( F \) be defined by (2.3). Then
\[
\begin{align*}
1^2 \varphi * F \in & C^0(\mathbb{R}^n) \quad \text{if} \quad C^2(\mathbb{R}^n) \\
& C^0(\mathbb{R}^n) \quad \text{if} \quad C^0(\mathbb{R}^n)
\end{align*}
\]

**Proof.** Obviously
\[
\begin{align*}
1^2 \varphi * F \in & C^0(\mathbb{R}^n) \quad \text{if} \quad C^2(\mathbb{R}^n) \\
& C^0(\mathbb{R}^n) \quad \text{if} \quad C^0(\mathbb{R}^n)
\end{align*}
\]
and Proposition 2.3(i) yields the result. \( \Box \)

**Lemma 2.8** (i) \( \mathcal{G} \in C^0(0, 1/2) \text{ and } \mathcal{G} \in C^0(0, 20) \), then
\[
1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n) \quad \text{if} \quad 1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n)
\]

\( \forall t \in [0, 1] \).

(ii) \( \mathcal{G} \in C^0(0, 1/2) \text{ and } \mathcal{G} \in C^0(0, 20) \), then
\[
1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n) \quad \text{if} \quad 1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n)
\]

\( \forall t \in [0, 1] \).

(iii) \( \mathcal{G} \in C^0(0, 1/2) \text{ and } \mathcal{G} \in C^0(0, 20) \), then
\[
1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n) \quad \text{if} \quad 1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n)
\]

\( \forall t \in [0, 1] \).

If, moreover, in cases (i), (ii), (iii) we assume \( \mathcal{G} \in C^0(0, 20) \), \( \mathcal{G} \in C^0(0, 20) \), respectively, then we get
\[
1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n) \quad \text{if} \quad 1^2 \varphi(\varphi * F) \in C^0(\mathbb{R}^n)
\]

\( \forall t \in [0, 1] \).

**Proof.** (i) We have for each \( (x, y) \in \mathbb{R}^n \) and \( \varphi \in [0, 1] \)
\[
1^2 \varphi(\varphi * F) = \int_{\mathbb{R}^n} \varphi(z) (z, \mathbb{R}^n) \left( F(x-z, y-z) \right) \varphi(z) \, dz
\]
and (2.18) yields the result. \( \Box \)

As the integral over \( \mathbb{R}^n \) of \( 1^2 \varphi \) vanishes, in the last integral we can replace \( F(x-t^{-1/2}z, y-t^{-1/2}w) \) by \( F(x-t^{-1/2}z, y-t^{-1/2}w) \).

\[
1^2 \varphi(\varphi * F) = \int_{\mathbb{R}^n} \varphi(z) (z, \mathbb{R}^n) \left( F(x-z, y-z) \right) \varphi(z) \, dz
\]
and Proposition 2.3(i) yields the result. \( \Box \)

where \( B(x, y, t^{-1/2}) = (z, w) \in \mathbb{R}^n \mid z^2 + (w-y)^2 < t \). A similar procedure applied to \( 1^2 \varphi(\varphi * F) \) leads to

\[
1^2 \varphi(\varphi * F) = \int_{\mathbb{R}^n} \varphi(z) (z, \mathbb{R}^n) \left( F(x-z, y-z) \right) \varphi(z) \, dz
\]
\[ \int_{\mathbb{R}^n} \psi(x,y) \leq c t^{-1} \int_{B(0,\epsilon t^{1/2})} \psi(x,y) \, dx. \]

(2.20)

As \( \mathbb{N} \) vanishes, we have
\[ \int_{\mathbb{R}^n} \psi(x,y) \leq c t^{-1} \int_{B(0,\epsilon t^{1/2})} \psi(x,y) \, dx. \]

(2.21)

and similarly
\[ \int_{\mathbb{R}^n} \psi(x,y) \leq c t^{-1} \int_{B(0,\epsilon t^{1/2})} \psi(x,y) \, dx. \]

(2.22)

hence by Lemma 1.16 we easily obtain
\[ \int_{\mathbb{R}^n} \psi(x,y) \leq c t^{-1} \int_{B(0,\epsilon t^{1/2})} \psi(x,y) \, dx, \]

and by Proposition 2.3 we get (ii). If moreover \( \mathbb{N} \) vanishes, then (2.17) follows easily by (2.21), (2.22) and Lemma 1.18.

As now \( \mathbb{N} \) instead of (2.18) we can write for each \( \psi(x,y) \in \mathbb{N} \) and \( \epsilon > 0 \) (with an obvious meaning of the notations)
\[ \int_{\mathbb{R}^n} \psi(x,y) \leq c t^{-1} \int_{B(0,\epsilon t^{1/2})} \psi(x,y) \, dx. \]

(2.23)

By Lemmas 2.6, 2.7 and 2.8 it follows that condition (2.10)(ii) holds.

The proof of condition (2.10)(iv) is a little more delicate. As in the case of \( \mathbb{N} \) (Lemma 2.6), it is easily seen that \( \mathbb{N} \) can be estimated in terms of \( \mathbb{N} \) and \( \mathbb{N} \), but the \( \mathbb{N} \)-norms of these functions are not controlled by the appropriate power of \( t \). The point is that, however, such derivatives appear in the expression of \( \mathbb{N} \) in a suitable combination, which can in fact be estimated by the required power of \( t \).

Let \( \mathbb{N} \) be a solution of the equation
\[ \frac{\partial}{\partial t} \mathbb{N} = \Delta \mathbb{N} - \mathbb{N} \cdot \mathbb{N} \]

(2.24)

where \( \mathbb{N} \) is a vector field. Then we have
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.25)

By Proposition 2.3, we obtain
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.26)

and consequently
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.27)

By Proposition 2.3, we obtain
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.28)

and consequently
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.29)

By Proposition 2.3, we obtain
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.30)

and consequently
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.31)

By Proposition 2.3, we obtain
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.32)

and consequently
\[ \int_{\mathbb{R}^n} \mathbb{N} \cdot \nabla \mathbb{N} \, dx \leq c t^{-1} \int_{\mathbb{R}^n} \mathbb{N} \, dx. \]

(2.33)
After the change of variables it is easily seen that
\[
I_w(t,x,y) = \frac{1}{2} \int \left[ \sum_{i=1}^n 2 \theta_i (x,z,w) + \sum_{i=1}^n 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
\[
\cdot \left[ F(x-t/2, y-t/2, w) \cdot \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial y} \right) \right] (z,v) \cdot \left( x-t, y-v \right)
\]
\[
\left[ 2 \theta_i (x,z,w) w \right]
\]
that \( \Phi_{N, 1}^{x, y} (\mathbb{R}^n) \); then to obtain (2.25) we have to do another
replacement in (2.27), namely the term \(- \frac{1}{1 + \alpha(x)} \left[ \phi(x) - \phi(x - t^{1/2} z) \right] \)
\( \cdot F(x - t^{1/2} z, t^{1/2} w) \) has to be replaced by
\[- \frac{1}{1 + \alpha(x)} \left[ \phi(x) - \phi(x - t^{1/2} z) \right] \frac{1}{2} \left[ F(x - t^{1/2} z, t^{1/2} w) + 2 \right] \frac{1}{1 + \alpha(x)} \left[ \phi(x) - \phi(x - t^{1/2} z) \right] F(x + t^{1/2} z, t^{1/2} w) \]; then we get
\[
|w_{t}(x, y)| \leq c t^{-1/2} \left[ \int \int_{\mathbb{R}^2} \Phi_{t}^{x, y} (x, y), t^{1/2} w) \right] F(x, y, t^{1/2} w) \]
and recalling Lemma 1.16 and Proposition 2.3(iii) we check (2.25).

(iii) As now \( \Phi_{N, 1}^{1} (\mathbb{R}^n) \), we proceed in a different way. We write
\[
\phi_t^{x, y} (\mathbb{R}^n) = \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x - t^{1/2} z, t^{1/2} w) \text{d}w \text{d}z,
\]
\[
\phi_t^{x, y} (\mathbb{R}^n) (x, y) = \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x - t^{1/2} z, t^{1/2} w) \text{d}w \text{d}z,
\]
thus, starting from (2.26) we easily get for each \((x, y) \in \mathbb{R}^n \)
and \(t \in [0, 1]\):
\[
w_{t}(x, y) = \frac{1}{2} t^{-1/2} \left[ \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x - t^{1/2} z, t^{1/2} w) - \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x - t^{1/2} z, t^{1/2} w) \right] + \left(1 - \frac{1}{1 + \alpha(x)} \left[ \phi(x) - \phi(x - t^{1/2} z) \right] F(x - t^{1/2} z, t^{1/2} w) \right)
\]
\[
\leq \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x - t^{1/2} z, t^{1/2} w) \text{d}w \text{d}z + \frac{3}{2} \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \phi_t^{x, y} (x, y), t^{1/2} w) \frac{1}{1 + \alpha(x)} \left[ \phi(x) - \phi(x - t^{1/2} z) \right] F(x - t^{1/2} z, t^{1/2} w) \text{d}w \text{d}z.
\]
As \((z, w) = \phi_t (z, w), t \text{ and } (z, w) = \phi_t (z, w) \) are odd functions, we can
replace \( \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x - t^{1/2} z, t^{1/2} w) \) and
\( \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x - t^{1/2} z, t^{1/2} w) \) respectively by
\[
\frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x + t^{1/2} z, t^{1/2} w) \text{d}w \text{d}z \]
and 
\[
|w_{t}(x, y)| \leq c t^{-1/2} \left[ \int \int_{\mathbb{R}^2} \Phi_{t}^{x, y} (\mathbb{R}^n) \right] F(x, y, t^{1/2} w) \text{d}w \text{d}z + \frac{3}{2} \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x + t^{1/2} z, t^{1/2} w) \text{d}w \text{d}z.
\]
and consequently
\[
|w_{t}(x, y)| \leq c t^{-1/2} \left[ \int \int_{\mathbb{R}^2} \Phi_{t}^{x, y} (\mathbb{R}^n) \right] F(x, y, t^{1/2} w) \text{d}w \text{d}z + \frac{3}{2} \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \phi_t^{x, y} (\mathbb{R}^n) F(x + t^{1/2} z, t^{1/2} w) \text{d}w \text{d}z.
\]
By Proposition 2.3(v) we get (iii). If in addition \( \Phi_{N, 1}^{1} (\mathbb{R}^n) \),
by (2.30) we also get (2.25).

By Lemma 2.9 condition (iv) of (2.10) is proved. This concludes
Step 2.

To complete the proof of Theorem 2.1 we have just to set
\[
\mathfrak{w} = \mathfrak{w} \mid [0, 1] \times \mathbb{R}^n.
\]
As \( \mathfrak{w} \) satisfies (2.10), it is clear that \( \mathfrak{w} \) satisfies (2.2). By
Definition 1.7 and Lemmas 1.10 and 1.11, this means
\( \mathfrak{w} \in C^0 (\mathbb{R}^n), C^0 (\mathbb{R}^n) \mid_{1-0} = \) (resp. \( \mathfrak{w} \in C^0 (\mathbb{R}^n), C^0 (\mathbb{R}^n) \mid_{1-0} = \))

Theorem 2.1 is proved. \( \mathfrak{w} \)

3. THE CASE OF A HALF-SPACE WITH A NON-TANGENTIAL BOUNDARY CONDITION

In this section we consider again the half-space \( \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, -1] \), with general (non-tangential) boundary conditions.
Let $c^2(\mathbb{R}^{n-1}, \mathbb{R})$ and $\mathcal{C}^0(\mathbb{R}^{n-1}, \mathbb{R}^n)$ be such that
\[ \alpha(x) \geq 0, \quad \beta_n(x) = (\beta(x)|\mathbf{w})^n \leq -\beta_0 < 0. \]

It is not restrictive to assume that $\beta_n(x) > 1 \forall x \in \mathbb{R}^{n-1}$. Again, we denote by $(x, y)$ the points of $\mathbb{R}_1^n (x \in \mathbb{R}^{n-1}, y \geq 0)$. Setting $\beta^n(x) = (\beta_0(x), \ldots, \beta_{n-1}(x))$, we have $\mathcal{C}^0(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ and we can write $\beta(x) = (\beta^n(x), -1)$.

Define the boundary operator
\[ (Du)(x) = \alpha(x) u(x, 0) + (\mathbf{f}(x)|Du(x, 1))_n, \quad \forall x \in \mathbb{R}^{n-1}. \quad (3.1) \]

We recall the definition of the spaces (see Definitions 1.13 and 1.14)
\[ (a) \mathcal{C}^k_B(\mathbb{R}^n) = \{f \in C^k(\mathbb{R}^n) : Bf = 0\}, \quad k = 1, 2; \]
\[ (b) \mathcal{C}^{1, 0}_B(\mathbb{R}^n) = \{f \in C^{1, 0}(\mathbb{R}^n) : Bf = 0\}, \quad h^{1, 0}_B(\mathbb{R}^n) = \{f \in C^{1, 0}(\mathbb{R}^n) : \partial x \in 0\}; \]
\[ (c) \mathcal{C}^{n-1}_B(\mathbb{R}^n) = \{f \in C^{n-1}(\mathbb{R}^n) : \partial x \in 0\}; \quad h^{n-1}_B(\mathbb{R}^n) = \{f \in C^{n-1}(\mathbb{R}^n) : \partial x \in 0\}. \]

Obviously when $\beta^n = 0$, i.e. $\mathbf{b} = -\mathbf{a}$, these spaces reduce to the spaces $C^{n-1}_1(\mathbb{R}^n)$, $h^{1, 0}_1(\mathbb{R}^n)$, $C^{n-1}_1(\mathbb{R}^n)$ of Section 2. In particular, they are all Banach spaces with the norms
\[ (a) \left\| f \right\|_{\mathcal{C}^k_B(\mathbb{R}^n)}; \quad (b) \left\| f \right\|_{\mathcal{C}^{1, 0}_B(\mathbb{R}^n)}; \quad (c) \left\| f \right\|_{\mathcal{C}^{n-1}_B(\mathbb{R}^n)} + \left\| f \right\|_{h^{n-1}_B(\mathbb{R}^n)}. \]

We want to prove the following results:

**Theorem 3.1** Let $\mathcal{C}^2(\mathbb{R}^{n-1}, \mathbb{R})$, $\mathcal{C}^0(\mathbb{R}^{n-1}, \mathbb{R}^n)$ with $\alpha > 0, \beta_0 < 0$,

and set $\mathbf{z} = (\mathbf{z}^0, -1)$. If $B$ is the operator defined in (3.1), the following continuous inclusions hold:
\[ (c^2_B(\mathbb{R}^n), \mathcal{C}^0(\mathbb{R}^n))_{1-\beta} \supseteq \{(0, 2\beta) \mathbf{f} \in [0, 1/2] \}
\]

\[ (c^2_B(\mathbb{R}^n), \mathcal{C}^0(\mathbb{R}^n))_{1-\beta} \supseteq \{(0, 1) \mathbf{f} \in [0, 1/2] \}
\]

\[ (c^2(\mathbb{R}^n), \mathcal{C}^0(\mathbb{R}^n))_{1-\beta} \supseteq \{(0, 2) \mathbf{f} \in [0, 1/2] \}
\]

\[ (c^2_B(\mathbb{R}^n), \mathcal{C}^0(\mathbb{R}^n))_{1-\beta} \supseteq \{(0, 1) \mathbf{f} \in [0, 1/2] \}
\]

**Proof.** We want to reduce ourselves to the situation of the preceding section, i.e. to the case $\beta = -\mathbf{a}$. Let $\mathcal{C}^0(\mathbb{R}^{n-1})$ be a function with support contained in $[0, 1]$ and such that
\[ \psi_0 = 1, \quad 0 \leq \psi_0(x) \leq 1 \forall x \geq 1 \]

\[ \psi_0(x) = \begin{cases} \psi_0(x) = 1, & 0 \leq \psi_0(x) \leq 1 \forall x \in [0, 1] \end{cases} \]

where $\psi_0 \in D^0_{1-1}$. Consider the function $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by
\[ \psi(x) = (\xi - \phi(x)|\mathbf{w})^n \phi(x), \quad \xi \in \mathbb{R}^{n-1}, \phi(x) \geq 0 \quad (3.2) \]

It is easy to see that $\psi$ is twice differentiable in $\mathbb{R}^n$ and is one-to-one. In addition $\psi^{-1}$ is also twice differentiable since the jacobian of $\psi$ is non-singular; indeed, we have
\[ (Du)(\xi, s) = \begin{pmatrix} \int_{\mathbb{R}^{n-1}} \frac{\xi}{\psi(\xi)} |\mathbf{w}| \\ 0 \end{pmatrix} - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) \]

\[ = \begin{pmatrix} \int_{\mathbb{R}^{n-1}} \frac{\xi}{\psi(\xi)} |\mathbf{w}| \\ 0 \end{pmatrix} - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) \]

\[ = \begin{pmatrix} \int_{\mathbb{R}^{n-1}} \frac{\xi}{\psi(\xi)} |\mathbf{w}| \\ 0 \end{pmatrix} - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) \]

\[ = \begin{pmatrix} \int_{\mathbb{R}^{n-1}} \frac{\xi}{\psi(\xi)} |\mathbf{w}| \\ 0 \end{pmatrix} - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) \]

\[ = \begin{pmatrix} \int_{\mathbb{R}^{n-1}} \frac{\xi}{\psi(\xi)} |\mathbf{w}| \\ 0 \end{pmatrix} - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) \]

\[ = \begin{pmatrix} \int_{\mathbb{R}^{n-1}} \frac{\xi}{\psi(\xi)} |\mathbf{w}| \\ 0 \end{pmatrix} - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) - \frac{\xi}{\psi(\xi)} |\mathbf{w}| \phi(x) \phi(x) \phi(x) \]
and consequently \( |\det(D\psi)(\xi,s)| = |\det(I_{n-1} - \int_0^s \phi(t)dt \cdot D\delta^0(t))| \); on the other hand we have
\[
\sup_{\xi, s} |\int \phi(t)dt \cdot D\delta^0(t)| = s|D\delta^0| < 1,
\]
\((\xi, s) \in B^4_+ \cap B^4_0\).

which implies \( |\det(D\psi)(\xi,s)| > 0 \) \( \forall (\xi,s) \in B^4_+ \).

Let us define now a mapping \( T: u \mapsto v(u) \) by
\[
v(\xi,s) = T(u)(\xi,s) = (u(u)(\xi,s), \quad u \in \mathcal{C}^2(B^4_+)).
\]

**PROPOSITION 3.2** - The transformation \( T \) maps isomorphically:

1. \( C^k(B^4_+) \rightarrow C^k(B^4_+), \quad k = 0, 2; \)
2. \( C^1(B^4_+) \rightarrow C^1(B^4_+), \quad C^2(B^4_+) \rightarrow C^2(B^4_+), \quad C^3(B^4_+) \rightarrow C^3(B^4_+), \quad C^4(B^4_+) \rightarrow C^4(B^4_+); \)
3. \( C^1(B^4_+) \rightarrow C^1(B^4_+), \quad C^2(B^4_+) \rightarrow C^2(B^4_+), \quad C^3(B^4_+) \rightarrow C^3(B^4_+), \quad C^4(B^4_+) \rightarrow C^4(B^4_+); \)
4. \( C^0(B^4_+) \rightarrow C^0(B^4_+), \quad C^1(B^4_+) \rightarrow C^1(B^4_+), \quad C^2(B^4_+) \rightarrow C^2(B^4_+), \quad C^3(B^4_+) \rightarrow C^3(B^4_+), \quad C^4(B^4_+) \rightarrow C^4(B^4_+); \)

Proof. It is clear that \( T \) preserves the regularity of type \( C^k, C^{k+\alpha}, C^{k+\beta}, C^{k+\gamma}, C^{k+\delta}, C^{k+\epsilon}, \) \( k = 0, 2; \) \( \alpha = 0, 1, \beta = 0, 1, \gamma = 0, 1, \delta = 0, 1, \) \( \epsilon = 0, 1 \).

We next prove that \( u \) is differentiable and \( \nu = 0 \). Then \( v \) also is differentiable and
\[
\nu(\xi,s) = u(\xi,s) - \frac{\partial}{\partial s} u(\xi,s) - \frac{\partial}{\partial \xi} u(\xi,s)
\]
\[
(\xi,s) \in B^4_+ \cap B^4_0
\]
\[
= 0 \quad \forall (\xi,s) \in B^4_+ \cap B^4_0.
\]

Thus we have proved all statements but (iii). Now let \( \mu \in \mathcal{C}^0_1(B^4_+) \); by Lemma 1.19, we have \( u = \mu \circ u \in \mathcal{C}^0_1(B^4_+) \) and
\[
|\nu(\xi,s)| \leq |u(\xi,s)| + \frac{1}{2} |u(\xi,s)|^{1/2} \leq 1/2 |u(\xi,s)|^{1/2}
\]
\[
C^0_1(B^4_+), \quad (3.3)
\]

In addition we have
\[
|\nu(\xi,s)| \leq |u(\xi,s)|^{1/2} \quad \forall (\xi,s) \in B^4_+ \cap B^4_0
\]
\[
(3.4)
\]

and hence
\[
|\nu(\xi,s)|^2 \leq |u(\xi,s)|^{1/2} \quad \forall (\xi,s) \in B^4_+ \cap B^4_0
\]
\[
(3.5)
\]

and therefore
\[
|\nu(\xi,s)| \leq |u(\xi,s)|^{1/2} \quad \forall (\xi,s) \in B^4_+ \cap B^4_0
\]
\[
(3.6)
\]

Thus this shows that \( \mu \in \mathcal{C}^0_1(B^4_+) \) and that
\[
|\nu| \leq |u|^{1/2}
\]
\[
(3.7)
\]

Suppose now that \( \mu \in \mathcal{C}^0_1(B^4_+) \); if \( |\xi| \geq 2 + |s| \), similarly to (3.3) we easily get
\[
|\nu(\xi,s)| \leq |u(\xi,s)|^{1/2} \quad \forall (\xi,s) \in B^4_+ \cap B^4_0
\]
\[
(3.8)
\]

hence \( \nu \in \mathcal{C}^0_1(B^4_+) \). Next, by (3.4) and (3.5) we deduce that
\[
|\nu(\xi,s)| \leq |u(\xi,s)|^{1/2} \quad \forall (\xi,s) \in B^4_+ \cap B^4_0
\]
\[
(3.9)
\]

and hence
\[
|\nu(\xi,s)| \leq |u(\xi,s)|^{1/2} \quad \forall (\xi,s) \in B^4_+ \cap B^4_0
\]
\[
(3.10)
\]
which implies $\varphi_{0}^{\ast,1}(E^{2})$. The proof is complete. \hfill \Box

Theorem 1.1 follows now easily. Fix $0 \leq r \leq 1$ and take accordingly $f$ in one of the spaces $C_{0}^{0,20}(\mathbb{R}^{n})$, $C_{0}^{1,20-1}(\mathbb{R}^{n})$, $C_{0}^{1,20}(\mathbb{R}^{n})$, $h_{\mathbb{R}^{n}}^{1,20}(\mathbb{R}^{n})$, $h_{\mathbb{R}^{n}}^{1,20-1}(\mathbb{R}^{n})$. Then $T_{f}x = \overline{x}$ is in the corresponding space, specified in Proposition 3.2, and the mapping is continuous. We can now apply Theorem 2.1, obtaining $T_{f}(c_{N}^{2}(\mathbb{R}^{n}), C_{0}^{0}(\mathbb{R}^{n}))_{1-\delta}$ with continuous inclusion. By definition, this means there exists a function $\varphi \in \mathcal{F}(\mathbb{R}^{n})$ satisfying

$$
\begin{align*}
\varphi(0,0,s) & = \text{flow}(s), \\
\varphi(0,1,s) & \in C_{\text{loc}}(\mathbb{R}^{n}) \quad \text{(resp. } \varphi \in C_{\text{loc}}^{0}(\mathbb{R}^{n}))
\end{align*}
$$

and

$$
\begin{align*}
\{ \varphi(x,t,s) \mid (x,t,s) \in \mathbb{R}^{n+2} \}
\end{align*}
$$

Hence if we set $u(t,x,y) = \varphi(t,x,y)$, we easily deduce

$$
\begin{align*}
u(0,0,s) = \varphi(0,0,s) = \text{flow}(s), \\
\varphi(0,1,s) & = \text{flow}(s) \quad \text{(resp. } \varphi \in C_{\text{loc}}^{0}(\mathbb{R}^{n}))
\end{align*}
$$

and

$$
\{ \text{flow}(x) \mid (x,t,s) \in \mathbb{R}^{n+2} \}
$$

Hence we have $\varphi \in C_{\text{loc}}^{0}(\mathbb{R}^{n})$ and the continuity of the inclusion follows. Theorem 3.1 is proved. \hfill \Box

4. THE CASE OF A BOUNDED CONNECTED OPEN SET

Let $\Omega$ be a bounded connected open set of $\mathbb{R}^{n}$, $n \geq 1$, with boundary $\partial \Omega$ of class $C^1$. For each $x \in \partial \Omega$, denote by $v(x)$ the unit exterior normal at $x$; then $v \in C^2(\partial \Omega, \mathbb{R}^n)$.

Let $v \in C^2(\partial \Omega, \mathbb{R}^n)$ and $w \in C^2(\Omega, \mathbb{R}^n)$ be such that $v(x) \geq 0$, $w(x) \geq 0$ for $x \in \partial \Omega$, and define the boundary operator

$$
[Bu](x) = v(x)u(x) + (v(x)w(x))_{\Omega}, \quad x \in \partial \Omega. \quad (4.1)
$$

We will use the spaces of functions (see Definitions 1.13 and 1.14) $C_{\Omega}^{2}(\partial \Omega), C_{\partial \Omega}^{1,2}(\partial \Omega), h_{\partial \Omega}^{1,2}(\partial \Omega), h_{\partial \Omega}^{1,2-1}(\partial \Omega)$, as well as the Hölder spaces $C^{0,\alpha}(\partial \Omega), h^{0,\alpha}(\partial \Omega)$.

Our goal is to prove the following result:

**THEOREM 4.1** - Let $\Omega \subset \mathbb{R}^{n}$ be a bounded connected open set, with $\partial \Omega$ of class $C^1$. Let $v \in C^2(\partial \Omega, \mathbb{R}^n)$, $w \in C^2(\Omega, \mathbb{R}^n)$ with $\alpha > 0$ and $(v)_{\Omega} \geq \delta > 0$. If $B$ is the operator defined in (4.1), then the following continuous inclusions hold:

$$
\begin{align*}
& C_{\Omega}^{2}(\partial \Omega), C_{\partial \Omega}^{1,2}(\partial \Omega) & \hookrightarrow & C_{\partial \Omega}^{0,\alpha}(\partial \Omega), C_{\partial \Omega}^{1,2-1}(\partial \Omega)
& C_{\partial \Omega}^{2}(\partial \Omega), C_{\partial \Omega}^{1,2}(\partial \Omega) & \hookrightarrow & C_{\partial \Omega}^{0,\alpha}(\partial \Omega), C_{\partial \Omega}^{1,2-1}(\partial \Omega)
& h_{\partial \Omega}^{0,\alpha}(\partial \Omega) & \hookrightarrow & h_{\partial \Omega}^{1,2-1}(\partial \Omega)
& h_{\partial \Omega}^{1,2-1}(\partial \Omega) & \hookrightarrow & h_{\partial \Omega}^{1,2-1}(\partial \Omega)
\end{align*}
$$

**Proof.** Fix $\delta > 0$ and, accordingly, take $f \in C_{\partial \Omega}^{0,2\alpha}(\partial \Omega), C_{\partial \Omega}^{1,2\alpha}(\partial \Omega)$ or $C_{\partial \Omega}^{1,2-1,2\alpha}(\partial \Omega)$ (resp. $h_{\partial \Omega}^{0,2\alpha}(\partial \Omega), h_{\partial \Omega}^{1,2-1,2\alpha}(\partial \Omega)$ or $h_{\partial \Omega}^{1,2-1,2\alpha}(\partial \Omega)$).
According with Definition 1.7 and Lemmata 1.10 and 1.11, we look for a function \( w(t,x) \in C^0([0,1] \times \Omega) \) such that:

\[
\begin{align*}
&\sup_{t \in [0,1]} t^{-3} |w(t,\cdot)|_C^0(\Omega) = 0, \\
&\sup_{t \in [0,1]} t^{-3} |\partial_t w(t,\cdot)|_C^0(\Omega) = 0.
\end{align*}
\]

Our method consists in transforming the given function \( f \), by an usual localization argument, into a finite set of functions \( \{F_j, G_j\} \) of two different kinds:

(a) a function \( F_j \), as smooth as \( f \), defined in a ball, with zero boundary conditions;

(b) a function \( G_j \), as smooth as \( f \), defined in the half space \( \mathbb{R}^n_+ \) and satisfying \( \partial_n G_j = 0 \) on the boundary whenever \( \partial_n f = 0 \) on \( \partial \Omega \); here \( \partial_n \) is a suitable first-order differential operator of type (4.1).

The localization argument is not completely standard, for it is carried on by the construction of a finite partition of unity in \( \overline{\Omega} \) with special properties along \( \partial \Omega \); namely, we need that the localization functions near \( \partial \Omega \) transform the boundary operator \( \partial_n \) into an operator \( \partial_j \) of the same kind. This is done by choosing the functions \( F_j \), localizing near \( \partial \Omega \), in such a way that at each \( x \in \partial \Omega \) their gradients \( \nabla F_j(x) \) are orthogonal to the vector \( \partial_n(x) \) appearing in (4.1). This construction is performed in the first step of the proof.

The second step is the verification that the localized functions \( F_j \) or \( G_j \) in fact satisfy the conditions stated either in case (a) or in case (b). In case (a), \( F_j \) has compact support in a ball \( U_j \), so we apply a result of Lunardi [13] to prove that \( F_j \in C^0_0(U_j), C^0(U_j), x \rightleftharpoons (\text{resp. } F_j \in C^0_0(U_j), C^0(U_j)) \) in case (b) we can apply the results of Section 3, obtaining \( G_j \in C^0_0(\mathbb{R}^n_+), C^0(\mathbb{R}^n_+) \) in case (a), we have \( F_j = w_j(0), G_j = \nu_j(0) \) for some suitable functions \( w_j(t), \nu_j(t) \).

In the final step we show that the function \( w(t) \), which is obtained by gluing together the functions \( w_j, \nu_j \), in fact satisfies (4.2).

**Step 1** - Here we will construct the required partition of unity. We start from the localizing functions near \( \partial \Omega \); for each \( x \in \partial \Omega \) the construction gives a suitable function \( \mu \), defined in a certain neighbourhood \( U \) of \( x \) in \( \overline{U} \); a compactness argument then yields a finite number of localizing functions \( \mu_j \) with the required properties. Next, we complete, in a standard way, the set of localizing functions by a finite number of suitable functions \( \eta_j \), defined in balls contained in \( U \). Finally we normalize the functions \( \eta_j, \mu_j \), obtaining the desired partition of unity.

To begin with, fix \( x_0 \in \partial \Omega \). Our first goal is to construct two neighbourhoods \( V, V' \) of \( x_0 \) in \( \overline{U} \) with \( V \subseteq V' \), and a function \( \mu: \overline{U} \to \mathbb{R} \) with the following properties:

\[
\begin{align*}
\mu &\geq 0 \text{ in } \overline{U}, \\
\mu &\geq 1 \text{ in } \overline{V}, \\
\mu &\leq 0 \text{ in } V' \setminus \overline{V}, \quad \mu \equiv 1 \text{ in } V.
\end{align*}
\]

And

\[
(\mathbf{u}(x), \partial_n \mathbf{u}(x)) = 0 \quad \forall x \in \partial \Omega.
\]

(4.4)
To do this, first of all note that, since \( \varphi \) is of class \( C^2 \), there exists a neighbourhood \( W \) of \( x_0 \) in \( \mathbb{R}^n \) and a diffeomorphism \( \psi: W \rightarrow \mathbb{R}^n \) of class \( C^3 \), such that, denoting \( W'\bar{\varphi} \) by \( W' \):

\[
\begin{align*}
\psi(W'\varphi) &= \mathbb{R}^n \\
\psi(W'\varphi) &= \mathbb{R}^n \\
\end{align*}
\]

\[
(4.5)
\]

Set

\[
\gamma_n(\xi) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\psi^{-1}(\xi))_i \psi^{-1}(\xi), \quad \xi \in W, \quad n = 1, \ldots, n; \tag{4.6}
\]

then \( \gamma \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) and it is easily seen that

\[
((D_u \psi^{-1}(\xi)) \varphi \psi^{-1}(\xi))_n = (D(\psi^{-1}(\xi)) \varphi(\psi^{-1}(\xi)))_n, \quad \forall \xi \in W, \quad \forall \varphi \in C^1(\mathbb{R}^n). \tag{4.7}
\]

Moreover, it is not difficult to verify that

\[
\gamma_n(\xi) = (\psi^{-1}(\xi))_n, \quad (D(\psi^{-1}(\xi)) \varphi^{-1}(\xi))_n = \frac{1}{(D(\psi^{-1}(\xi)) \varphi^{-1}(\xi))} \psi^{-1}(\xi), \quad \forall \xi \in W, \quad \forall \varphi \in C^1(\mathbb{R}^n),
\]

which implies

\[
\gamma_n(\xi) \leq \frac{1}{\delta} < 0, \quad \forall \xi \in W. \tag{4.8}
\]

Thus, the non-tangential vector \( \beta(\xi) \) is transformed by \( \psi \) into the non-tangential vector \( \gamma(\xi) \) given by (4.6). Hence, we can define

\[
\lambda_n(\xi) = -\gamma_n(\xi), \quad \forall \xi \in W, \quad n = 1, \ldots, n. \tag{4.9}
\]

Thus \( \lambda \) is twice differentiable, and, by (4.7)

\[
((D_u \psi^{-1}(\xi)) \varphi \psi^{-1}(\xi))_n = (D(\psi^{-1}(\xi)) \varphi(\psi^{-1}(\xi)))_n, \quad \forall \xi \in W, \quad \forall \varphi \in C^1(\mathbb{R}^n). \tag{4.10}
\]

Next, we want to transform the vector \( \lambda_n \), defined in (4.9), into \(-e^j\); hence we need to construct a function \( \omega: \mathbb{R}^n \rightarrow \mathbb{R}^n \) just as in (3.2) of Section 3, with \( \xi \) replaced here by \( \lambda \). To be more precise, write \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \xi = (\xi_1, \ldots, \xi_n) \), so that

\[
\xi = (\xi_1, \ldots, \xi_n), \quad \lambda = (\lambda_1, \ldots, \lambda_n), \quad \xi_0 = (\xi_0, \xi_1, \ldots, \xi_n), \quad \lambda_0 = (\lambda_0, \lambda_1, \ldots, \lambda_n), \quad \xi_0 = (\xi_0, 0, 0, \ldots, 0), \quad \lambda_0 = (\lambda_0, 0, 0, \ldots, 0).
\]

Then \( \omega: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is twice differentiable and one-to-one, with \( \omega^{-1} \) also twice differentiable; note that

\[
\omega(\xi) = \xi_0 - \lambda_0, \quad \omega(\xi) = (\xi_0, 0), \quad \forall \xi \in W. \tag{4.11}
\]

Now let \( \xi \in (0, \epsilon) \) be such that \( 0 < \epsilon < 1 \), \( \xi_0 \) outside \( (0, 1) \), \( \epsilon = 1 \) in \([0, 1/2)\), and set

\[
\xi(\xi) = (\xi_0), \quad \xi_0 \in \mathbb{R}^n.
\]

Then, clearly, \( \xi \in C^0(\mathbb{R}^n) \), \( 0 < \epsilon \leq 1, \) \( \xi_0 \) outside \( B^+(0, 1) \), \( \xi_0 \) in \( B^+(0, 1/2) \), and moreover

\[
\frac{\partial}{\partial \xi} \xi_0 = 0, \quad \xi_0 \in \mathbb{R}^n. \tag{4.12}
\]

i.e. the gradient of \( \xi \) is orthogonal to \(-e^j\). By (4.12) and (4.11) we easily get

\[
(D(\psi^{-1}(\xi)) \lambda(\xi))_n = 0, \quad \forall \xi \in W. \tag{4.13}
\]

We are finally ready to define the desired function \( \psi \) satisfying (4.3) and (4.4); set

\[
\]
\[
\mu(x) = \begin{cases} 
\text{conj}(\phi(x)) & \text{if } x \in S, \\
0 & \text{if } x \notin \overline{S},
\end{cases}
\]  
(4.14)

then it is easily seen that (4.3) holds with

\[
V' = \phi^{-1}(u(B^+(0,1/2))), \quad V'' = \phi^{-1}(u(B^+(0,1/2))),
\]

while by (4.13), (4.9) and (4.7) we get (4.4).

Up to now, for each fixed \( x_0 \in \overline{S} \), we have constructed the function \( \mu \) given by (4.14), which satisfies (4.3) and (4.4), as well as the sets \( V', V'' \), verifying \( V' \subseteq V'' \subseteq V \) and the functions \( \psi, \gamma, \lambda \) given respectively by (4.5), (4.6), (4.9), for which (4.8) and (4.10) hold. By the compactness of \( S \), we can select a finite number of points \( x_0, x_1, \ldots, x_k \), such that the corresponding neighbourhoods \( V_{x_0}, V_{x_1}, \ldots, V_{x_k} \), cover \( S \). Accordingly, we have also selected the corresponding functions \( \psi_{x_0}, \gamma_{x_0}, \lambda_{x_0} \) and the sets \( V_{x_0}, V_{x_1} \); in particular, for \( j = 1, \ldots, k \), we have \( V' \subseteq V_j \subseteq V'' \) and:

\[
\psi_j(V_j) = \mathbb{R}^n, \quad \psi_j^{-1}(\mathbb{R}^n) = \xi_j, \quad \psi_j^{-1} \in C^3(\mathbb{R}^n).
\]  
(4.15)

\[
\gamma_j(\xi) = (D\psi_j)(\psi_j^{-1}(\xi)) \cdot \psi_j^{-1}(\xi), \quad \forall \xi \in \xi_j.
\]  
(4.16)

\[
(x_j(\xi) = \gamma_j(\xi)|x|_n \leq \delta < 0, \quad \forall \xi \in \xi_j.
\]  
(4.17)

\[
\lambda_j(\xi) = \frac{1}{(\gamma_j(\xi)|x|_n - \delta_j)}, \quad \forall \xi \in \xi_j.
\]  
(4.18)

\[
\{\text{soft}_{\lambda_j(\xi)}(\xi) \} \cdot (\text{def}(x) + \text{soft}_{\lambda_j(\xi)}(\xi) \cdot (\gamma_j(\xi)(\xi)|x|_n \cdot \text{def}(x)) \} \cdot (\gamma_j(\xi)(\xi)|x|_n \cdot \text{def}(x))
\]  
(4.19)

\[
\psi_j \in C^\infty, \quad \psi_j(x) \in \psi_j(V_j). 
\]  
(4.20)

\[
\mu_j \in C^2, \quad 0 < \mu_j < 1
\]

The functions \( \mu_j \) are the required localizing functions near \( S \).

To complete the set of localizing functions, we are to find \( \psi_j \subseteq \psi_j(V_j) \) such that \( \cup_j \psi_j(V_j) \) is a finite set of open balls \( \{ U_j, 1 \leq j \leq m \} \), such that \( U_j \subseteq U_j \cdot V_j \), and select another family of open sets \( \{ U_j, 1 \leq j \leq m \} \) satisfying \( \cup_j U_j(1 \leq j \leq m) \). Finally, we take for \( j = 1, \ldots, m \) a function \( \eta_j \in C^2(\mathbb{R}^n) \) such that \( \eta_j = 0 \) outside \( U_j \), \( \eta_j = 1 \) in \( U_j \).

The set \( \{ x_j, 1 \leq j \leq m \} \) is the required set of localizing functions. To conclude Step 1, we have only to localize the functions \( \psi_j, \eta_j \) in order to get a partition of unity. Thus, we set:

\[
\sigma_j(x) = \sum_{i=1}^m \eta_j(x) \cdot \psi_j(x)
\]  
(4.22)

Clearly \( \sigma_j \in C^2(\mathbb{R}^n) \) for \( i = 1, \ldots, m; \ j = 1, \ldots, k \), and:

\[
\begin{cases}
\sigma_j(x) x_i \in \Omega_j, \quad \text{in } \overline{\Omega_j}, \\
\sigma_j(x) x_i \in \Omega_j, \quad \text{in } \Omega_j,
\end{cases}
\]  
(4.23)

in addition, by (4.21) it follows that:

\[
(\sigma_j(x) x_i | D\psi_j(x) = 0 \quad \forall \Omega_j, \quad j = 1, \ldots, k.
\]  
(4.24)

Step 1 is finished.

Step 2. Here we reduce our problem to the cases (a) and (b)
mentioned at the beginning of the proof. Let
\[
\begin{align*}
C^0,20(\Omega) \text{ (resp. } h^0,20(\Omega)) & \quad \text{if } \theta \in ]0,1/2[ \\
C^0,1(\Omega) \text{ (resp. } h^0,1(\Omega)) & \quad \text{if } \theta = 1/2 \\
C^{1,20-1}(\Omega) \text{ (resp. } h^{1,20-1}(\Omega)) & \quad \text{if } \theta \in ]1/2,1[ \\
\end{align*}
\]
we localize \( f \) by setting
\[
\bar{f}_k(x) = \begin{cases} 
\gamma_k(x)f(x) & \text{if } x \in U_k \\
0 & \text{if } x \notin U_k
\end{cases}
\]
(4.25)
\[
G_j(\xi) = \sum_{j=1}^{k} \frac{C^0,20(\xi_j)}{(\gamma_j)_n(\xi)} a_{\gamma_j}(\xi_j) [\partial u_j(\xi)], \; \xi \in \mathbb{R}^n, \; j = 1, \ldots, k.
\]
(4.26)
Define
\[
\Lambda_j u_j(\xi) = \frac{a_{\gamma_j}^{-1}(\xi)}{(\gamma_j)_n(\xi)} u(\xi) + \gamma_j(\xi) [\partial u_j(\xi)], \; \xi \in \mathbb{R}^n, \; j = 1, \ldots, k.
\]
(4.27)
then we have:

**PROPOSITION 4.2** For \( i = 1, \ldots, m \) and \( j = 1, \ldots, k \) we have:
\[
\bar{f}_k|_{U_k} \in \begin{cases} 
C^0,28(\mathbb{R}^n) \text{ (resp. } h^0,28(\mathbb{R}^n)) & \text{if } \theta \in ]0,1/2[ \\
C^{0,1}(\mathbb{R}^n) \text{ (resp. } h^{0,1}(\mathbb{R}^n)) & \text{if } \theta = 1/2 \\
C^{1,28-1}(\mathbb{R}^n) \text{ (resp. } h^{1,28-1}(\mathbb{R}^n)) & \text{if } \theta \in ]1/2,1].
\end{cases}
\]
\[
G_j \in \begin{cases} 
C^0,20(\mathbb{R}^n) \text{ (resp. } h^0,20(\mathbb{R}^n)) & \text{if } \theta \in ]0,1/2[ \\
C^{0,1}(\mathbb{R}^n) \text{ (resp. } h^{0,1}(\mathbb{R}^n)) & \text{if } \theta = 1/2 \\
C^{1,20-1}(\mathbb{R}^n) \text{ (resp. } h^{1,20-1}(\mathbb{R}^n)) & \text{if } \theta \in ]1/2,1].
\end{cases}
\]

Moreover:
\[
\sum_{i=1}^{m} \bar{f}_k|_{U_k} + \sum_{j=1}^{k} G_j \leq c k.
\]

in the corresponding norms.

**Proof.** Obviously, \( f \) has compact support in \( U_k \), and its regularity follows easily (for the case \( \theta = 1/2 \) recall Lemma 1.18), as well as the required estimates.

About \( G_j \), the result is obvious if \( \theta \in ]0,1/2[ \). Suppose now \( \theta = 1/2 \): by Lemmata 1.19 and 1.18 it is clear that \( G_j \in C^{0,1}(\mathbb{R}^n) \) (resp. \( G_j \in \mathcal{L}^1(\mathbb{R}^n) \)) and \( G_j \in C^{0,1}(\mathbb{R}^n) \) (resp. \( G_j \in \mathcal{L}^1(\mathbb{R}^n) \)) so we have only to show that
\[
\sup_{\xi \in \mathbb{R}^n} \left| \frac{G_j(\xi - \gamma_j(\xi)) - G_j(\xi)}{\xi} \right|_{C^{0,1}} \leq c k.
\]
(4.28)
and
\[
\lim_{c \to 0^+} \frac{G_j(\xi - \gamma_j(\xi)) - G_j(\xi)}{\xi} = - \frac{a_{\gamma_j}^{-1}(\xi)}{(\gamma_j)_n(\xi)} G_j(\xi)
\]
(4.29)
First, we observe that \( \psi_j^{-1} \in C^3(\mathbb{R}^n) \) and that, by (4.16) and (4.18), we have
\[
D\psi_j^{-1}(\xi) \cdot \gamma_j(\xi) = - \frac{1}{(\gamma_j)_n(\xi)} g(\psi_j^{-1}(\xi)) \quad \forall \xi \in \mathbb{R}^n,
\]
(4.30)
and hence by Taylor's formula
\[
\psi_j^{-1}(\xi - \gamma_j(\xi)) - \psi_j^{-1}(\xi) = \sum_{i=1}^{m} \frac{1}{(\gamma_j)_n(\xi)} \left[ D\psi_j^{-1}(\xi) \right] \gamma_j \left[ D^2\psi_j^{-1}(\xi) \right] \gamma_j \left[ D^3\psi_j^{-1}(\xi) \right] \gamma_j \left[ D^4\psi_j^{-1}(\xi) \right] \gamma_j \left[ D^5\psi_j^{-1}(\xi) \right] \gamma_j
\]
(4.31)
\( \forall \xi \in \mathbb{R}^n \).

Next, by (4.30) and (4.24),
\[
(D\gamma_j^{-1})(\xi) \left[ D\gamma_j^{-1}(\xi) \right] = (D\gamma_j^{-1})(\xi) \left[ D\gamma_j^{-1}(\xi) \right] \gamma_j \left[ D^2\gamma_j^{-1}(\xi) \right] \gamma_j \left[ D^3\gamma_j^{-1}(\xi) \right] \gamma_j \left[ D^4\gamma_j^{-1}(\xi) \right] \gamma_j \left[ D^5\gamma_j^{-1}(\xi) \right] \gamma_j \left[ D^6\gamma_j^{-1}(\xi) \right] \gamma_j \left[ D^7\gamma_j^{-1}(\xi) \right] \gamma_j \left[ D^8\gamma_j^{-1}(\xi) \right] \gamma_j
\]
(4.32)
\( \forall \xi \in \mathbb{R}^n \).
Now observe that, by (4.17) and Lemma 1.15, for each \( \sigma \in \mathbb{R} \) the point \( \psi_j^{-1}(\xi) + \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \) certainly lies in \( \mathbb{R}_+ \), provided \( \sigma \) is sufficiently small, say \( \sigma \in (0, \sigma_0) \). Consequently we can write

\[
G_j(\xi - \psi_j^{-1}(\xi)) = \frac{1}{\sigma} \left[ \rho_j \psi_j^{-1}(\xi - \psi_j^{-1}(\xi)) - \rho_j \psi_j^{-1}(\xi) \right] \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) + \right.
\]

\[
+ \rho_j \psi_j^{-1}(\xi) \cdot \frac{1}{\sigma} \left[ f(\psi_j^{-1}(\xi)) - f(\psi_j^{-1}(\xi)) + \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right] \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
(4.33)
\]

Hence, if \( \xi \in \mathbb{R}_+^{1,1} \) by (4.31) we easily deduce that

\[
\frac{G_j(\xi - \psi_j^{-1}(\xi)) - G_j(\xi)}{\sigma} \geq \begin{cases} 1 - \frac{1}{2} \| \psi_j^{-1}(\xi) \|_{1,1} & \text{if } \sigma \geq 1 \\
0 & \text{if } \sigma < 1 \end{cases}
\]

\[
\forall \sigma \in (0, \sigma_0), \forall \xi \geq 0.
\]

By Remark 1.4(iii), this yields (4.26), so that \( G_j \in C^{1,2}_{\mathbb{R}_+} \) and the required estimate holds.

Suppose now, in addition, that \( \mathbb{R}_+^{1,1} \) lies in \( \mathbb{R}_+^{1,1} \); then by (4.33), (4.32) and (4.31), recalling that in particular \( \mathbb{R}_+^{1,1/2} \) we get as \( \sigma \to 0^+ \)

\[
G_j(\xi - \psi_j^{-1}(\xi)) - G_j(\xi) = \lim_{\sigma \to 0^+} \left[ \rho_j \psi_j^{-1}(\xi) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
+ \rho_j \psi_j^{-1}(\xi) \cdot \frac{1}{\sigma} \left[ f(\psi_j^{-1}(\xi)) - f(\psi_j^{-1}(\xi)) + \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right] \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
\forall \sigma \in (0, \sigma_0).
\]

This proves (4.29), and hence \( G_j \in C^{1,2}_{\mathbb{R}_+} \). The proof for the case \( \sigma = 1/2 \) is complete.

Suppose finally \( \sigma = 1/2, \xi \) it is easy to see that

\[
G_j \in C^{1,2}_{\mathbb{R}_+} \quad \text{(resp. } G_j \in C^{1,2}_{\mathbb{R}_+} \text{)}.
\]

The required estimate holds. Hence it is enough to verify that \( A_j = 0 \), with \( A_j \) given by (4.27). Now if \( \xi \in \mathbb{R}_+ \) we get, by (4.19), (4.24) and (4.32)

\[
A_j G_j(\xi) = \frac{1}{(\gamma_j^{-1}(\xi)) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
- (\gamma_j^{-1}(\xi) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
- (\gamma_j^{-1}(\xi) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
= - \frac{1}{(\gamma_j^{-1}(\xi)) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
= - \frac{1}{(\gamma_j^{-1}(\xi)) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \left( \frac{\sigma}{(\gamma_j^{-1}(\xi) + \beta(\psi_j^{-1}(\xi)) \sigma} \right) \right]
\]

\[
0.
\]

This shows that \( G_j \in C^{1,2}_{\mathbb{R}_+} \) (resp. \( G_j \in C^{1,2}_{\mathbb{R}_+} \)), and the proof is complete. •

By a result of Lunardi ([13], Proposition 2.5) we have for \( i = 1, \ldots, m \)

\[
\frac{d}{dt} \left[ G_j(C_0^0(U_j), C_0^0(U_j))_{L^2} \right] = \left( \frac{d}{dt} \left[ G_j(C_0^0(U_j), C_0^0(U_j))_{L^2} \right] \right)
\]

while by Theorem 3.1 we get for \( j = 1, \ldots, k \)

\[
G_j \in C^{1,2}_{\mathbb{R}_+} \quad \text{(resp. } G_j \in C^{1,2}_{\mathbb{R}_+} \text{)},
\]

and in addition
Hence by Definition 1.7 there exist functions \( z_1, v_j \) satisfying
\[
\begin{cases}
  z_1 \in C^0((0,1), C^0(\overline{U}_1)), & z_1(0) = f_1 \\
  v_j \in C^0((0,1), C^0(\overline{U}_j)), & v_j(0) = g_j
\end{cases}
\]
\begin{align*}
&z_1 \in C_{-\epsilon}^1((0,1), C^0(\overline{U}_1)) \quad \text{(resp. } z_1 \in C_{-\epsilon}^1((0,1), C^0(\overline{U}_1)) \text{)} \quad \text{if } \emptyset \in \{0,1/2\} \\
&z_1 \in C_{1-\epsilon}^1((0,1), C^0(\overline{U}_1)) \quad \text{(resp. } z_1 \in C_{1-\epsilon}^1((0,1), C^0(\overline{U}_1)) \text{)} \quad \text{if } \emptyset = 1/2
\end{align*}

and
\begin{align*}
&v_j \in C^0((0,1), C^0(\overline{U}_j)) \\
&v_j \in C_{-\epsilon}^1((0,1), C^0(\overline{U}_j)) \quad \text{(resp. } v_j \in C_{-\epsilon}^1((0,1), C^0(\overline{U}_j)) \text{)} \quad \text{if } \emptyset < 1 \\
&v_j \in C_{1-\epsilon}^1((0,1), C^0(\overline{U}_j)) \quad \text{(resp. } v_j \in C_{1-\epsilon}^1((0,1), C^0(\overline{U}_j)) \text{)} \quad \text{if } \emptyset = 1
\end{align*}

Finally we extend the functions \( z_1(t, \cdot) \) to the whole \( \Omega \) by defining
\[
u_0(t, x) = z_1(t, x) \quad \text{if } t \in [0,1], x \in \overline{U}_1 \\
0 \quad \text{if } t \in [0,1], x \not\in \overline{U}_1
\]
where for \( i=1, \ldots, m \), \( \nu_i \in C^0(\Omega, \mathbb{R}) \) is a function with support contained in \( U_i \) and such that \( \nu_i \in U_i^1 \). (Compare with (4.22) and (4.25)). It is clear that
\[
\begin{cases}
  \nu_0 \in C^0((0,1), C^0(\overline{U})) \\
  \nu_0 \in C_{1-\epsilon}^1((0,1), C^0(\overline{U})) \quad \text{(resp. } \nu_0 \in C_{1-\epsilon}^1((0,1), C^0(\overline{U})) \text{)} \quad \text{if } \emptyset < 1/2 \\
  \nu_0 \in C_{1/2}^1((0,1), C^0(\overline{U})) \quad \text{if } \emptyset = 1/2
\end{cases}
\]

This concludes Step 2.

Step 3. In order to construct a function \( w(t, \cdot) \) satisfying (4.2) we just glue together the functions (4.37) and (4.36), setting
\[
\begin{align*}
  w(t, x) &= \sum_{i=1}^{m} \nu_i(t, x) + \nu_0(t, \cdot), & t \in [0,1], x \in \Omega. \\
  w(t, x) &= f(x), & x \in \Omega.
\end{align*}
\]

We have to show that (4.2) holds. By (4.25), (4.26), (4.37) and (4.36), recalling (4.22) and (4.23), it is clear that \( w \in C^0([0,1], \Omega) \) and \( w(0, x) = f(x) \) \( \forall x \in \Omega \).

By (4.38) it is also clear that \( w \in C^0((0,1), C^0(\overline{U})) \) (resp. \( C_{-\epsilon}^1((0,1), C^0(\overline{U})) \)) and, by (4.34)
\[
\sup_{t \in [0,1]} \epsilon^q \|w(t, \cdot)\|_{C^{0}(\overline{U})} + \sup_{t \in [0,1]} \epsilon^{q-1} \|w(t, \cdot)\|_{C^{1}(\overline{U})} < \infty
\]

\[
\begin{cases}
  \epsilon^{q-1} \|w(t, \cdot)\|_{C^{1}(\overline{U})} \leq \epsilon^{q-1} \|w(t, \cdot)\|_{C^{0}(\overline{U})} \quad \text{if } \emptyset < 1 \\
  \epsilon^{q-1} \|w(t, \cdot)\|_{C^{1}(\overline{U})} \leq \epsilon^{q-1} \|w(t, \cdot)\|_{C^{0}(\overline{U})} \quad \text{if } \emptyset = 1
\end{cases}
\]

It remains to verify that \( w(t, \cdot) = 0 \) \( \forall t \in [0,1] \). Now, for each \( x \in \Omega \) we get
\[ [Bw(t, \cdot)](x) = \alpha(x)w(t, x) + (8(x)[Dw(x)]_n = \]
\[ k = \sum_{j=1}^{k} p_j(x)[(\alpha(x)v_j(t, x) + (8(x)[Dv_j(t, x)])]_n + \]
\[ \sum_{j=1}^{k} (8(x)[Dp_j(x)]_n, v_j(t, \cdot)) \]
and recalling (4.36), (4.24) and (4.27)
\[ \frac{[Bw(t, \cdot)](x)}{[Bv(t, \cdot)](x)} = \sum_{j=1}^{k} p_j(x)[(\alpha(x)v_j(t, x) - (8(x)[Dv_j(t, x)])]_n \]
\[ \cdot (8(x)(\alpha(x)[v_j(t, x)])_n \cdot (8(x)[Dv_j(t, x)])_n, v_j(t, \cdot)) \]
\[ = \sum_{j=1}^{k} \left[ (8(x)[(v_j(t, x)])_n \cdot (8(x)[Dv_j(t, x)])_n, v_j(t, \cdot)) \right] \frac{v_j(t, x)}{v_j(x)} = 0 \]

since \( v_j(t, \cdot) \in C^2(\mathbb{R}^n) \).

Thus we have shown that the function \( w \) defined in (4.37) satisfies (4.2). Hence \( \mathcal{E} \in (C^2(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta}, \phi \in (C^2(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta}, \) and
\[
\begin{cases}
\text{if } & C^{0,1-\theta}(\overline{\Omega}) \quad \text{if } \theta \in [0, 1/2] \\
\text{if } & C^{0,\theta}(\overline{\Omega}) \quad \text{if } \theta = 1/2 \\
\text{if } & C^1(\overline{\Omega}) \quad \text{if } \theta \in ]1/2, 1[.
\end{cases}
\]

Theorem 4.1 is proved.

5. THE REVERSE INCLUSION

Let \( \Omega \) be a bounded open set of \( \mathbb{R}^n \), \( n \geq 1 \), with boundary \( \partial \Omega \) of class \( C^2 \). Consider the differential operator with complex-valued coefficients, defined by

\[ [Bw(t, \cdot)](x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial w}{\partial x_i}(x) + c(x)w(x), \quad (5.1) \]

under the following assumptions:
\[ \text{(A.1) - (Strong uniform ellipticity). There exists } \nu > 0 \text{ such that } \]
\[ \Re \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall \Omega. \]

\[ \text{(A.2) - The coefficients } a_{ij}, b_i, c \text{ belong to } C^0(\overline{\Omega}, \mathbb{C}) \text{ for } i,j=1, \ldots, n. \]

Consider also the boundary differential operator with real-valued coefficients, defined by

\[ [Bw(t, \cdot)](x) = \sum_{i=1}^{n} a_i(x)u_i(x) + \sum_{i=1}^{n} b_i(x)u_i(x) + c(x)u(x), \quad (5.2) \]

under the following assumptions:
\[ \text{(B.1) - Denote by } v(x) \text{ the unit exterior normal vector at } x \in \partial \Omega; \therefore \text{ exists } \delta > 0 \text{ such that } \]
\[ \alpha(x) \geq \delta, \quad (8(x)[v(x)])_n \geq \delta \quad \forall x \in \partial \Omega. \]

\[ \text{(B.2) - The coefficients } a_i, b_i \text{ belong to } C^2(\Omega, \mathbb{R}) \text{ for } i=1, \ldots, n. \]

Under these hypotheses the pair \( [\mathbb{R}, B] \) is a very special example of the situation considered by Stewart [20]. Hence if we set
\[
\begin{cases}
D(A) = \{u \in C^0(\overline{\Omega}), 8u \in C^0(\overline{\Omega}), 8u = 0\} \\
\text{with } \{ A u = [Bw(t, \cdot)](x) \},
\end{cases}
\]

by Theorem 1 of [20] we get that \( A \) is the infinitesimal generator of an analytic semi-group in the space \( C^0(\overline{\Omega}) \); more precisely we have:

**Lemma 5.1** - Let \( \Omega \) be a bounded connected open set of \( \mathbb{R}^n \), with
Suppose that the operators $A, B$ defined by (5.1) and (5.2) satisfy (A.1), (A.2), (B.1) and (B.2). There exists $k_0 > 0$ such that for each $x, t \in H$ the following estimate holds:

$$
\sup \{ \left| \langle A, x \rangle \right| : x \in X \} \leq k_0 \| x \|_H \quad \forall x \in H,
$$

where $\| x \|_H$ is the norm in the intermediate space $H$. For each $x, t \in H$, we have

$$
\| x \|_H \leq \| x \|_H + \| x \|_G.
$$

By Remark 1.9 it follows that a norm in the intermediate space $D(A) \cap C_0^0(H)$ is given by

$$
\| x \|_H = \sup \left\{ \frac{\| A^* x \|}{\| x \|} : x \in C_0^0(H) \right\},
$$

where

$$
\| x \|_G = \sup \left\{ \frac{\| A x \|}{\| x \|} : x \in C_0^0(H) \right\}.
$$

In addition, if we set

$$
\| x \|_M = \sup \left\{ \frac{\| A^* x \|}{\| x \|} : x \in C_0^0(H) \right\},
$$

then

$$
D(A) = C_0^0(H) \quad \text{and} \quad \lim_{n \to \infty} \| x \|_M = 0.
$$

Obviously we have also

$$
\| x \|_M \leq \| x \|_H \leq \| x \|_G.
$$

This section is concerned with the proof of the following result:

**Theorem 5.2.** Under the assumptions of Lemma 5.1, let $\mathcal{D}(A, B, C)$ be defined by (5.3); moreover if $\varphi = \varphi(t)$ then exists $c_0 > 0$.

**Proof.** See Theorem 1 of [20]; see also Acquistapace-Terrani [1], Lemma 7.7. C

By Remark 1.9 it follows that a norm in the intermediate space $D(A) \cap C_0^0(H)$ is given by

$$
\| x \|_H = \sup \left\{ \frac{\| A^* x \|}{\| x \|} : x \in C_0^0(H) \right\},
$$

where

$$
\| x \|_G = \sup \left\{ \frac{\| A x \|}{\| x \|} : x \in C_0^0(H) \right\}.
$$

In addition, if we set

$$
\| x \|_M = \sup \left\{ \frac{\| A^* x \|}{\| x \|} : x \in C_0^0(H) \right\},
$$

then

$$
D(A) = C_0^0(H) \quad \text{and} \quad \lim_{n \to \infty} \| x \|_M = 0.
$$

Obviously we have also

$$
\| x \|_M \leq \| x \|_H \leq \| x \|_G.
$$

Finally, we consider the auxiliary function defined by

$$
\nu(t) = \begin{cases} 
\frac{1}{e} R(A, t) f(t) & \text{if } t \in [0, t_0], \\
0 & \text{if } t = 0. 
\end{cases}
$$

In the next proposition we list the main properties of the function (5.4).

**Proposition 5.3.** Under the assumptions of Lemma 5.1, let $\mathcal{D}(A, B, C)$ be defined by (5.3); then:

1. $\mathcal{D}(A, B, C)$ is continuously differentiable and $u(t) = R(A, t) f(t)$ whenever $t \in [0, t_0]$.

2. $u(t) = R(A, t) f(t)$ is continuously differentiable and $u(t) = R(A, t) f(t)$ whenever $t \in [0, t_0]$. 

$$
\begin{align*}
\lim_{t \to 0} \| u(t) \| &= 0, \\
\lim_{t \to 0} \| u'(t) \| &= 0.
\end{align*}
$$
(iii) If \( q \geq 0 \) there exists \( k_1 > 0 \) such that
\[
\| u(t, x) \|_{L^q(\Omega)} \leq \frac{k_1}{\varepsilon} \| u(t) \|_{C^1(\overline{\Omega})} + \frac{k_2}{\varepsilon^2} \| \nabla u(t) \|_{L^2(\Omega)} + \frac{k_3}{\varepsilon^3} \| \Delta u(t) \|_{L^1(\Omega)} \quad \forall t \in [0, \tau].
\]
(iv) If \( q > 0 \) there exists \( k_2 > 0 \) such that
\[
\| u(t, x) \|_{L^q(\Omega)} \leq \frac{k_2}{\varepsilon} \| u(t) \|_{C^2(\overline{\Omega})} + \frac{k_3}{\varepsilon^2} \| \nabla u(t) \|_{L^2(\Omega)} + \frac{k_4}{\varepsilon^3} \| \nabla^2 u(t) \|_{L^1(\Omega)} \quad \forall t \in [0, \tau].
\]
(v) If \( q = 0 \), \( \lambda \in (0, 1) \), then we have
\[
\| u(t, x) \|_{L^q(\Omega)} \leq \frac{k_5}{\varepsilon} \| u(t) \|_{C^3(\overline{\Omega})} + \frac{k_6}{\varepsilon^2} \| \nabla u(t) \|_{L^2(\Omega)} + \frac{k_7}{\varepsilon^3} \| \nabla^2 u(t) \|_{L^1(\Omega)} \quad \forall t \in [0, \tau].
\]

Proof. (i) For each \( \tau \in (0, 1) \) we have \( \frac{\tau}{\varepsilon} \geq \lambda \), so that \( R(1/t, A) \)

is defined; as \( f \in D(0, q, \lambda) \subseteq \mathbb{C}(0, q, \lambda) \), the result is an obvious con-
sequence of the properties of the resolvent.

(iii) It follows easily by a straightforward computation.

(iii) Set \( \lambda = \varepsilon^2 \), \( \tau = \varepsilon \), and the estimate fol-

\[
\| u(t, x) \|_{L^q(\Omega)} \leq \frac{k_1}{\varepsilon} \| u(t) \|_{C^1(\overline{\Omega})} + \frac{k_2}{\varepsilon^2} \| \nabla u(t) \|_{L^2(\Omega)} + \frac{k_3}{\varepsilon^3} \| \Delta u(t) \|_{L^1(\Omega)} \quad \forall t \in [0, \tau].
\]

and the result follows.

(v) For each \( \varepsilon \), \( x \in \mathbb{R}^d \) we get by (iii)
\[
\| u(t, x) \|_{L^q(\Omega)} \leq \frac{k_1}{\varepsilon} \| u(t) \|_{C^1(\overline{\Omega})} + \frac{k_2}{\varepsilon^2} \| \nabla u(t) \|_{L^2(\Omega)} + \frac{k_3}{\varepsilon^3} \| \Delta u(t) \|_{L^1(\Omega)} \quad \forall t \in [0, \tau].
\]

and (v) follows.

(vi) Let \( \tau \in (0, 1) \), \( x \in \mathbb{R}^d \). By (iv)
\[
\| u(t, x) \|_{L^q(\Omega)} \leq \frac{k_1}{\varepsilon} \| u(t) \|_{C^1(\overline{\Omega})} + \frac{k_2}{\varepsilon^2} \| \nabla u(t) \|_{L^2(\Omega)} + \frac{k_3}{\varepsilon^3} \| \Delta u(t) \|_{L^1(\Omega)} \quad \forall t \in [0, \tau].
\]

hence, by (iii)
\[
\| u(t, x) \|_{L^q(\Omega)} \leq \frac{k_1}{\varepsilon} \| u(t) \|_{C^1(\overline{\Omega})} + \frac{k_2}{\varepsilon^2} \| \nabla u(t) \|_{L^2(\Omega)} + \frac{k_3}{\varepsilon^3} \| \Delta u(t) \|_{L^1(\Omega)} \quad \forall t \in [0, \tau].
\]
which implies (vi).

(vii) We have by (iv) for $0 < t < t_0$

$$\int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \leq k_2 \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{2/3} \leq \frac{k_2}{2} \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{1/3},$$

and (vii) is proved.

(viii) Let $\epsilon > 0$, $t$. We have

$$D^2 u(t, x) = \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right) dx$$

and therefore by (iii)

$$D^2 u(t, x) - D^2 u(t, x) \leq k_2 \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{1/2} \leq \frac{k_2}{2} \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{1/3},$$

and the result is proved.

Now we are ready to prove Theorem 5.2. We distinguish three cases:

1. $t < 1/2$, $\omega < 1/2$ and $\epsilon > 1/2$.

Case 1. $t < 1/2$. Let $u(x, y)$, choose $\epsilon > 0$ such that $e \in (0, 1/2)$, where $\omega$, $\Omega$ are the numbers defined in Lemma 1.16, and take $x, y \in \Omega$ with $|x - y| \leq \epsilon/2$. Then the points $x, y$ satisfy the assumptions of Lemma 1.16, and hence there exists a continuously differentiable path $\Gamma: [0, 1] \to \Omega$ such that

$$\Gamma(0) = x, \quad \Gamma(1) = y, \quad \Gamma'(t) \leq t |x - y|.$$  

(Clearly, if $\Omega$ is convex we can take as $\Gamma$ the segment joining $x$ and $y$.)

Set $t_0 = 1/2$, so that $t < 0, t_0$. If $u$ is the function (5.4),

by Proposition 5.3 (v)-(vi) we have

$$|\frac{d}{dt} u(t, x)| \leq \frac{1}{2} |\frac{d}{dt} u(t, x)|^2 + |u(t, x) - u(t, y)| + |u(t, y) - \frac{d}{dt} u(t, y)|$$

$$\leq k_2 \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{1/2} \leq \frac{k_2}{2} \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{1/3},$$

and (vii) is proved.

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$$\Gamma(0) = x, \quad \Gamma(1) = y, \quad \Gamma'(t) \leq t |x - y|.$$  

(Clearly, if $\Omega$ is convex we can take as $\Gamma$ the segment joining $x$ and $y$.)

Set $t_0 = 1/2$, so that $t < 0, t_0$. If $u$ is the function (5.4),

by Proposition 5.3 (v)-(vi) we have

$$|\frac{d}{dt} u(t, x)| \leq \frac{1}{2} |\frac{d}{dt} u(t, x)|^2 + |u(t, x) - u(t, y)| + |u(t, y) - \frac{d}{dt} u(t, y)|$$

$$\leq k_2 \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{1/2} \leq \frac{k_2}{2} \left( \int_0^t \frac{d^2 u(t, s)}{ds^2} \, ds \right)^{1/3},$$

and (vii) is proved.
\[ u(t,x) + u(t,y) - 2u(t, \frac{x+y}{2}) \leq \frac{1}{\alpha, \Omega, (x, y, t)^{1/2}} \int_{\Omega} (\psi'(\lambda) - \psi'_{1}(\lambda)) d\lambda \leq c_{0} \int_{\Omega} \psi_{1}(\lambda) d\lambda \]

By Proposition 5.3 (v)

\[ \int_{\Omega} \psi_{1}(\lambda) d\lambda \leq c_{0} \int_{\Omega} \psi_{1}(\lambda) d\lambda \]

On the other hand

\[ u(t,x) = \int_{0}^{1} \frac{\partial}{\partial \lambda} [u(t, \psi(\lambda)) + u(t, \psi'(\lambda))] d\lambda = \int_{0}^{1} \psi_{1}(\lambda) d\lambda \]

Since

\[ \int_{\Omega} \psi_{1}(\lambda) d\lambda = c_{0} \int_{\Omega} \psi_{1}(\lambda) d\lambda \]

we can rewrite (5.8) as

\[ u(t,x) + u(t,y) - 2u(t, \frac{x+y}{2}) \leq c_{0} \int_{\Omega} \psi_{1}(\lambda) d\lambda \]

Now recall that, by Lemma 1.20, \( Du(t) \) is Hölder continuous with any exponent \( \alpha \in (0, 1) \), and moreover if \( p = n(1-\alpha)^{-1} \)

\[ \| Du(t) \|_{L^p(\Omega(x_0, \delta))} \leq c_{0} \int_{\Omega} \psi_{1}(\lambda) d\lambda \]

hence for \( \alpha \in (0, 1) \) fixed, and \( \frac{1}{2} \leq \alpha \leq \frac{3}{2} \), by (5.9) we get

\[ \int_{\Omega} \psi_{1}(\lambda) d\lambda \leq c_{0} \int_{\Omega} \psi_{1}(\lambda) d\lambda \]

which yields, since \( \epsilon \) is arbitrarily small,

\[ \lim_{\epsilon \to 0^+} \sup_{x \in \Omega} |f| = 0, \text{ i.e. } f \in C^{1,1}(\Omega) \]

Next, we have to prove that \( f \in C^{1,1}(\Omega) \) (or \( f \in C^{1,1}(\Omega) \) if \( f \in C^{1,1}(\Omega) \));

thus, we have to estimate the quantity \( \sigma^{-1} \int_{\Omega} f(x - \sigma \delta(x)) - f(x) \) when
\[ x \in \mathbb{R}, \quad \sigma > 0, \quad x = \sigma \varphi (x) e^{\eta}. \]

By Lemma 1.15, it suffices to consider small values of \( \sigma \), say \( \sigma = \varepsilon \sigma_0, \varepsilon_0 \), with \( \sigma_0 \geq \sigma \). Choose \( \varepsilon > 0 \) such that \( \varepsilon < \sigma_0^2 \| \varphi \|_\infty \), and take \( x \in \mathbb{R} \); we can suppose that \( x - \sigma \varphi (x) e^{\eta} \) with \( \varepsilon_0 = \varepsilon_{1/2} \). By (5.15),

\[
\left| f(x - \sigma \varphi (x)) - f(x) \right| \leq \left| f(x - \sigma \varphi (x)) - u(t, x - \sigma \varphi (x)) \right| + \left| u(t, x - \sigma \varphi (x)) - f(x) \right| \leq \left| f(x - \sigma \varphi (x)) - u(t, x - \sigma \varphi (x)) \right| + \left| u(t, x - \sigma \varphi (x)) - f(x) \right|.
\]

By Proposition 5.3(v)

\[
\left| f(x - \sigma \varphi (x)) - u(t, x - \sigma \varphi (x)) \right| + \left| u(t, x - \sigma \varphi (x)) - f(x) \right| \leq \frac{c |x|}{t^{1/2}}.
\]

On the other hand,

\[
u(t, x - \sigma \varphi (x)) = \frac{\sigma}{\sigma} \left| u(t, x - \sigma \varphi (x)) \right| \leq \frac{c |x|}{t^{1/2}},
\]

and, recalling (5.13), we have \( f \leq \frac{c |x|}{t^{1/2}} \) on the other hand, if \( f \) is in addition \( \mathcal{C}^1 (\varphi) \), we obtain similarly

\[
\left| \frac{f(x - \sigma \varphi (x)) - f(x)}{\sigma} - u(t, x - \sigma \varphi (x)) \right| \leq \frac{c |x|}{t^{1/2}},
\]

As \( \varepsilon \) is arbitrarily small, the left hand side of (5.21) tends to 0 as \( \varepsilon \to 0^+ \); this shows that \( F \in \mathcal{C}^1 (\varphi) \).

**Case 3.** \( \varepsilon \leq \varepsilon_{1/2} \). Let \( \mathcal{C}^1 (\varphi) \). First note that, if \( u \) is the function (5.4), then by Proposition 5.3(i)- (vii)

\[
u(t) + \varphi, \quad Du(t) \in C^0 (\varphi) \quad t \geq 0,
\]

which means \( \mathcal{C}^1 (\varphi) \) and \( Df(x) = Du(0) \) in \( C^0 (\varphi) \). In particular, as (5.17) holds for small positive \( t \), we get

\[
a(x) f(x) + \delta(x) |Df(x)| = 0 \quad \forall x \in \mathbb{R},
\]
i.e. \( \mathcal{C}^1 (\varphi) \); in addition by Proposition 5.3(vii)-(ii)

\[
\left| Du(t) \right| \leq \left| Du(0) \right| + \left| Du(t) \right| \leq c |\varphi| \leq 0 \quad \forall x \in \mathbb{R}.
\]
and as \( t \to 0^+ \) we clearly obtain

\[
1_{f_1} \leq c |f|_g, \tag{5.22}
\]

Thus we have only to show that \( \text{Dir}^0,\overline{20} - 1_0(\Omega) \) (or \( \text{Dir}^0,\overline{2} - 1_0(\Omega) \) if \( \text{Dir}^0(\Omega) \)). Choose \( \varepsilon \in [0, \varepsilon_0] \), and take \( x, y \in \Omega \) with \( |x-y| < \varepsilon^{1/2} \).

Set \( t = |x-y|^2 \), so that \( t \in [0, \varepsilon] \); then if \( p = \frac{1}{2}n(1-\varepsilon)_1 \) and \( q > p \), by Lemma 1.20 and Proposition 5.3(vii)-(viii) we have

\[
|\text{Dir}(x) - \text{Dir}(y)| \leq |\text{Dir}(x) - \text{Du}(t, x)| + |\text{Du}(t, x) - \text{Du}(t, y)| + |\text{Du}(t, y) - f(y)|
\]

\[
\leq \varepsilon^{0-1/2} |f| \left( \frac{t}{2\varepsilon} \right)^{1-1/2} |x-y|^{28-1} + \frac{n}{2q}(q-p)
\]

\[
\leq \varepsilon^{0-1/2} |f| \left( \frac{t}{2\varepsilon} \right)^{1-1/2} |x-y|^{28-1} + \frac{n}{2q}(q-p)
\]

\[
\leq \frac{n}{2q}(q-p) - 1 + \varepsilon \leq 0
\]

As \( \frac{n}{2q}(q-p) - 1 + \varepsilon \leq 0 \), we get

\[
|\text{Dir}(x) - \text{Dir}(y)| \leq |x-y|^{28-1} |f| \left( \frac{t}{2\varepsilon} \right)^{1-1/2} \leq 1_{f_1}. \tag{5.23}
\]

Hence \( \text{Dir}^0,\overline{20} - 1_0(\Omega) \) and, recalling (5.22), \( 1_{f_1} \leq c f_1, \overline{20} - 1_0(\Omega) \). If in addition \( \text{Dir}^0(\Omega) \), choosing in (5.23) \( t = |x-y|^2 \), \( c f_1, \overline{20} - 1_0(\Omega) \) we obtain

\[
|\text{Dir}(x) - \text{Du}(t, x)| + |\text{Du}(t, x) - \text{Du}(t, y)| + |\text{Du}(t, y) - f(y)|
\]

and since \( \varepsilon \) is arbitrarily small, we deduce that

\[
\lim_{t \to 0^+} \sup_{x \in \Omega} \text{Dir}(x) - \text{Du}(t, x) = 0. \tag{5.24}
\]

5.2 is completely proved. \( Q \)

6. CONCLUSIONS

Collecting the results of the preceding sections, we have proved the following result:

**THEOREM 6.1.** Let \( \Omega \) be a bounded connected open set of \( \mathbb{R}^n \), \( n \geq 1 \), with boundary \( \partial \Omega \) of class \( C^2 \). Let \( B, B^* \) be the differential operators respectively defined by (5.1) and (5.2), and suppose that conditions (A.1), (A.2), (B.1), (B.2) of Section 5 hold. Let \( A \) be the abstract operator defined by (5.3) in the space \( C^0(\Omega) \).

Then the following equalities hold (with equivalent norms):

\[
D_A(\partial\Omega) = (C^0_B(\Omega), \mathcal{C}^0(\Omega)) \quad \text{and} \quad \mathcal{D}_A(\partial\Omega) = (C^2_B(\Omega), \mathcal{C}^2(\Omega)).
\]

**REMARK 6.2.** The case of Dirichlet boundary conditions, i.e. \( B(\Omega) \subset (5.2) \), can also be studied with our method; however, the extension procedure given by (2.3) in the case \( \Omega = \mathbb{R}^n \) has to be replaced by the odd extension method. It can be seen that in this case the treatment developed in Sections 2, 3, 4 still...
works; on the other hand the reverse inclusion of Section 5 can be proved in the same way, by applying the estimates of Stewart [19] instead of [20]. Hence we find again a known result of Lunardi, which had been proved in [12] with a slight strengthening of assumptions. Namely, we have:

**THEOREM 6.3.** Let $\Omega$ be a bounded, connected open set of $\mathbb{R}^n$, $n \geq 1$, with boundary $\partial \Omega$ of class $C^2$. Let $A$ be the differential operator defined by (5.1), and suppose that conditions (A.1) and (A.2) of Section 5 hold. Let $A$ be the abstract operator defined in the space $C^0_c(\Omega)$ by

$$D(A) = \{ u \in C^2(\Omega) : Du \in C^0_c(\Omega), u|_{\partial \Omega} = 0 \},$$

where

$$A u = [Du](\cdot).$$

Then the following equalities hold (with equivalent norms):

$$D_{\lambda}(\theta, \gamma) = (C^0_\theta(\Omega), C^0_\gamma(\Omega))_{1-\theta, \gamma} = \begin{cases} C^{0,2,0}_{0,0}(\Omega) & \text{if } \theta \in [0,1/2], \\ C^{0,1,0}_{0,0}(\Omega) & \text{if } \theta = 1/2, \\ C^{0,1,1}_{0,0}(\Omega) & \text{if } \theta \in [1/2,1]. \end{cases}$$

$$D_{\lambda}(\theta) = (C^0_\theta(\Omega), C^0_0(\Omega))_{1-\theta} = \begin{cases} h^{0,2,0}_{0,0}(\Omega) & \text{if } \theta \in [0,1/2], \\ h^{0,1,0}_{0,0}(\Omega) & \text{if } \theta = 1/2, \\ h^{1,2,1}_{0,0}(\Omega) & \text{if } \theta \in [1/2,1]. \end{cases}$$

**REFERENCES**


