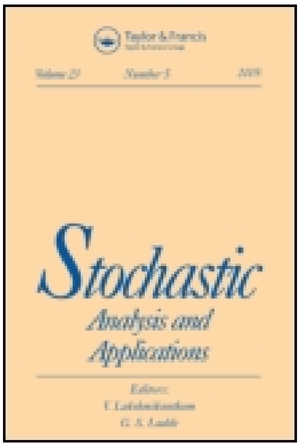

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An approach to Ito linear equations in Hilbert spaces by approximation of white noise with coloured noise

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AN APPROACH TO ITO LINEAR EQUATIONS IN HILBERT SPACES BY APPROXIMATION OF WHITE NOISE WITH COLOURED NOISE

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ABSTRACT

We consider the stochastic problem $du(t)=[A(t)u(t) + 1/2\, B^2u(t) + f(t)]dt + Bu(t)dW_t$, $u(0)=X$, in a Hilbert space H, where f,X are prescribed data, W_t is a real Brownian motion, and A(t), B generate an analytic semigroup and a strongly continuous group respectively. The domains D(A(t)) may vary with t and we only require $D(A(t))\subseteq D(B)$ for each t. A unique generalized solution is constructed as the pathwise uniform limit of solutions of suitable approximating deterministic problems, which are obtained by approaching the white noise dW_t with a sequence of regular coloured noises $W_n'(t)$.

0. INTRODUCTION

Let (Ω,ε,P) be a probability space, let H be a real separable Hilbert space. We look for a solution of the following stochastic problem:

$$\begin{cases} du(t) = [C(t)u(t) + f(t)]dt + Bu(t)dW_t, \quad t\in[0,T] \\ u(0)=X \end{cases} \qquad (S_0)$$

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where $C(t)$ and B are closed linear operators on H, with domains $D(C(t))$ and $D(B)$, W_t is a real Brownian motion on Ω, and $f:[0,T]\times\Omega\to H$, $X:\Omega\to H$ are prescribed data. Problems of this kind arise in a lot of applications, as for example filtering theory, control theory, population dynamics, hydrodynamics, theoretical physics,etc. (see, among others, Zakai [26], Lipster-Shiryayev [17], Curtain-Pritchard [6], Krylov-Rozowskii[16]).

One among the most fruitful methods for the study of Problem (S_0) is based upon semi-group theory: following this approach several results have been obtained by a large number of authors (Dawson [10], Balakrishnan [3], Metivier-Pistone [18], Curtain [5], Krylov-Rozovskii[15], Chojnowska Michalik [4], Kotelenez [14]). In all these papers it is assumed that B is bounded and $C(t)$ generates a strongly continuous semi-group, and existence and uniqueness of the solution are proved by the contraction principle.

The case of unbounded B has been studied with variational methods by Pardoux [19],[20] and Krylov-Rozovskii [16], and from the semi-group point of view, by Curtain-Pritchard [6], Ichikawa [12], Da Prato-Iannelli-Tubaro [8],[9].

The method employed in [9] consists in solving (S_0) path by path, by transforming (S_0) into an equivalent deterministic problem; this one is in turn studied using the classical theory of Tanabe [23] about linear abstract evolution equations. In [9] it is supposed that B generates a strongly continuous group while $C(t)\equiv C$ is a closed linear operator with domain $D(C)\subseteq D(B^2)$ such that

$C - \frac{B^2}{2}$ generates an analytic semi-group.

The method of [9] can be adapted to cover also the case of a family of operators $C(t)$, provided $D(C(t))$ is constant and contained into $D(B^2)$ and, for each $t\in[0,T]$, $C(t)-\frac{B^2}{2}$ generates an analytic semigroup.

In this paper we study problem (S_0) from the same point of view of [9], but we allow $D(C(t))$ to vary with t. The method of [9] cannot be directly extended to this case; in fact, the transformation into an equivalent deterministic problem leads to a non-autonomous evolution equation where operators $\tilde{C}(t)$ with variable domains appear: in this case the classical theory of Kato-Tanabe [13] requires, for solvability, a differentiability condition in t for the analytic semi-group generated by $\tilde{C}(t)$. Now, this condition does not hold, since the Brownian notion has non-differentiable sample paths.

In order to overcome this difficulty, we will consider for each $n\in\mathbb{N}$ and for a.a. $\omega\in\Omega$ the deterministic problem

$$\begin{cases} u_n'(t) = [C(t) - \frac{B^2}{2}]u_n(t) + f(t)-W_n'(t)Bu_n(t), t\in[0,T] \\ u_n(0)=X \end{cases} \qquad (S_{n,0})$$

where $W_n(t)$, $n\in\mathbb{N}$, are regular functions converging uniformly, as $n\to\infty$, to the paths of the Brownian motion. Now it is well known the following phenomenon (see Wong-Zakai [24], Sussmann [22]): given in $\mathbb{R}^m$ the stochastic problem

$$\begin{cases} du = g(u)dt + h(u)dW_t, & t\in[0,T] \\ u(0) = X \end{cases} \qquad (0.1)$$

where W_t is a real Brownian motion, if we approximate uniformly the paths of the Brownian motion by regular functions $W_n(t)$, then for a.a. $\omega\in\Omega$ the solutions u_n of the corresponding deterministic problems (with fixed ω)

$$\begin{cases} \frac{du_n}{dt} = g(u_n) + h(u_n)W_n', & t\in[0,T], \\ u_n(0) = X \end{cases}$$

converge uniformly **pathwise** as $n\to\infty$ to the solution of (0.1) in the sense of Stratonovich [21], i.e. to the solution - in the classical sense of Itô- of the problem

$$\begin{cases} du = (g(u)+\frac{1}{2} < h'(u), h(u) >) \, dt + h(u) \, dW_t \\ u(0) = X \end{cases}$$

where the extra deterministic term $\frac{1}{2}<h'(u),h(u)>dt$ appears. Note that if $h(u)=Bu$, where B is a mxm matrix, then $\frac{1}{2} <h'(u); h(u)> = \frac{1}{2} B^2u$.

This is also the case in our situation. We will show that the solution u_n of $(S_{n,0})$ converge uniformly pathwise as $n\to\infty$ to the solution, in the sense of Stratonovich, of

$$\begin{cases} du(t)=[[C(t)-\frac{1}{2}B^2]u(t) + f(t)]dt + Bu(t) \, dW_t, & t\in[0,T] \\ u(0) = X \end{cases}$$

i.e. to the solution of (S_0) in the sense of Itô.

Thus existence and uniqueness of the solution of (S_0) will be proved, generalizing the result of [9]; in addition this solution will be obtained as the uniform limit, path by path, of the solutions of the deterministic problems driven by a suitable coloured noise $W'_n(t)$ approaching, as $n\to\infty$, the white noise dW_t.

If we set $A(t) = C(t) - \frac{1}{2} B^2$, problem (S_0) can be rewritten as follows:

$$\begin{cases} du(t)=[A(t)u(t)+\frac{1}{2}B^2u(t)+f(t)]dt+Bu(t)\ dW_t,\ t\in[0,T] \\ u(0) = X \end{cases} \qquad (S_1)$$

where B generates a strongly continuous group and for each $t\in[0,T]$ $A(t)$ generates an analytic semi-group. Problem (S_1) is exactly equivalent to (S_0) provided we assume that $D(A(t))\equiv D(C(t))\subseteq D(B^2)$ for each $t\in[0,T]$; however this formulation allows us to weaken slightly the hypotheses about $D(A(t))$: we will require only that $D(A(t))\subseteq D(B)$ for each $t\in[0,T]$.

1. NOTATIONS AND ASSUMPTIONS

Let us introduce some notations.

Let H be a Hilbert space. We will consider the following Banach spaces:

a) $C^0([0,T],H)=\{u:[0,T]\to H \text{ continuous}\}$, with norm

$$\|u\|_{C^0([0,T]H)} = \sup_{t\in[0,T]} \|u(t)\|_H ,$$

b) for each $\theta\in]0,1]$, the θ-Holder space $C^{0,\theta}([0,T],H) = \{u:[0\ T]\to H : \|u(t)-u(s)\|_H = 0(|t-s|^\theta)\}$, with norm

$$\|u\|_{C^{0,\theta}([0,T],H)} = \|u\|_{C^0([0,T],H)} + \sup_{t\neq s} \frac{\|u(t)-u(s)\|_H}{|t-s|^\theta}$$

c) $C^1([0,T],H)=\{u:[0,T]\to H$ strongly differentiable with $u'\in C^0([0,T],H)\}$, with norm

$$\|u\|_{C^1([0,T],H)} = \|u\|_{C^0([0,T],H)} + \|u'\|_{C^0([0,T],H)}$$

d) for each $p\in[1,\infty]$, $L^p(0,T,H)=\{u:]0,T[\to H$ strongly measurable with $\|u(\cdot)\|_H \in L^p(0,T)\}$, with norm

$$\|u\|_{L^p(0,T,H)} = \begin{cases} \left[\int_0^T \|u(t)\|_H^p dt\right]^{1/p} & \text{if } p<\infty \\ \operatorname{esssup}_{t\in]0,T[} \|u(t)\|_H & \text{if } p=\infty \end{cases}$$

We denote by $\mathcal{L}(H)$ the Banach space of bounded linear operators $H\to H$, with norm

$$\|A\|_{\mathcal{L}(H)} = \sup_{x\neq 0} \frac{\|Ax\|_H}{\|x\|_H}$$

if more generally A is a linear operator on H, we denote by D(A) its domain and by R(A) its range ; $\rho(A)$ is the resolvent set of A, $\sigma(A)$ its spectrum, and the resolvent operator $(\lambda-A)^{-1}$ is denoted by $R(\lambda,A)$. If B is another linear operator, we write $[A,B] = AB-BA$ whenever the right-hand side is defined. Now let $\{W_t\}_{t\geq 0}$ be a real Brownian motion on the probability space $(\Omega,\mathcal{E},P)$ and let $\mathcal{F}=\{\mathcal{F}_t\}_{t\geq 0}$ be an increasing family of σ-algebras contained into $\mathcal{E}$, non-anticipating with respect to $\{W_t\}_{t\geq 0}$, and such that $(\Omega,\mathcal{F}_0,P)$ is a complete measure space.

We denote by $C^0_{\mathcal{F}}([0,T],H)$ (resp. $C^{0,\theta}_{\mathcal{F}}([0,T],H)$) the class of processes $u:[0,T]\times\Omega\to H$ adapted to $\mathcal{F}$, and such that $t\to u(t,\omega)$ is continuous (resp. θ-Holder continuous) for a.e. $\omega\in\Omega$. $C^1_{\mathcal{F}}([0,T],H)$ is the class of processes $u:[0,T]\times\Omega\to H$ adapted to $\mathcal{F}$ and such that $t\to u(t,\omega)$ is strongly differentiable with $t\to \frac{\partial u}{\partial t}(t,\omega)$ continuous, for a.e. $\omega\in\Omega$.

Finally $L^p_F(0,T,H)$, $1\le p\le\infty$, is the class of processes $u:[0,T]\times\Omega\to H$ adapted to F, and such that $t\to u(t,\omega)$ belongs to $L^p(0,T,H)$ for a.e. $\omega\in\Omega$, and $L_{F_0}(H)$ is the class of all H-valued F_0-measurable random variables.

Let us list now our assumptions.

Let W_t be a real Brownian motion on the probability space (Ω,E,P), and let $\{F_t\}_{t\ge 0}$ be an increasing family of σ-algebras contained into E, non-anticipating with respect to $\{W_t\}_{t\ge 0}$ and such that (Ω,F_0,P) is a complete measure space.

Let H be a separable real Hilbert space. Let $\{A(t)\}_{t\in[0,T]}$, B be operator on H satisfying the following conditions:

HYPOTHESIS I B is a closed linear operator on H with domain D(B), which generates a strongly continuous group $\{e^{\xi B}\}_{\xi\in R}$; in particular

i) there exists $\eta>0$ such that $\rho(B)\supseteq\{\lambda\in\mathbb{C}: |Re\lambda|>\eta\}=:\Sigma_B$,

ii) there exists N>0 such that

$$\|[R(\lambda,B)]^n\|_{L(H)}\le\frac{N}{[\,|Re\lambda|-\eta]^n}\qquad\forall n\in\mathbb{N}\ ,\ \forall\lambda\in\Sigma_B.$$

HYPOTHESIS II For each $t\in[0,T]$ A(t) is a closed linear operator on H with domain D(A(t)), which generates an analytic semi-group $\{e^{\xi A(t)}\}_{\xi\ge 0}$; moreover:

(i) there exists $\theta_0\in]\frac{\pi}{2},\pi[$ such that $\rho(A(t))\supseteq$
$\supseteq\{\lambda\in C: |\arg\lambda|<\theta_0\}\cup\{0\}=:\Sigma_{\theta_0}\quad\forall t\in[0,T]$;

(ii) there exists M>0 such that

$$\|A(t)^{-1}\|_{L(H)}\le M,\quad \|R(\lambda,A(t))\|_{L(H)}\le\frac{M}{|\lambda|}$$

$\forall\lambda\in\Sigma_{\theta_0}-\{0\}$, $\forall t\in[0,T]$;

(iii) $t\to R(\lambda,A(t))x\in C^1([0,T],H)$ $\forall x\in H$, $\forall\lambda\in\Sigma_{\theta_0}$ and

there exist $K>0$ and $\alpha\in]0,1]$ such that

$$\left\|\frac{d}{dt}A(t)^{-1}\right\|_{\mathcal{L}(H)}\leq K,\quad \left\|\frac{\partial}{\partial t}R(\lambda,A(t))\right\|_{\mathcal{L}(H)}\leq\frac{K}{|\lambda|^{\alpha}}$$

$\forall\lambda\in\Sigma_{\theta_0}-\{0\}$, $\forall t\in[0,T]$.

HYPOTHESIS III

(i) $D(A(t))\subseteq D(B)$ $\forall t\in[0,T]$.

(ii) For each $t\in[0,T]$ there exist $\lambda_0(t)\in\mathbb{C}$, $L(t)\in\mathcal{L}(H)$ such that:

(a) $\lambda_0\in C([0,T],\mathbb{C})$, $L\in C([0,T],\mathcal{L}(H))$;

(b) $D(B)\subseteq\{x\in H : B\,R(\lambda_0(t),A(t))x\in D(A(t))\}$

(c) $[\lambda_0(t)-A(t)]\,B\,R(\lambda_0(t),A(t))x=Bx+L(t)x$ $\forall x\in D(B)$.

In view of Remark 1.2 below, we shall assume $\lambda_0(t)\equiv 0$.

HYPOTHESIS IV

$t\to BA(t)^{-1}x\in C([0,T],H)$ $\forall x\in H$; in particular there exists $E>0$ such that

$$\|BA(t)^{-1}\|_{\mathcal{L}(H)}\leq E\quad\forall t\in[0,T].$$

REMARK 1.1 Hypothesis II is classical in the theory of analytic semi-groups with variable domain (see Kato-Tanabe [13], Acquistapace-Terreni [1]. In the following we shall use the results of [1], where however condition

(iii) of Hypothesis II is replaced by a slightly stronger one, namely

(iii)' $t \to R(\lambda, A(t)) \in C^1([0,T], \mathcal{L}(H))$ $\forall \lambda \in \Sigma_{\theta_0}$ and there exist $K>0$ and $\alpha \in]0,1]$ such that

$$\left\| \frac{d}{dt} A(t)^{-1} \right\|_{\mathcal{L}(H)} \leq K, \left\| \frac{\partial}{\partial t} R(\lambda, A(t)) \right\|_{\mathcal{L}(H)} \leq \frac{K}{|\lambda|^{\alpha}}$$

$\forall \lambda \in \Sigma_{\theta_0} - \{0\}$, $\forall t \in [0,T]$.

Hence we have to verify that all results of [1] still hold under Hypothesis II. Indeed, this is true with essentially the same proofs: in fact, some of the proofs in [1] use only the estimates about $\frac{\partial}{\partial t} R(\lambda, A(t))$, so that no change is needed; in all other cases the operators $\frac{\partial}{\partial t} R(\lambda, A(t))$ are always evaluated at a fixed vector or at a continuous function $g(t)$, and therefore condition (iii) of Hypothesis II guarentees the continuity of the composition, which is all what is really needed.

REMARK 1.2 Hypothesis III arises from a similar (and apparently weaker) assumption of Da Prato-Iannelli-Tubaro [9], where an analogous situation (with $A(t) \equiv A$) is considered. They suppose there that condition (ii) of Hypothesis III holds for all x in a dense (in the graph norm) subspace $V \subseteq D(B)$ (and not possibly for all $x \in D(B)$). But we shall see in the Appendix that a similar condition in the case $A(t) \not\equiv A$ (i.e. the existence of a family $\{V(t)\}_{t \in [0,T]}$ of dense subspaces of $D(B)$ such that (ii) holds for all $x \in V(t)$) in fact implies that (ii) is satisfied in the whole $D(B)$.

It is also easy to see that if Hypothesis III holds, then for each $t\in[0,T]$ and $\lambda\in\Sigma_{\theta_0}$ we have $D(B)\subseteq\{x\in H: BR(\lambda,A(t))x\in \in D(A(t))\}$ and there exists an operator $L_\lambda(t)$ such that $L_\lambda\in C([0,T], \mathcal{L}(H))$ and

$$[\lambda-A(t)]B\,R(\lambda,A(t))x = Bx + L_\lambda(t)x \quad \forall x\in D(B)$$

(one has simply to take $L_\lambda(t)= L(t)[\lambda_0(t)-A(t)]\ R(\lambda,A(t))$). This shows that it is not restrictive to assume $\lambda_0(t)\equiv 0$ in Hypothesis III.

2. AUXILIARY RESULTS

In this section we collect a list of results which will be used throughout Some of them are almost obvious, but we state them for further reference.

PROPOSITION 2.1 $D(B)$, $D(B^2)$, $D(A(t))$ (for each $t\in[0,T]$) are dense in H.

Proof See e.g. Yosida [25].

PROPOSITION 2.2. If $\phi\in C^0([0,T],H)$ define for each $k\in\mathbb{N}$ $\zeta_k(t)=kR(k,B)\phi(t)$, $Z_k(t)=kR(k,A(0))\phi(t)$. Then $\zeta_k\in C^0([0,T], D(B))$, $Z_k\in C^0([0,T],D(A(0)))$ and $\zeta_k\to\phi$, $Z_k\to\phi$ in $C^0([0,T],H)$ as $k\to\infty$.

Proof. It follows by straightforward compactness arguments.

PROPOSITION 2.3. (i) There exist $N>0$ and $\omega\in\mathbb{R}$ such that

$$\|e^{\sigma B}\|_{\mathcal{L}(H)} \le Ne^{\omega|\sigma|} \qquad \forall\sigma\in\mathbb{R}$$

(ii) $x\in D(B) \Rightarrow \|(e^{\sigma B}-1)x\|_H \le C|\sigma|\,\|Bx\|_H \quad \forall\sigma\in\mathbb{R}$

(iii) $\left\|\frac{e^{\sigma B}-1}{\sigma} R(\lambda,B)\right\|_{\mathcal{L}(H)} \le C \quad \forall\sigma\neq 0, \quad \forall\lambda\in\Sigma_B$

(iv) <u>For each</u> $\sigma>0$ <u>and</u> $t\in[0,T]$ <u>we have</u> $e^{\sigma A(t)} =$

$$= \frac{1}{2\pi i}\int_\gamma e^{\sigma\lambda} R(\lambda,A(t))d\lambda, \text{ where } \gamma=\gamma_0\cup\gamma_+\cup\gamma_-,$$

$\gamma_0 = \{\lambda\in\mathbb{C} : |\lambda|=1, |\arg\lambda|<\theta\}$

$\gamma_\pm = \{\lambda\in\mathbb{C} : |\lambda|\ge 1, \arg\lambda=\pm\theta\} \quad \theta\in\,]\pi/2,\theta_0[\,;$

<u>in particular</u> $\|e^{\sigma A(t)}\|_{\mathcal{L}(H)} \le C \ \forall\sigma\ge 0, \ \forall t\in[0,T]$

(v) $\|A(t)e^{\sigma A(t)}\|_{\mathcal{L}(H)} \le \frac{C}{\sigma} \quad \forall\sigma>0, \quad \forall t\in[0,T]$

(vi) $x\in D(A(0)) \Rightarrow \|A(t)e^{\sigma A(t)}x\|_H \le C\|A(0)x\|_H \ \forall\sigma>0, \ \forall t\in[0,T]$

(vii) $x\in H \Rightarrow \lim_{\sigma\to 0^+} \|\sigma A(t)e^{\sigma A(t)}x\|_H=0 \quad \forall t\in[0,T]$

(viii) $\|\frac{\partial}{\partial t}(e^{\sigma A(t)})\|_{\mathcal{L}(H)} \le \frac{c}{\sigma^{1-\alpha}} \ \forall\sigma>0, \ \forall t\in[0,T]$

<u>Proof</u> (i)-(ii) Standard.

(iii) It follows by (ii) since $\|BR(\lambda,B)\|_{\mathcal{L}(H)} \le C \ \forall\lambda\in\Sigma_B$

(iv)-(viii) See [1], formula (1.1), Lemma 1.5 and formula (1.3).

PROPOSITION 2.4 <u>For each</u> $t\in[0,T]$ <u>and</u> $\lambda\in\Sigma_{\theta_0}$ <u>we have</u>

$$[R(\lambda,A(t)),B]x=R(\lambda,A(t))L(t)A(t)R(\lambda,A(t))x \quad \forall x\in D(B),$$

<u>consequently the operator</u> $[R(\lambda,A(t)),B]$ <u>has a unique extension to an element</u> $T_{\lambda,t}\in\mathcal{L}(H)$, <u>which satisfies</u>

$$\|T_{0,t}\|_{\mathcal{L}(H)}\le C, \ \|T_{\lambda,t}\|_{\mathcal{L}(H)} \le \frac{C}{|\lambda|} \quad \forall\lambda\in\Sigma_{\theta_0}-\{0\}, \ \forall t\in[0,T]$$

Proof By Hypothesis III we have

$$BA(t)^{-1}x=A(t)^{-1}Bx+A(t)^{-1}L(t)x \qquad \forall x\in D(B);$$

now if $x\in D(B)$ and $\lambda\in\Sigma_{\theta_0}$ we get

$$\begin{aligned}BR(\lambda,A(t))x&=\lambda\, BR(\lambda,A(t))A(t)^{-1}x-BA(t)^{-1}x =\\ &=\lambda\, BR(\lambda,A(t))A(t)^{-1}x - A(t)^{-1}Bx - A(t)^{-1}L(t)x=\\ &=\lambda\, BR(\lambda,A(t))A(t)^{-1}x- \lambda R(\lambda,A(t))A(t)^{-1}Bx +\\ &+R(\lambda,A(t))Bx - A(t)^{-1}L(t)x =\\ &=\lambda\, BR(\lambda,A(t))A(t)^{-1}x-\lambda\, R(\lambda,A(t))[BA(t)^{-1}x -\\ &-A(t)^{-1}L(t)x\,]+R(\lambda,A(t))Bx-A(t)^{-1}L(t)x =\\ &=\lambda[B,R(\lambda,A(t))]A(t)^{-1}x + R(\lambda,A(t))Bx+\\ &+[\lambda\, R(\lambda,A(t)) - 1]\, A(t)^{-1}\, L(t)x\end{aligned}$$

which implies

$$[B,R(\lambda,A(t))]\,(1 - \lambda A(t)^{-1})x = R(\lambda,A(t))L(t)x \quad \forall x\in D(B).$$

Now $x\in D(B)$ if and only if $y:=(1-\lambda A(t)^{-1})x\in D(B)$; hence

$$\begin{aligned}[B,R(\lambda,A(t))]y &= R(\lambda,A(t))L(t)\,[1-\lambda A(t)^{-1}]^{-1}y =\\ &=-R(\lambda,A(t))L(t)\, A(t)R(\lambda,A(t))y \quad \forall y\in D(B)\end{aligned}$$

The operator $T_{\lambda,t}=-R(\lambda,A(t))L(t)\, R(\lambda,A(t))$ is obviously in $\mathcal{L}(H)$, with norm bounded by $\frac{c}{|\lambda|}$, and the result follows.

COROLLARY 2.5 For each $t\in[0,T]$ and $\lambda\in\Sigma_{\theta_0}$ the operator $R(\lambda,A(t))B$ can be uniquely extended to an element of $\mathcal{L}(H)$ with norm bounded independently of t,λ.

Proof We have $R(\lambda,A(t))Bx = BR(\lambda,A(t))x+[R(\lambda,A(t)),B]x$ $\forall x\in D(B)$. The result follows by Hypothesis IV and Proposition 2.4.

PROPOSITION 2.6

(i) $\|Be^{\sigma A(t)}\|_{\mathcal{L}(H)} \leq \frac{C}{\sigma}$ $\forall\sigma>0$, $\forall t\in[0,T]$

(ii) $x\in H \Rightarrow \lim_{\sigma\to 0^+} \|\sigma Be^{\sigma A(t)}x\|_H = 0$ $\forall t\in[0,T]$

(iii) $x\in D(B) \Rightarrow \|Be^{\sigma A(t)}x\|_H \leq C\{\|x\|_H+\|Bx\|_H\}$ $\forall\sigma\geq 0, \forall t\in[0,T]$

(iv) $x\in D(B) \Rightarrow A(t)e^{\sigma A(t)}x\in D(B)$ and $\|BA(t)e^{\sigma A(t)}x\|_H \leq$
$\leq \frac{C}{\sigma}\{\|x\|_H + \|Bx\|_H\}$ $\forall\sigma>0$, $\forall t\in[0,T]$

(v) $x\in D(B) \Rightarrow Be^{\sigma A(t)}x\in D(A(t))$ and $\|A(t)Be^{\sigma A(t)}x\|_H \leq$
$\leq \frac{C}{\sigma}\{\|x\|_H + \|Bx\|_H\}$ $\forall\sigma>0$, $\forall t\in[0,T]$.

Proof (i) We have $Be^{\sigma A(t)}x=BA(t)^{-1}A(t)e^{\sigma A(t)}x$ and the result follows by Hypothesis IV and Proposition 2.3(v).

(ii) If $x\in D(B)$ we can write by Proposition 2.3 (iv)

$$Be^{\sigma A(t)}x = \frac{1}{2\pi i}\int_\gamma e^{\sigma\lambda}[B,R(\lambda,A(t))]x\,d\lambda + e^{\sigma A(t)}Bx$$

and the conclusion follows by Proposition 2.4. The general case follows by (i) and Proposition 2.1.

(iii) We proceed as in (ii), applying again Proposition 2.4.

(iv) We have

$$BA(t)e^{\sigma A(t)}x = \frac{1}{2\pi i}\int_\gamma \lambda e^{\sigma\lambda}[B,R(\lambda,A(t))]x\,d\lambda + A(t)e^{\sigma A(t)}Bx,$$

and Proposition 2.4 gives the result.

(v) As $A(t)e^{\sigma A(t)}x\in D(A(t))\subseteq D(B)$, we can write by Hypothesis III

$$Be^{\sigma A(t)}x=BA(t)^{-1}\,A(t)e^{\sigma A(t)}x\in D(A(t)),$$

and

$$A(t)Be^{\sigma A(t)}x = [B+L(t)]\ A(t)e^{\sigma A(t)}x;$$

thus the conclusion follows by (iv) and Proposition 2.3 (v).

PROPOSITION 2.7 *$D(A(t))\cap D(B^2)$ is dense in H for each $t\in[0,T]$.*

Proof Let $x\in H$; by Proposition 2.1 for each $\varepsilon>0$ there exists $y\in D(B)$ such that $\|x-y\|_H<\varepsilon$. Since $D(A(t))$ is dense in H, we have $\lim_{\sigma\to 0}\|e^{\sigma A(t)}y-y\|_H=0$ so that there exists $\delta>0$ such that $\|e^{\delta A(t)}y-x\|_H<2\varepsilon$. By Proposition 2.6 (v), $e^{\delta A(t)}y\in D(A(t))\cap D(B^2)$ and the result is proved.

PROPOSITION 2.8 *For each $t\in[0,T]$ and $\sigma\in\mathbb{R}$ we have:*

$$e^{\sigma B}(D(A(t)))\subseteq D(A(t)) \text{ and}$$

$$A(t)e^{\sigma B}A(t)^{-1}x = e^{\sigma(B+L(t))}x \quad \forall x\in H.$$

Proof See Da Prato-Iannelli-Tubaro [9] , proof of Proposition 1.

PROPOSITION 2.9 *For each $t\in[0,T]$ and $\xi\in\mathbb{R}$ we have:*

$$\|e^{\xi(B+L(t))}-e^{\xi B}\|_{\mathcal{L}(H)}\le|\xi|e^{\omega|\xi|}\sup_{|\eta|\le|\xi|}\|e^{\eta(B+L(t))}\|_{\mathcal{L}(H)}\cdot$$

$$\cdot\|L(t)\|_{\mathcal{L}(H)}.$$

Proof See Da Prato-Iannelli-Tubaro [9], proof of Proposition 1.

COROLLARY 2.10. *For each $t\in[0,T]$, $\xi\in\mathbb{R}$ and $\sigma>0$ we have:*

$$[A(t),e^{\xi B}]e^{\sigma A(t)}=[e^{\xi[B+(t)]}-e^{\xi B}]A(t)e^{\sigma A(t)}\in\mathcal{L}(H)$$

and

$$\|[A(t),e^{\xi B}]e^{\sigma A(t)}\|_{\mathcal{L}(H)}\le C\frac{|\xi|}{\sigma}e^{C|\xi|}$$

Proof. Immediate consequence of Propositions 2.8 and 2.9.

PROPOSITION 2.11 For each $t,r\in[0,T]$, $\xi\in\mathbb{R}$ we have:

$$\|e^{\xi[B+L(t)]}-e^{\xi[B+L(r)]}\|_{\mathcal{L}(H)}\le\|L(t)-L(r)\|_{\mathcal{L}(H)}\cdot$$

$$\cdot\ C|\xi|\exp(\exp C|\xi|)$$

Proof For each $x\in H$ we have (see [9], proof of Proposition 1)

$$e^{\xi[B+L(t)]}x=e^{\xi B}+\int_0^\xi \frac{\partial}{\partial s}[e^{(\xi-s)B}e^{s[B+L(t)]}x]\,ds =$$

$$=e^{\xi B}+\int_0^\xi e^{(\xi-s)B}L(t)\ e^{s[B+L(t)]}x\ ds,$$

which implies

$$e^{\xi[B+L(t)]}x-e^{\xi[B+L(r)]}x=\int_0^\xi e^{(\xi-s)B}[L(t)e^{s[B+L(t)]}-$$

$$-\ L(r)e^{s[B+L(r)]}]x\ ds.$$

Hence

$$\|e^{\xi[B+L(t)]}x - e^{\xi[B+L(r)]}x\|_H \le$$

$$\le\|\int_0^\xi e^{(\xi-s)B}[L(t)-L(r)]e^{s[B+L(t)]}x\ ds\|_H +$$

$$+\|\int_0^\xi e^{(\xi-s)B}L(r)\ [e^{s[B+L(t)]}-e^{s[B+L(r)]}]x\ ds\|_H.$$

Set $\phi_{t,r}(\xi)= \|e^{\xi[B+L(t)]}x - e^{\xi[B+L(r)]}x\|_H$ and

$\Lambda = \|L\|_{C^0([0,T],\mathcal{L}(H))}$. Then we deduce:

$$\phi_{t,r}(\xi) \le |\xi|\|L(t)-L(r)\|_{\mathcal{L}(H)}\|x\|_H\ \frac{e^{(\omega+\Lambda)|\xi|}}{\Lambda} +$$

$$+\ \Lambda\ e^{\omega|\xi|}\left|\int_0^\xi \phi_{t,r}(s)\,ds\right|$$

By a Gronwall-type argument (see e.g. Amann [2], Corollary 2.4) we check

$$\phi_{t,r}(\xi) \leq \|L(t)-L(r)\|_{L(H)}\|x\|_H \left[\frac{|\xi| \; e^{(\omega+\Lambda)|\xi|}}{\Lambda} + \right.$$
$$\left. + \left|\int_0^\xi e^{\Lambda e^{\omega|\xi|}(\xi-s)} \frac{|s| \; e^{(\omega+\Lambda)|s|}}{\Lambda} ds\right|\right]$$

and the result follows easily.

3. APPROXIMATION OF THE STOCHASTIC PROBLEM

Let $f \in L^1_F(0,T,H)$ and $x \in L_{F_0}(H)$. Consider the following linear stochastic problem:

$$du(t)=[A(t)u(t)+\tfrac{1}{2}B^2u(t) + f(t)]dt + Bu(t)dW_t \qquad (S)$$
$$u(0)=X$$

DEFINITION 3.1 We say that $u \in C^0_F([0,T],H)$ is a strict solution of (S) if:

(i) $u(t) \in D(A(t))$ $\forall t \in [0,T]$ w.p.1,

(ii) $t \to A(t)u(t) \in L^1_F(0,T,H)$;

(iii) $t \to B^2u(t) \in L^1_F(0,T,H)$;

(iv) $t \to Bu(t) \in L^2_F(0,T,H)$;

(v) $u(t)=x + \int_0^t [A(s)u(s)+\tfrac{1}{2}B^2u(s)+f(s)]ds + \int_0^t Bu(s)dW_s$ $\forall t \in [0,T]$, w.p.1,

where the stochastic integral in (v) is in the sense of Ito.

DEFINITION 3.2 We say that $u \in C^0_F([0,T],H)$ is a generalized solution of (S) if there exist $\{u_k\}_{k\in N} \subseteq C^0_F([0,T],H)$, $\{f_k\}_{k\in N} \subseteq L^1_F(0,T,H)$, and $\{x_k\} \subseteq L_{F_0}(H)$ such that:

i) u_k is a strict solution of

$$du_k(t)=[A(t)u_k(t) + \frac{1}{2}B^2u_k(t) + f_k(t)]dt + Bu_k(t)dW_t,$$

$$u_k(0) = x_k$$

ii) for each $\varepsilon>0$ we have

$$\lim_{k\to\infty} P\{\sup_{t\in[0,T]} \|u_k(t)-u(t)\|_H > \varepsilon\} = 0$$

$$\lim_{k\to\infty} P\{\int_0^T \|f_k(t)-f(t)\|_H dt > \varepsilon\} = 0$$

$$\lim_{k\to\infty} P\{\|x_u-x\|_H > \varepsilon\} = 0$$

We will consider now a deterministic problem which is, in some sense, an approximation of (S); it is obtained by approaching pathwise the white noise dW_t by a suitable Wiener process $\zeta_n(t)$ (coloured noise), namely the stationary Ornstein-Uhlenbeck process defined by

$$\begin{cases} d\zeta_n(t) = -n\zeta_n(t)dt + ndW_t \\ \zeta_n(0) = 0; \end{cases}$$

then it is well known that

$$\zeta_n(t) = n\int_0^t e^{-n(t-s)}dW_s.$$

Define $W_n(t) = \int_0^t \zeta_n(s)ds$, then we have:

<u>LEMMA 3.3.</u>

(i) $W_n\in C^1[0,T]$, $W_n(0)=0$ w.p. 1;

(ii) $W_n(t)\to W_t$ <u>as</u> $n\to\infty$, <u>uniformly in</u> $[0,T]$, w.p. 1;

(iii) $\|W_n(\cdot)\|_{C^{0,\beta}[0,T]} \le K_\beta < \infty$ w.p. 1 $\forall\beta\in]0,1/2[$

Proof By Ito's formula (i) follows easily and in particular

$$\frac{\partial}{\partial t} W_n(t)=\zeta_n(t) = nW_t-\int_0^t n^2 W_s e^{-n(t-s)}ds \quad \text{w.p. } 1;$$

hence

$$W_n(t) = \int_0^t [nW_s- \int_0^s n^2 W_\sigma e^{-n(s-\sigma)}d\sigma] ds =$$

$$= n \int_0^t W_s ds - \int_0^t [\int_\sigma^t n^2 W_\sigma e^{-n(s-\sigma)}ds] d\sigma =$$

$$= n \int_0^t W_\sigma e^{-n(t-\sigma)}d\sigma, \qquad \text{w.p.1},$$

and again Ito's formula gives

$$W_n(t) = W_t - \int_0^t e^{-n(t-s)}dW_s \qquad \text{w.p.1},$$

which proves (ii).

To prove (iii) let $t,\tau\in[0,T]$ with $\tau<t$. Then

$$|W_n(t)-W_n(\tau)|=|n \int_0^t W_\sigma e^{-n(t-\sigma)}d\sigma-n \int_0^\tau W_\sigma e^{-n(\tau-\sigma)}d\sigma| \leq$$

$$\leq n \int_\tau^t |W_\sigma-W_\tau|e^{-n(t-\sigma)}d\sigma+n| W_\tau | \int_\tau^t e^{-n(t-\sigma)}d\sigma +$$

$$+ \int_0^\tau [e^{-n(t-\sigma)}-e^{-n(\tau-\sigma)}] d\sigma|+ n \int_0^\tau |W_t-W_\sigma|[e^{-n(\tau-\sigma)} -$$

$$- e^{-n(t-\sigma)}]d\sigma+n|W_t-W_\tau|\int_0^\tau [e^{-n(\tau-\sigma)}-e^{-n(t-\sigma)}]d\sigma \quad \text{w.p.1}.$$

Recalling that W_t is β-Holder continuous w.p.1 $\forall\beta\in]0,\frac{1}{2}[$, integration by parts yields

$$|W_n(t)-W_n(\tau)|\leq C\int_\tau^t (\sigma-\tau)^\beta ne^{-n(t-\sigma)}d\sigma+C\tau^\beta|1-e^{-n(t-\tau)} +$$

$$+ e^{-n(t-\tau)}-e^{-nt}-1+e^{-n\tau}|+C[1-e^{-n(t-\tau)}]\int_0^\tau (t-\sigma)^\beta ne^{-n(\tau-\sigma)}d\sigma+$$

$$+C(t-\tau)^\beta[1-e^{-n\tau}-e^{-n(t-\tau)}+e^{-nt}]=C[(t-\tau)^\beta-\int_\tau^t \frac{\beta}{(\sigma-\tau)^{1-\beta}} \cdot$$

$$\cdot\, e^{-n(t-\sigma)}d\sigma] + C\tau^{\beta}[e^{-n\tau}-e^{nt}] + C[1-e^{-n(t-\tau)}][(t-\tau)^{\beta} -$$

$$-t^{\beta}e^{-n\tau}+\int_0^{\tau} \frac{\beta}{(t-\sigma)^{1-\beta}} e^{-n(\tau-\sigma)}d\sigma] + C(t-\tau)^{\beta}[1-e^{-n(t-\tau)}] \cdot$$

$$\cdot [1-e^{-n\tau}] \le C(t-\tau)^{\beta} + C(\tau^{\beta}-t^{\beta})(e^{-n\tau}-e^{-nt}) + C(t-\tau)^{\beta} +$$

$$+ C\, \frac{1-e^{-n(t-\tau)}}{n(t-\tau)}\, \beta(t-\tau)^{\beta}(1-e^{-n\tau}) + C(t-\tau)^{\beta} \le C(3+\beta)(t-\tau)^{\beta}$$

w.p.1.

Now denote by N the subset of Ω such that

$$\begin{cases} P(N)=0, \text{ and for each } \omega\in N^c: \\ t \to f(t,\omega) \in C^0([0,T], H) \\ t \to W_n(t,\omega) \text{ satisfies the properties stated in} \\ \text{Lemma 3.3. for each } n\in\mathbb{N}. \end{cases}$$

Now for each (fixed) $\omega\in N^c$ and $n\in\mathbb{N}$, consider the deterministic problem

$$\begin{cases} v'(t)-A(t)v(t)-W_n'(t)Bv(t)=f(t), & t\in[0,T] \\ v(0)=x. & (S_n(\omega)) \end{cases}$$

<u>DEFINITION 3.4.</u> We say that $v\in C^1([0,T],H)$ is a <u>strict solution</u> of $(S_n(\omega))$ if $v(t)\in DA(t)$ $\forall t\in[0,T]$, $A(\cdot)v(\cdot)\in C^0([0,T],H)$ and $v(0)=x$, $v'-A(\cdot)v(\cdot)-W_n'Bv(\cdot)=f$ in $[0,T]$.

<u>REMARK 3.5</u> If v is a strict solution of $(S_n(\omega))$, then $Bv(\cdot)\in C^0([0,T],H)$ by Hypothesis IV and by the identity $Bu(t)=BA(t)^{-1}(A(t)u(t))$.

<u>DEFINITION 3.6.</u> We say that $v\in C^0([0,T],H)$ is a <u>strong solution</u> of $(S_n(\omega))$ if there exists $\{v_k\}_{k\in N} \subseteq C^1([0,T],H)$ such that:

$$v_k(t)\in DA(t) \quad \forall t\in[0,T], \quad \forall k\in\mathbb{N}$$

$$v_k'-A(\cdot)v_k(\cdot)\in C^0([0,T],H)$$

$$v_k'-A(\cdot)v_k(\cdot)-W_n'Bv_k \triangleq f_k \to f \text{ in } C^0([0,T],H)$$

$$v_k(0) \triangleq x_k \to x \text{ in } H$$

$$v_k \to v \text{ in } C^0([0,T],H).$$

We shall find a strong solution $v(t,\omega)\equiv v_n(t,\omega)$ of $(S_n(\omega))$ for each $f\in C^0_{\mathcal{F}}([0,T],H)$ and $x\in L_{\mathcal{F}_0}(H)$. We shall see that as $n\to\infty$ v_n converges to a process $u(t,\omega)$ which will turn out to be a generalized solution of (S), or, equivalently, a solution of

$$\begin{cases} du(t)=[A(t)u(t)+f(t)]dt + Bu(t)dW_t \\ u(0)=x \end{cases} \qquad (S')$$

where the stochastic integral is in the sense of Stratonovich.

To solve $(S_n(\omega))$, we will transform it into an equivalent one. Set

$$u(t) = e^{-W_n(t)B}v(t),$$

then, formally, u solves

$$\begin{cases} u'(t)=e^{-W_n(t)B}A(t)e^{W_n(t)B}u(t)+e^{-W_n(t)B}f(t), t\in[0,T], \\ u(0)=x. \end{cases} \qquad (P_n(\omega))$$

Define

$$\begin{cases} D(A_n(t))=D(A(t)) \\ A_n(t)z=e^{-W_n(t)B}A(t)e^{W_n(t)B}z \end{cases}$$

Then Problem $(P_n(\omega))$ can be written as

$$\begin{cases} u'(t)-A_n(t)u(t)=F(t), \ t\in[0,T] \\ u(0)=x \end{cases}$$

where $F(t)=e^{-W_n(t)B}f(t)$.

Let us verify that Problems $(S_n(\omega))$ and $(P_n(\omega))$ are indeed the same:

LEMMA 3.7. <u>v is a strict (resp. strong) solution of $(S_n(\omega))$ if and only if u is a strict (resp. strong) solution of $(P_n(\omega))$ in the sense of</u> [1].

<u>Proof</u> By definition if v is a strict solution of $(S_n(\omega))$ we have

$$\begin{cases} v\in C^1([0,T],H), \\ v(t)\in D(A(t)) \ \forall t\in[0,T] \\ A(\cdot)v(\cdot)\in C^0([0,T],H) \\ v(0)=x, \ v'-A(\cdot)v(\cdot)-W_n'Bv(\cdot)\equiv f \ \text{ in } [0,T], \end{cases}$$

so we immediately deduce that

$$\begin{cases} u(t)\in D(A(t)) \quad \forall t\in[0,T] \quad \text{(Proposition 2.8)} \\ A_n(\cdot)u(\cdot)\in C^0([0,T],H) \\ u\in C^1([0,T],H) \\ u(0)=x, \ u'-A_n(\cdot)u(\cdot)\equiv F \ \text{ in } [0,T], \end{cases}$$

i.e. u is a strict solution of $(P_n(\omega))$ in the sense of [1]. The converse is quite similar. The case of strong solutions is analogous.

We want to apply to Problem $(P_n(\omega))$ the results of Acquistapace-Terreni [1]. We have to verify that all hypotheses of [1] hold in the present situation. First of all, we have:

LEMMA 3.8. $\rho(A_n(t))=\rho(A(t))$ for each $n\in\mathbb{N}$ and $t\in[0,T]$, and there exists $C=C(\omega)$ such that

$$\|A_n(t)^{-1}\|_{\mathcal{L}(H)}\le C,\ \|R(\lambda,A_n(t))\|_{\mathcal{L}(H)}\le \frac{C}{|\lambda|}\quad \forall\lambda\in\Sigma_{\theta_0}-\{0\},\ \forall n\in\mathbb{N},\ \forall t\in[0,T].$$

Proof For each $\lambda\in\rho(A(t))$

$$R(\lambda,A_n(t))=e^{-W_n(t)B}R(\lambda,A(t))e^{W_n(t)B},$$

hence the result follows.

LEMMA 3.9. For each $\lambda\in\Sigma_{\theta_0}$ and $x\in H$ the function $t\to R(\lambda,A_n(t))x$ is in $C^1_F([0,T],H)$ and its derivative is given by

$$\frac{\partial}{\partial t}R(\lambda,A(t))x=e^{-W_n(t)B}\frac{\partial}{\partial t}R(\lambda,A(t))e^{W_n(t)B}x+W_n'(t)e^{-W_n(t)B}\cdot$$
$$\cdot[R(\lambda,A(t)),B]e^{W_n(t)B}x;$$

in addition for each $n\in\mathbb{N}$ there exists $C_n=C_n(\omega)$ such that

$$\|\frac{d}{dt}A_n(t)^{-1}\|_{\mathcal{L}(H)}\le C_n,\ \|\frac{\partial}{\partial t}R(\lambda,A_n(t))\|_{\mathcal{L}(H)}\le \frac{C_n}{|\lambda|^\alpha}\quad \forall\lambda\in\Sigma_{\theta_0}-\{0\},\ \forall n\in\mathbb{N}.$$

Proof A straightforward computation yields, as $\tau\to t$

$$\frac{R(\lambda,A_n(t))x-R(\lambda,A_n(\tau))x}{t-\tau}\to e^{-W_n(t)B}\frac{\partial}{\partial t}R(\lambda,A(t))e^{W_n(t)B}x+$$
$$+e^{-W_n(t)B}[R(\lambda,A(t)),B]\ W_n'(t)e^{W_n(t)B}x,$$

and it is clear that $t\to\frac{\partial}{\partial t}R(\lambda,A_n(t))x\in C^0_F([0,T],H)$. Moreover by Proposition 2.4.

$$\left\|\frac{\partial}{\partial t}R(\lambda,A_n(t))\right\|_{L(H)}\leq \frac{K}{|\lambda|^{\alpha}} +\|[B,R(\lambda,A(t))]\|_{L(H)} \cdot$$

$$\cdot \|W_n'\|_{C^0([0,T],H)} \leq \frac{K}{|\lambda|^{\alpha}} + \frac{C_n}{|\lambda|} \leq \frac{C_n}{|\lambda|^{\alpha}} .$$

Taking into account Proposition A.1 of the Appendix, we can apply the results of Acquistapace-Terreni [1], obtaining that Problem $(P_n(\omega))$ has a unique strong solution $u_n(t)$, which in addition satisfies

$$\|u_n(t)\|_H\leq C_n(\omega)\{\|x\|_H+ \int_0^t\|e^{-W_n(s)B}f(s)\|_H ds\} \quad \forall t\in[0,T]$$

Hence Problem $(S_n(\omega))$ has a unique strong solution too, given by $v_n(t) = e^{W_n(t)B}u_n(t)$, which satisfies

$$\|v_n(t)\|_H \leq C_n(\omega)\ [\|x\|_H + \int_0^t\|f(s)\|_H ds] .$$

REMARK 3.10. The function $u_n(t)$, strong solution of $P_n(\omega)$, has its own representation formula in terms of the semi-group $\{e^{\xi A_n(t)}\}_{\xi\geq 0}$ (see [1], formula (4.1.)); consequently a representation formula in terms of $\{e^{\xi A_n(t)}\}_{\xi\geq 0}$ does exist also for the function $v_n(t)$. But we need another formula for $v_n(t)$ in terms of $\{e^{\xi A(t)}\}_{\xi\geq 0}$ and $\{e^{\xi B}\}_{\xi\in R}$, in order to be able later to "pass to the limit" and generalize it to the stochastic case.

THEOREM 3.11. *For each* $n\in\mathbb{N}$, *for each* $x\in H$ *and* $f\in C^0([0,T],H)$, *Problem* $(S_n(\omega))$ *has a unique strong solution given by*

$$v_n(t)=e^{W_n(t)B}e^{tA(t)}x + \int_0^t e^{[W_n(t)-W_n(s)]B}e^{(t-s)A(t)}g_n(s)ds,$$

(3.1)

where $g_n(t)$ is the unique solution of the integral equation

$$g_n(t)+\int_0^t R_n(t,s)g_n(s)ds = f(t)-R_n(t,0)dx \qquad (3.2)$$

whose kernel $R_n(t,s) = R_n(t,s,\omega)$ is

$$R_n(t,s)=e^{[W_n(t)-W_n(s)]B}\,[\frac{\partial}{\partial t}e^{\xi A(t)}]_{\xi\ t-s}-[A(t),\ e^{[W_n(t)-W_n(s)]B}]\ e^{(t-s)A(t)},\quad 0\leq s<t\leq T \qquad (3.3)$$

Proof First of all we prove some lemmata about the integral equation (3.2).

LEMMA 3.12. For each $\sigma\in]0,\alpha]\cap]0,\frac{1}{2}[$ there exists $M_\sigma=M_\sigma(\omega)$ such that

$$\|R_n(t,s)\|_{L(H)}\leq \frac{M_\sigma}{(t-s)^{1-\sigma}}\qquad \forall n\in N,\ 0\leq s<t\leq T,\quad \text{w.p.1}$$

Proof It is an evident consequence of Prop. 2.3(i)-(v)-(viii), Lemma 3.3(i) and Corollary 2.10.

LEMMA 3.13. Consider the integral operator defined by

$[R_n\phi](t)=[R_n(\omega)\phi](t)=\int_0^t R_n(t,s)\phi(s)ds,\quad \phi\in C^0([0,T],H)$ or

$L^p(0,T,H)$, $p\in[1,\infty]$.

Then $(1+R_n)$ is invertible in $C^0([0,T],H)$ and $L^p(0,T,H)$, $1\leq p\leq\infty$, and

$$\|(1+R_n)^{-1}\|_{L(C^0([0,t_0],H))}\leq M'=M'(\omega)$$

$$\qquad\qquad \forall n\in\mathbb{N},\ \forall t_0\in]0,T]$$

$$\|(1+R_n)^{-1}\|_{L(L^p(0,t_0,H))}\leq M'_p=M'_p(\omega)$$

Proof As in [1], Proposition 3.6(i), taking into account Proposition A.1. of the Appendix.

LEMMA 3.14. For each $n\in N$ we have:

(i) $x\in H\Rightarrow R_n(\cdot,0)x\in C^0(]0,T],H)\cap L^p(0,T,H)\quad \forall p\in[1,2\wedge\frac{1}{1-\alpha}[$,

(ii) $x\in D(A(0))\Rightarrow R_n(\cdot,0)x\in C^0([0,T],H)$ and $R_n(0,0)x=0$.

Proof (i) By Lemma 3.12 we get $R_n(\cdot,0)x\in L^p(0,T,H)$ $\forall p\in[1,2\wedge\frac{1}{1-\alpha}[$. Let us show continuity in $]0,T]$: we have

$$R_n(t,0)x=e^{W_n(t)B}[\frac{\partial}{\partial t}e^{\xi A(t)}]_{\xi=t}x-[e^{W_n(t)[B+L(t)]}-$$

$$-e^{W_n(t)B}]A(t)e^{tA(t)}x;$$

the first term is the composition of a strongly continuous operator with the function $t\to[\frac{\partial}{\partial t}e^{\xi A(t)}]_{\xi=t}x$ which is continuous in $]0,T]$ (see [1], Prop. 3.3(i)); hence it is continuous in $]0,T]$.
Similarly the second term is continuous in $[0,T]$ since it is the composition of a strongly continuous operator with the function $t\to A(t)e^{tA(t)}x$, which is continuous in $]0,T]$ ([1], Prop. 3.4(i)).
(ii) if $x\in D(A(0))$ then $t\to[\frac{\partial}{\partial t}e^{\xi A(t)}]_{\xi=t}x$ and $t\to A(t)e^{tA(t)}x$ are continuous in $[0,T]$ and the first vanishes at $t=0$ ([1], Proposition 3.3(iii) and 3.4(v)). By Proposition 2.9 the result follows easily.
The preceding lemmata imply in particular that equation (3.2) is uniquely solvable in $L^p(0,T,H)$, $p\in[1,2\wedge\frac{1}{1-\alpha}[$ and its solution g_n satisfies

$$\|g_n\|_{L^p(0,T,H)}\le C_p=C_p(\omega)\quad \forall n\in N,\ \forall p\in[1,2\wedge\frac{1}{1-\alpha}[.$$

In addition we have:

LEMMA 3.15. *For each* $n\in N$, $g_n\in C^0(]0,T],H)$; *in addition if* $x\in DA(0)$ *then* $g_n\in C^0([0,T],H)$ *and* $g_n(0)=f(0)$.

Proof As in [1], Prop. 3.6(i)-(iii).

We have thus proved that equation (3.2) has a unique solution $g_n\in C^0(]0,T],H)\cap L^p(0,T,H)$ $\forall p\in[1,2\frac{1}{1-\alpha}[$.

Now we will verify that the function $v_n(t)$ given by (3.1) is a strong solution of $(S_n(\omega))$.

First, $v_n\in C^0([0,T],H)$, due to the strong continuity of the group $\{e^{\xi B}\}_{\xi\in R}$ and of the function $t\to e^{tA(t)}$ (see Propositions 3.4(iii) and 3.7(i) in [1]).

Let us construct the regular data x_k, f_k approximating x,f. As $\{x_k\}$ we take any sequence contained in $D(A(0))$ and converging to x. To construct f_k, define

$$\psi_k(t) = (1+R_n)^{-1}(f-R_n(\cdot,0)x_k)(t);$$

then $\psi_k\in C^0([0,T],H)$ by Lemma 3.14(ii) and Lemma 3.12; moreover as $k\to\infty$ $\psi_k\to g_n$ in $L^p(0,T,H)$ for each $p\in[1,2\wedge\frac{1}{1-\alpha}[$, due to Lemma 3.12 and 3.13. Define ψ_k out of $[0,T]$ setting

$$\begin{cases} \psi_k(t) = \psi_k(0), & t<0 \\ \psi_k(t) = \psi_k(T), & t>T. \end{cases}$$

Next, set

$$\phi_k(t) = \theta_k * \psi_k(t) = \int_{\mathbb{R}} \psi_n(t-s)\theta_k(s)ds,$$

where $\theta_k(s)=k\theta(ks)$ is a mollifier: then $\phi_k\in C^1([0,T],H)$ and $\phi_k-\psi_k\to 0$ in $C^0([0,T],H)$ as $k\to\infty$. Now recalling Proposition 2.2, for each $k\in\mathbb{N}$ there exists $h_k\in\mathbb{N}$ such that the function $\xi_k(t)=h_kR(h_k,B)\phi_k(t)$ satisfies

$$\xi_k \in C^1([0,T],H)$$

$$\xi_k(t)\in D(B) \quad \forall t\in[0,T], \ B\xi_k(\cdot)\in C^1([0,T],H) \qquad (3.4)$$

$$\|\xi_k-\phi_k\|_{C^0([0,T],H)} \le \frac{1}{k}$$

Define finally the desired functions f_k by

$$f_k=(1+R_n)\xi_k+R_n(\cdot,0)x_k;$$

then $f_k\in C^0([0,T],H)$ and $f_k\to f$ in $C^0([0,T],H)$ as $k\to\infty$, since

$$f_k-f=(1+R_n)\xi_k+R_n(\cdot,0)x_k-f=(1+R_n)[\xi_k-\psi_k]+(1+R_n)\psi_k+R_n(\cdot,0)\cdot$$

$$\cdot x_k-f=(1+R_n)(\xi_k-\phi_k)+(1+R_n)(\phi_k-\psi_k) \to 0 \text{ as } k\to\infty$$

We have thus constructed the approximating data x_k, f_k.

Now set

$$u_k(t)=e^{W_n(t)B}e^{tA(t)}x_k+\int_0^t e^{[W_n(t)-W_n(s)]B}e^{(t-s)A(t)}\Big|\cdot$$
$$\cdot\xi_k(s)ds; \qquad (3.5)$$

we shall verify that $u_k\to v_n$ in $C^0([0,T],H)$ as $k\to\infty$, and that u_k is the strict solution of

$$u_k'(t)-A(t)u_k(t)-W_n'(t)Bu_k(t)=f_k(t)$$
$$u_k(0)=x_k; \qquad (3.6)$$

this will prove that v_n is the strong solution of $(S_n(\omega))$.

It is clear that

$$\sup_{t\in[0,T]}\|u_k(t)-v_n(t)\|_H\le C\|x_k-x\|_H+C\int_0^T\|\xi_k(s)-g_n(s)\|_H ds\to 0$$

as $k\to\infty$,

since $\xi_k \to g_n$ in $L^p(0,T,H)$ $\forall p \in [1, 2\wedge \frac{1}{1-\alpha}[$.

Let us show that u_k solves (3.6). Let us compute $A(t)\cdot$ $\cdot u_k(t)$: to begin with, the first term in (3.5) is in $D(A(t))$ (Proposition 2.8) and

$$A(t)e^{W_n(t)B}e^{tA(t)}x_k = e^{W_n(t)[B+L(t)]}A(t)e^{tA(t)}x_k; \quad (3.7)$$

clearly it is a continuous function of t (see Proposition 2.11 and the proof of Lemma 3.14(ii)).

The second term in (3.5) can be written as:

$$\int_0^t e^{[W_n(t)-W_n(s)]B}e^{(t-s)A(t)}\xi_k(s)ds = A(t)^{-1}\cdot$$

$$\cdot\{\int_0^t e^{[W_n(t)-W_n(s)][B+L(t)]}A(t)e^{(t-s)A(t)}[\xi_k(s)-\xi_k(t)]ds +$$

$$+\int_0^t [e^{[W_n(t)-W_n(s)][B+L(t)]}-e^{[W_n(t)-W_n(s)]B}]A(t)e^{(t-s)A(t)}\cdot$$

$$\cdot\xi_k(t)ds + \int_0^t [e^{[W_n(t)-W_n(s)]B}-1]A(t)e^{(t-s)A(t)}\xi_k(t)ds +$$

$$+(e^{t(A(t)}-1)\xi_k(t)\}, \quad (3.8)$$

and all integrals do converge (by (3.4), Proposition 2.9 and Proposition 2.3(iii)-(v));

hence this term belongs to $D(A(t)$ and is a continuous function of t, as it can be easily seen by a repeated use of Lebesgue's Theorem. This shows that $u_k(t)\in DA(t)$ $\forall t\in[0,T]$ and that $A(\cdot)u_k(\cdot)\in C^0([0,T],H)$. Let us compute now $u_k'(t)$. It is easy to verify (see also [1], Propositions 3.4(i) and 3.7(iv)) that if $t\in]0,T]$ we have

$$\frac{d}{dt}e^{W_n(t)B}e^{tA(t)}x_k = W_n'(t)Be^{W_n(t)B}e^{tA(t)}x_k + e^{W_n(t)B}.$$

$$\cdot A(t)e^{tA(t)}x_k+e^{W_n(t)B}\left[\frac{\partial}{\partial t}e^{\xi A(t)}\right]_{\xi=t}x_k,$$

and

$$\frac{d}{dt}\int_0^t e^{[W_n(t)-W_n(s)]B}e^{(t-s)A(t)}\xi_k(s)ds=\xi_k(t)+$$

$$+\int_0^t W_n'(t)Be^{[W_n(t)-W_n(s)]B}e^{(t-s)A(t)}\xi_k(s)ds+$$

$$+\int_0^t e^{[W_n(t)-W_n(s)]B}A(t)e^{(t-s)A(t)}[\xi_k(s)-\xi_k(t)]ds+$$

$$+\int_0^t[e^{[W_n(t)-W_n(s)]B}-1]A(t)e^{(t-s)A(t)}\xi_k(t)ds+$$

$$+(e^{tA(t)}-1)\xi_k(t)+\int_0^t e^{[W_n(t)-W_n(s)]B}\left[\frac{\partial}{\partial t}e^{\xi A(t)}\right]_{\xi=t-s}\xi_k(s)ds.$$

Taking into account Hypothesis IV, it is seen that these functions are continuous in $]0,T]$; this shows that $u_k'\in C^0(]0,T],H)$ and summing up we get $\forall t\in]0,T]$:

$$u_k'(t)=A(t)u_k(t)+W_n'(t)Bu_k(t)+\xi_k(t)+e^{W_n(t)B}\left[\frac{\partial}{\partial t}e^{\xi A(t)}\right]_{\xi=t}x_k-$$

$$-[e^{W_n(t)[B+L(t)]}-e^{W_n(t)B}]A(t)e^{tA(t)}x_k+\int_0^t e^{[W_n(t)-W_n(s)]B}\cdot$$

$$\cdot\left[\frac{\partial}{\partial t}e^{\xi A(t)}\right]_{\xi=t-s}\xi_k(s)ds-\int_0^t[e^{[W_n(t)-W_n(s)][B+L(t)]}-$$

$$-e^{[W_n(t)-W_n(s)]B}]A(t)e^{(t-s)A(t)}\xi_k(s)ds=A(t)u_k(t)+W_n'(t)B\cdot$$

$$\cdot u_k(t)+\xi_k(t)+R_n(t,0)x_k+\int_0^t R_n(t,s)\xi_k(s)ds=$$

$$=A(t)u_k(t)+W_n'(t)Bu_k(t)+f_k(t).$$

On the other hand, as $t\to 0^+$ we have (see Lemma 3.3.(iii) of [1]):

$$u_k'(t)\to A(0)x_k+W_n'(0)Bx_k+\xi_k(0)=A(0)x_k+W_n'(0)Bx_k+f_k(0)$$

and this shows that $u_k' \in C^0([0,T],H)$ and that u_k solves (3.6). The proof of Theorem 3.11 is complete.

4. CONVERGENCE OF THE SOLUTIONS

Let $x \in L_{\mathcal{F}_0}(H)$, $f \in C^0_{\mathcal{F}}([0,T],H)$. For a.e. $\omega \in \Omega$ and for each $n \in \mathbb{N}$ we can solve the deterministic problem $(S_n(\omega))$ with data $x(\omega) \in H$ and $f(\cdot,\omega) \in C^0([0,T],H)$; its strong solution $v_n(\cdot,\omega)$ is then given by (3.1). In this section we will show that the sequence $\{v_n\}$ converges uniformly in $[0,T]$ w.p.1. More precisely we have:

THEOREM 4.1. Let $x \in L_{\mathcal{F}_0}(H)$, $f \in C^0_{\mathcal{F}}([0,T],H)$ and let $v_n(t,\omega)$ be given by (3.1). Then as $n \to \infty$ $v_n \to u$ uniformly in $[0,T]$ w.p.1, where $u \in C^0_{\mathcal{F}}([0,T],H)$ is defined by

$$u(t) = e^{W_t B} e^{tA(t)} x + \int_0^t e^{(W_t - W_s)B} e^{(t-s)A(t)} g(s)\,ds, \qquad (4.1)$$

$g(t)$ being the unique solution of the Volterra integral equation

$$g(t) + \int_0^t R(t,s) g(s)\,ds = f(t) - R(t,0)x \qquad (4.2)$$

whose kernel $R(t,s)$ is given by

$$R(t,s) = e^{(W_t - W_s)B} \left[\frac{\partial}{\partial t} e^{\xi A(t)} \right]_{\xi = t-s} - [A(t), e^{(W_t - W_s)B}] \cdot$$
$$\cdot\, e^{(t-s)A(t)} \qquad (4.3)$$

Proof We need some preliminary lemmata.

LEMMA 4.2. For each $\sigma \in]0,\alpha] \cap]0,1/2[$ there exists $M_\sigma = M_\sigma(\omega)$ such that

$$\| R(t,s) \|_{\mathcal{L}(H)} \leq \frac{M_\sigma}{(t-s)^{1-\sigma}} \quad 0 \leq s < t \leq T, \text{ w.p.1.}$$

Proof. As in Lemma 3.12.

LEMMA 4.3. For each $p \in [1, \frac{1}{1-\alpha} \wedge 2[$ we have as $n \to \infty$

$$R_n(\cdot,0)x \to R(\cdot,0)x \quad \text{in } L^p(0,T,H) \quad \text{w.p.1.}$$

Proof It is a simple application of Lebesgue's Theorem.

LEMMA 4.4. For each $p \in [1,\infty]$ define

$$R\phi(t) = [R(\omega)\phi](t) = \int_0^t R(t,s)\phi(s)ds, \quad \phi \in L^p(0,T,H) \text{ or}$$
$$\phi \in C^0([0,T],H). \tag{4.4}$$

Then $R \in \mathcal{L}(L^p(0,T,H)) \cap \mathcal{L}(C([0,T],H))$ $\forall p \in [1,\infty]$, w.p.1.
If in addition R_n is the integral operator whose kernel is $R_n(t,s)$, the we have for each $\phi \in L^p(0,T,H)$, $p \in [1,\infty[$,

$$R_n\phi \to R\phi \quad \text{in } L^p(0,T,H) \quad \text{w.p.1.}$$

Proof. The boundedness of R can be proved as in [1], Proposition 3.5(i). Next, if $0 \leq s < t \leq T$ we have

$$\lim_{n\to\infty} \| [R_n(t,s) - R(t,s)]\phi(s) \|_H = 0 \text{ w.p.1.}$$

Hence by Lemma 3.12, Lemma 4.2 and Lebesgue's Theorem we get

$$\lim_{n\to\infty} \int_0^t \| [R_n(t,s) - R(t,s)]\phi(s) \|_H^p ds = 0 \quad \forall t \in [0,T] \text{ w.p.1.}$$

On the other hand,

$$\int_0^t \| [R_n(t,s) - R(t,s)]\phi(s) \|_H^p ds \leq C\, T^{\sigma(p-1)} \int_0^t \| \phi(s) \|_H^p ds,$$

and applying again Lebesgue's Theorem we obtain the result.

LEMMA 4.5.

(i) $(1+R)$ has bounded inverse in $C^0([0,T],H)$ and in $L^p(0,T,H)$ for each $p\in[1,\infty]$, w.p.1.

(ii) For each $p\in[1,\infty[$ and $\phi\in L^p(0,T,H)$ we have as $n\to\infty$

$$(1+R_n)^{-1}\phi \to (1+R)^{-1}\phi \quad \text{in } L^p(0,T,H), \quad \text{w.p.1.}$$

Proof.

(i) As in Lemma 3.13.

(ii) Set $\Psi_n=(1+R_n)^{-1}\phi$, $\Psi=(1+R)^{-1}\phi$; then $\Psi_n,\Psi\ L^p(0,T,H)$ and

$$\Psi_n-\Psi=(1+R_n)^{-1}(R_n-R)\Psi;$$

hence Lemmata 3.13 and 4.4 yield the result.

The preceding lemmata imply that the integral equation (4.2) has a unique solution g belonging to $L^p(0,T,H)$ $\forall p\in]0,\frac{1}{1-\alpha}\wedge 2[$ w.p.1; namely $g(t)=(1+R)^{-1}(f-R(\cdot,0)x)$. In addition we have:

LEMMA 4.6. $R(\cdot,0)x$ and g belong to $C^0(]0,T],H)$ w.p.1. If in addition $x\in L_{\mathcal{F}_0}(D(A(0)))$ then w.p.1 $R(\cdot,0)x$, $g\in C^0([0,T],H)$ and $R(0,0)x=0$, $g(0)=f(0)$.

Proof. As $t\mapsto W_t$ is β-Hölder continuous $\forall\beta\in]0,1/2[$ w.p.1, it suffices to repeat the proof of Lemmata 3.14 and 3.15.

Now we are able to prove Theorem 4.1. In what follows we fix ω out of the exceptional set whose P-measure is 0 and where all the preceding lemmata may fail to be true. Let $\varepsilon>0$. Because of Proposition 2.1 there exists $\delta_\varepsilon>0$ such that

$$|\sigma|<\delta_\varepsilon \Rightarrow \|(e^{\sigma B}-1)x\|_H<\varepsilon;$$

$$0<t<\delta_\varepsilon \Rightarrow \|(e^{tA(t)}-1)x\|_H<\varepsilon.$$

Set $K = \sup_{t,n}|W_n(t)|$, fix $\lambda_0\in\rho(B)$ and define

$M_\varepsilon = \sup_{t\geq\delta_\varepsilon} \|(\lambda_0-B)e^{tA(t)}x\|_H$ (note that $M_\varepsilon \leq \frac{c}{\delta_\varepsilon}$ by Proposition 2.6(i)). Next, take $n_\varepsilon\in\mathbb{N}$ such that

$$n\geq n_\varepsilon \Rightarrow |W_n(t)-W_t|<\delta_\varepsilon\wedge\varepsilon\wedge\frac{\varepsilon}{M_\varepsilon} \quad \forall t\in[0,T].$$

Then by Proposition 2.3(i) we have, for each $t\in[0,\delta_\varepsilon]$

$$\|(e^{W_n(t)B}-e^{W_tB})e^{tA(t)}x\|_H\leq \|(e^{W_n(t)B}-e^{W_tB})(e^{tA(t)}-1)x\|_H+$$

$$+\|e^{W_tB}(e^{(W_n(t)-W_t)B}-1)x\|_H\leq 2Ne^{\omega K}\varepsilon+Ne^{\omega K}\varepsilon\leq C\varepsilon \quad \forall n\geq n_\varepsilon;$$

on the other hand for each $t\in[\delta_\varepsilon,T]$ we have by Proposition 2.3(iii)

$$\|(e^{W_n(t)B}-e^{W_tB})e^{tA(t)}x\|_H\leq \|e^{W_tB}[e^{(W_n(t)-W_t)B}-1]\cdot$$

$$\cdot R(\lambda_0,B)(\lambda_0-B)e^{tA(t)}x\|_H\leq Ne^{\omega K}\sup_{0\leq s\leq 2K}\left\|\frac{e^{sB}-1}{s}R(\lambda_0,B)\right\|_{\mathcal{L}(H)}\cdot$$

$$\cdot|W_n(t)-W_t|\cdot M_\varepsilon\leq Ne^{\omega K}C\varepsilon \quad \forall n\geq n_\varepsilon.$$

This proves that as $n\to\infty$

$$e^{W_n(t)B}e^{tA(t)}x \to e^{W_tB}e^{tA(t)}x \quad \text{uniformly in } [0,T] \text{ w.p.1.}$$

We shall prove now that as $n\to\infty$

$$\int_0^t e^{[W_n(t)-W_n(s)]B}e^{(t-s)A(t)}g_n(s)\,ds\to\int_0^T e^{(W_t-W_s)B}e^{(t-s)A(t)}\cdot$$

$\cdot g(s)ds$ uniformly in $[0,T]$ w.p.1.

By Lemmata 4.3, 3.12 and 4.4 we have $g_n\to g$ in $L^1(0,T,H)$;

thus it is enough to show that as $n\to\infty$

$$\sup_{t\in[0,T]} \| \int_0^t [e^{[W_n(t)-W_n(s)]B}-e^{(W_t-W_s)B}]e^{(t-s)A(t)}g(s)ds\|_H \to 0$$

w.p.1.

Since $g\in L^1(0,T,H)\cap C^0(]0,T],H)$, for each $\varepsilon>0$ we can choose $\delta_\varepsilon>0$ such that $\int_t^{t+\delta_\varepsilon} \|g(s)\|_H ds<\varepsilon \quad \forall t\in[0,T-\delta_\varepsilon]$; set

$H_\varepsilon = \sup_{t\in[\delta_\varepsilon,T]} \|g(t)\|_H$ and take $n_\varepsilon\in\mathbb{N}$ such that $|W_n(t)-W_t|< \frac{\varepsilon\cdot\delta_\varepsilon}{H_\varepsilon}$ for each $n\geq n_\varepsilon$ and $t\in[0,T]$. Then it follows that, if $t\in[0,\delta_\varepsilon]$,

$$\|\int_0^t [e^{[W_n(t)-W_n(s)]B}-e^{(W_t-W_s)B}]e^{(t-s)A(t)}g(s)ds\|_H \leq$$

$$\leq C\int_0^{\delta_\varepsilon} \|g(s)\|_H ds \leq C\varepsilon \quad \forall n\in\mathbb{N},$$

while if $t\in]\delta_\varepsilon,T]$ by Proposition 2.3(iii) and 2.6 (i) we have

$$\|\int_0^t [e^{[W_n(t)-W_n(s)]B}-e^{(W_t-W_s)B}]e^{(t-s)A(t)}g(s)ds\|_H \leq$$

$$\leq \|\int_0^{\delta_\varepsilon} \ldots ds\|_H + \|\int_{\delta_\varepsilon}^{t-\delta_\varepsilon} \ldots ds\|_H + \|\int_{t-\delta_\varepsilon}^{t} \ldots ds\|_H \leq$$

$$\leq 2C\varepsilon + \|\int_{\delta_\varepsilon}^{t-\delta_\varepsilon} e^{(W_t-W_s)B}[e^{(W_n(t)-W_n(s)-W_t+W_s)B}-1] \cdot$$

$$\cdot R(\lambda_0,B)(\lambda_0-B)\, e^{(t-s)A(t)}g(s)ds\|_H \leq 2C\varepsilon +$$

$$+C\int_{\delta_\varepsilon}^{t-\delta_\varepsilon}[\,|W_n(t)-W_t|+|W_n(s)-W_s|\,]\,\frac{ds}{t-s}\cdot H_\varepsilon \leq 2C\varepsilon+2C\cdot$$

$$\cdot \frac{\varepsilon\delta_\varepsilon}{H_\varepsilon}\,\frac{1}{\delta_\varepsilon}\,T\,H_\varepsilon = C\varepsilon .$$

To complete the proof of Theorem 4.1 it remains to show that $u \in C^0_{\mathcal{F}}([0,T],H)$, i.e. $\omega \to u(t,\omega)$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$. First of all we have:

LEMMA 4.7. For each $y \in H$ and $t \in [0,T]$, the function $\omega \to e^{W_t(\omega)B} y$ is $\mathcal{F}_t$ measurable.

Proof. The following equality holds:

$$e^{W_t(\omega)B}y = \begin{cases} y \quad \text{if } W_t(\omega)=0 \\ \lim\limits_{k\to\infty} [\frac{k}{W_t(\omega)} R(\frac{k}{W_t(\omega)}, B)]^k y \text{ if } W_t(\omega)>0 \\ \lim\limits_{k\to\infty} [(\frac{k}{W_t(\omega)} R(\frac{k}{W_t(\omega)}, -B)]^k y \text{ if } W_t(\omega)<0 \end{cases}$$

Define

$$\phi_k(\omega) = y\chi_{\{W_t=0\}} + [\frac{k}{W_t(\omega)} R(\frac{k}{W_t(\omega)}, B)]^k y \cdot \chi_{\{0<W_t<\frac{k}{\eta}\}} +$$

$$+ [\frac{k}{W_t(\omega)} R(\frac{k}{W_t(\omega)}, -B)]^k y \cdot \chi_{\{-\frac{k}{\eta} < W_t < 0\}};$$

then $\phi_k(\omega) \to e^{W_t(\omega)B}y$ as $k \to \infty$ w.p.1.

Since H is separable, it is enough to prove that for each $k \to \mathbb{N}$ the function

$$\omega \to [\frac{k}{W_t(\omega)} R(\frac{k}{W_t(\omega)}, B)]^k y \cdot \chi_{\{0<W_t<\frac{k}{\eta}\}}$$

is $\mathcal{F}_t$-measurable. Consider the functions

$$\psi : \{|s|>\eta\} \to H, \qquad \psi(s) = [sR(s,B)]^k y$$

$$F : \mathbb{R} - \{0\} \to \mathbb{R}, \qquad F(\tau) = \frac{k}{\tau}$$

we have to show that $\omega \to (\psi \circ F)(W_t(\omega)) \cdot \chi_{\{0<W_t<\frac{k}{n}\}} =: G(\omega)$

is $\mathcal{F}_t$-measurable. Now if $A \subseteq H$ is a Borel set, we have

$$\{G\in A\}=\begin{cases}[\{0<W_t<\frac{k}{n}\}\cap\{W_t\in F^{-1}(\psi^{-1}(A))\}]\cup\{W_t\le 0\}\cup\{W_t\ge\frac{k}{n}\} & \text{if } 0\in A;\\ \{0<W_t<\frac{k}{n}\}\cap\{W_t\in F^{-1}(\psi^{-1}(A))\} & \text{if } 0\notin A.\end{cases}$$

As $\omega\to W_t(\omega)$ is $\mathcal{F}_t$-measurable and $F^{-1}(\psi^{-1}(A))$ is a Borel set of $\mathbb{R}$, we conclude that $\{G\in A\}\in\mathcal{F}_t$.

LEMMA 4.8. Let $t\in[0,T]$ and consider the kernel $R(t,s,\omega)$ and the operator $R(\omega)$ defined in (4.3) and (4.4). Then we have:

(i) If $x\in L_{\mathcal{F}_0}(H)$, then the function $\omega\to R(t,s,\omega)x(\omega)$ is $\mathcal{F}_t$-measurable for each $s\in[0,t[$.

(ii) If $\phi\in L^1_{\mathcal{F}}(0,T,H)$ then $\omega\to[R(\omega)\phi](t,\omega)$ is $\mathcal{F}_t$-measurable.

(iii) If $\phi\in L^1_{\mathcal{F}}(0,T,H)$ then $\omega\to[1+R(\omega)]^{-1}\phi(t,\omega)$ is $\mathcal{F}_t$-measurable.

Proof. (i) As W_t-W_s is $\mathcal{F}_t$-measurable for each $s\in[0,t[$, the result is an easy consequence of Lemma 4.7.

(ii) Set $\psi(s,\omega)=R(t,s,\omega)\phi(s,\omega)$; then by (i), $\omega\to\psi(s,\omega)$ is $\mathcal{F}_t$-measurable for each $s\in[0,t[$. Thus there exists a sequence of functions ψ_k, having the form

$$\psi_k(s,\omega) = \sum_{i=1}^{n_k} \psi(s_{i-1}^k,\omega)\chi_{[s_{i-1}^k,s_i^k[}(s), \quad 0=s_0^k<\ldots<s_{n_k}^k=t,$$

such that as $k\to\infty$

$$\psi_k(s,\omega)\to\psi(s,\omega) \quad \text{for a.e. } s\in[0,t[\quad \text{w.p.1},$$

$$\int_0^t \psi_k(s,\omega)ds \to [R(\omega)\phi](t,\omega) \qquad \text{w.p.1.}$$

Since $\omega \to \int_0^t \psi_k(s,\omega)ds = \sum_{i=1}^{n_k} \psi(s_{i-1}^k,\omega)(s_i^k - s_{i-1}^k)$ is F_t-mea surable, the conclusion follows.

(iii) From the identity $[1+R(\omega)]^{-1}\phi = \sum_{n=0}^{\infty} [R(\omega)]^n \phi$ we deduce by induction the result, since each term in the series is F_t-measurable by (ii).

By Lemmata 4.6, 4.7 and 4.8 we conclude that the function $u(t,\omega)$ defined in (4.1) belongs to $C_F^0([0,T],H)$; Theorem 4.1 is completely proved.

5. THE STOCHASTIC PROBLEM: EXISTENCE

Let us go back to the stochastic problem (S) introduced at the beginning of Section 3. We want to show that the function $u(t)$ defined in (4.1) is a generalized solution of (S). We will first consider the particular case in which $x \in L_{F_0}(D(A(0)) \cap D(B^2))$ and the integral equation (4.2) has a solution g having suitable regularity properties. More precisely we have:

THEOREM 5.1. *Let* $x \in L_{F_0}(D(A(0)) \cap D(B^2))$, *and let* $f \in C_F^0([0,T],H)$ *have the form*

$$f(t) = [(1+R)g](t) + R(t,0)x, \tag{5.1}$$

with $g \in C_F^1([0,T],H)$ *such that* $g(t) \in D(A(0)) \cap D(B^2)$ $\forall t \in [0,T]$ *and* $B^2 g(\cdot) \in C_F^0([0,T],H)$. *Then the function* $u(t)$ *defined in* (4.1) *is a strict solution of* (S) (*see Definition* 3.1)

Proof. Let us verify that $u(t)\in D(A(t))\cap D(B^2)$ $\forall t\in[0,T]$ w.p.1. As in the proof of Theorem 3.11, we have $u(t)\in D(A(t))$ w.p.1 and (compare with (3.7), (3.8)):

$$A(t)u(t)=e^{W_t[B+L(t)]}A(t)e^{tA(t)}x+\int_0^t e^{(W_t-W_s)[B+L(t)]}\cdot A(t)e^{(t-s)A(t)}[g(s)-g(t)]ds+\int_0^t[e^{(W_t-W_s)[B+L(t)]}-e^{(W_t-W_s)B}]A(t)e^{(t-s)A(t)}g(t)ds+\int_0^t[e^{(W_t-W_s)B}-1]\cdot A(t)e^{(t-s)A(t)}g(t)ds+(e^{tA(t)}-1)g(t), \qquad (5.2)$$

moreover it can be seen that $A(\cdot)u(\cdot)\in C^0_{\mathcal{F}}([0,T],H)$, by using arguments which are similar to those employed in Theorems 3.11 and 4.1.

Thus, in particular, $u(t)\in D(B)$ $\forall t\in[0,T]$ w.p.1 and

$$t\to Bu(t) = BA(t)^{-1}A(t)u(t)\in C^0([0,T],H);$$

but we need now a different expression for $Bu(t)$, namely

$$Bu(t)=e^{W_tB}[\frac{1}{2\pi i}\int_\gamma e^{t\lambda}[B,R(\lambda,A(t))]xd\lambda]+e^{W_tB}e^{tA(t)}Bx+\int_0^t e^{(W_t-W_s)B}[\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}[B,R(\lambda,A(t))]d\lambda]g(s)ds+\int_0^t e^{(W_t-W_s)B}e^{(t-s)A(t)}Bg(s)ds. \qquad (5.3)$$

Let us show now that $u(t)\in D(B^2)$ $\forall t\in[0,T]$ w.p.1. By (5.3) we see that the first term in (4.1) belongs to $D(B^2)$ and, by Proposition 2.4,

$$B^2e^{W_tB}e^{tA(t)}x=Be^{W_tB}[\frac{1}{2\pi i}\int_\gamma e^{t\lambda}[B,R(\lambda,A(t))]xd\lambda+e^{tA(t)}Bx]=$$

$$=e^{W_tB}[-\frac{1}{2\pi i}\int_\gamma e^{t\lambda}BR(\lambda,A(t))L(t)A(t)R(\lambda,A(t))xd\lambda+\frac{1}{2\pi i}\int_\gamma e^{t\lambda}[B,R(\lambda,A(t))]Bxd\lambda+e^{tA(t)}B^2x]=$$

$$=e^{W_tB}[-\frac{1}{2\pi i}\int_\gamma e^{t\lambda}BR(\lambda,A(t))L(t)[\lambda(R(\lambda,A(t))-R(\lambda,A(0)))x+$$

$$+A(0)R(\lambda,A(0))x]\,d\lambda+\frac{1}{2\pi i}\int_\gamma e^{t\lambda}[B,R(\lambda,A(t))]Bx\,d\lambda+e^{tA(t)}B^2x]=$$

$$=e^{W_tB}[\frac{1}{2\pi i}\int_\gamma e^{t\lambda}BR(\lambda,A(t))L(t)[\int_0^t(-\lambda\frac{\partial}{\partial s}R(\lambda,A(s))\;[A(0)^{-1}-$$

$$-A(s)^{-1}]A(0)x\,ds-\int_0^t(\frac{\partial}{\partial s}R(\lambda,A(s))+\frac{d}{ds}A(s)^{-1})A(0)x\,ds+$$

$$+\int_0^t\lambda R(\lambda,A(s))\frac{d}{ds}A(s)^{-1}A(0)x\,ds-R(\lambda,A(0))A(0)x]\,d\lambda]+$$

$$+e^{W_tB}(\frac{1}{2\pi i}\int_\gamma e^{t\lambda}[B,R(\lambda,A(t))]Bx\,d\lambda)+e^{W_tB}e^{tA(t)}B^2x. \quad (5.4)$$

It is not difficult to see that all integrals converge and that the last equality in (5.4) defines an element of $C^0_F([0,T],H)$.

Again by (5.3) and Propositions 2.4, 2.8 we have that the second term in (4.1) is in $D(B^2)$ and

$$B^2\int_0^t e^{(W_t-W_s)B}e^{(t-s)[illegible]}[illegible](s)\,ds=B[\int_0^t e^{(W_t-W_s)B}\,\cdot$$

$$\cdot[-\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}R(\lambda,A(t))L(t)A(t)R(\lambda,A(t))g(s)\,d\lambda+$$

$$+e^{(t-s)A(t)}Bg(s)]\,ds]=\int_0^t BA(t)^{-1}e^{(W_t-W_s)[B+L(t)]}\,\cdot$$

$$\cdot[-\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}A(t)R(\lambda,A(t))L(t)A(t)R(\lambda,A(t))g(s)\,d\lambda]\,ds+$$

$$+\int_0^t e^{(W_t-W_s)B}[\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}[B,R(\lambda,A(t))]Bg(s)\,d\lambda]\,ds+$$

$$+\int_0^t e^{(W_t-W_s)B}e^{(t-s)A(t)}B^2g(s)\,ds= \quad (5.5)$$

$$=\int_0^t BA(t)^{-1}e^{(W_t-W_s)[B+L(t)]}\;[\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}A(t)R(\lambda,A(t))\cdot$$

$$\cdot L(t)[\int_0^t (-\lambda\frac{\partial}{\partial\sigma}R(\lambda,A(\sigma))[A(0)^{-1}-A(\sigma)^{-1}]A(0)g(s)d\sigma-$$

$$-\int_0^t (\frac{\partial}{\partial\sigma}R(\lambda,A(\sigma))+\frac{d}{d\sigma}A(\sigma)^{-1})A(0)g(s)d\sigma+\int_0^t \lambda R(\lambda,A(\sigma))\cdot$$

$$\cdot\frac{d}{d\sigma}\cdot A(\sigma)^{-1}A(0)g(s)d\sigma-R(\lambda,A(0))A(0)g(s)]d\lambda]ds+\int_0^t e^{(W_t-W_s)B}\cdot$$

$$\cdot[\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}[B,R(\lambda,A(t))]Bg(s)d\lambda]ds+\int_0^t e^{(W_t-W_s)B}\cdot$$

$$\cdot e^{(t-s)A(t)}B^2g(s)ds;$$

again it is seen that the last equality defines a function belonging to $C^0_{\mathcal{F}}([0,T],H)$.

We have thus proved that $u(t)\in D(A(t))\cap D(B^2)$ for each $t\in[0,T]$ w.p.1, and that the functions $t\to A(t)u(t)$, $t\to Bu(t)$, $t\to B^2u(t)$ belong to $C^0_{\mathcal{F}}([0,T],H)$. We have now to verify that

$$u(t)=x+\int_0^t [A(s)u(s)+\frac{1}{2}B^2u(s)+f(s)]ds+\int_0^t Bu(s)dW_s \quad \forall t\in[0,T] \text{ w.p.1.} \tag{5.6}$$

Let us compute first the Ito integral $\int_0^t Bu(s)dW_s$. We recall Ito's Formula:

<u>LEMMA 5.2.</u> <u>Let</u> $G=G(y,r):\mathbb{R}\times[0,T]\to H$ <u>be a continuous function such that</u> $\frac{\partial G}{\partial y}$, $\frac{\partial^2 G}{\partial y^2}$, $\frac{\partial G}{\partial r}$ <u>are continuous</u>. <u>Then</u>

$$G(W_t,t)=G(0,0)+\int_0^t [\frac{\partial G}{\partial r}(W_s,s)+\frac{1}{2}\frac{\partial^2 G}{\partial y^2}(W_s,s)]ds+$$

$$+\int_0^t \frac{\partial G}{\partial y}(W_s,s)dW_s.$$

<u>Proof</u>. See Friedman [11] page 81.

We will apply Lemma 5.2 with suitable choices of the function $G(y,r)$.

Suppose first

$$G(y,r)= e^{yB}\ e^{rA(r)}x;$$

then

$$\frac{\partial G}{\partial y}(y,r)=Be^{yB}e^{rA(r)}x,\ \frac{\partial^2 G}{\partial y^2}(y,r)=B^2e^{yB}e^{rA(r)}x,$$

$$\frac{\partial G}{\partial r}(y,r)=e^{yB}[A(r)e^{rA(r)}x+[\frac{\partial}{\partial r}e^{\xi A(r)}]_{\xi=r}x]$$

which implies

$$\int_0^t Be^{W_sB}e^{sA(s)}xdW_s=e^{W_tB}e^{tA(t)}x-x-\int_0^t [e^{W_sB}[A(s)e^{sA(s)}x +$$

$$[\frac{\partial}{\partial s}e^{\xi A(s)}]_{\xi=s}x]+\frac{1}{2}B^2e^{W_sB}e^{sA(s)}x]\,ds \tag{5.7}$$

Set now

$$G(y,r)=\int_0^r e^{(y-W_\sigma)B}e^{(r-\sigma)A(r)}g(\sigma)d\sigma;$$

then it is easily seen that

$$\frac{\partial G}{\partial y}(y,r)=B\int_0^r e^{(y-W_\sigma)B}e^{(r-\sigma)A(r)}g(\sigma)d\sigma,\frac{\partial^2 G}{\partial y^2}(y,r)= B^2\cdot$$

$$\cdot\int_0^r e^{(y-W_\sigma)B}e^{(r-\sigma)A(r)}g(\sigma)d\sigma,$$

and(compare with (5.2))

$$\frac{\partial G}{\partial r}(y,r)=e^{(y-W_r)B}g(r)+\int_0^r e^{(y-W_\sigma)B}[\frac{\partial}{\partial r}e^{\xi A(r)}]_{\xi=r-\sigma}\cdot$$

$$\cdot g(\sigma)d\sigma+\int_0^r e^{(y-W_\sigma)B}A(r)e^{(r-\sigma)A(r)}[g(\sigma)-g(r)]\,d\sigma\ +$$

$$+\int_0^r[e^{(y-W_\sigma)B}-e^{(y-W_r)B}]A(r)e^{(r-\sigma)A(r)}g(r)d\sigma+$$

$$+\ e^{(y-W_r)B}[e^{rA(r)}-1]\,g(r).$$

Thus we deduce that

$$\int_0^t B\left[\int_0^s e^{(W_s-W_\sigma)B} e^{(s-\sigma)A(s)} g(\sigma)\,d\sigma\right] dW_s = \int_0^t e^{(W_t-W_\sigma)B} \cdot$$

$$\cdot e^{(t-\sigma)A(t)} g(\sigma)\,d\sigma - \int_0^t \Big[g(s) + \int_0^s e^{(W_s-W_\sigma)B} \Big[\frac{\partial}{\partial s} e^{\xi A(s)}\Big]_{\xi=s-\sigma} \cdot$$

$$\cdot g(\sigma)\,d\sigma + \int_0^s e^{(W_s-W_\sigma)B} A(s) e^{(s-\sigma)A(s)} [g(\sigma)-g(s)]\,d\sigma +$$

$$+ \int_0^s [e^{(W_s-W_\sigma)B}-1] A(s) e^{(s-\sigma)A(s)} g(s)\,d\sigma + (e^{sA(s)}-1) g(s) +$$

$$+ \frac{1}{2} B^2 \int_0^s e^{(W_s-W_\sigma)B} e^{(s-\sigma)A(s)} g(\sigma)\,d\sigma \Big]\,ds. \tag{5.8}$$

By (5.7) and (5.8) we get, recalling (4.1),(4.2),(4.3) and (5.2):

$$\int_0^t Bu(s)\,dW_s = u(t) - x - \int_0^t \Big[R(s,0)x + e^{W_s[B+L(s)]} A(s) e^{sA(s)} x +$$

$$+ g(s) + \int_0^s R(s,\sigma) g(\sigma)\,d\sigma + \int_0^s e^{(W_s-W_\sigma)[B+L(s)]} A(s) e^{(s-\sigma)A(s)} \cdot$$

$$\cdot [g(\sigma)-g(s)]\,d\sigma + \int_0^s [e^{(W_s-W_\sigma)[B+L(s)]} - e^{(W_s-W_\sigma)B}] A(s) \cdot$$

$$\cdot e^{(s-\sigma)A(s)} g(s)\,d\sigma + \int_0^s (e^{(W_s-W_\sigma)B}-1) A(s) e^{(s-\sigma)A(s)} g(s)\,ds +$$

$$+ [e^{sA(s)}-1] g(s) + \frac{1}{2} B^2 u(s)\Big]\,ds = u(t) - x - \int_0^t [f(s) + A(s)u(s) +$$

$$+ \frac{1}{2} B^2 u(s)]\,ds.$$

This proves that u(t) is a strict solution of (S).

Let us consider now the case of general data x,f. We have:

<u>THEOREM 5.3.</u> <u>Let</u> $x \in L_{\mathcal{F}_0}(H)$ <u>and</u> $f \in C^0_{\mathcal{F}}([0,T],H)$, <u>and let</u> u <u>be the function defined</u> in (4.1). <u>Then</u> u <u>is a generalized solution of</u> (S) (<u>see Definition</u> 3.2).

Proof. Let $\{x_k\}\subseteq L_{\mathcal{F}_0}(D(A(0))\cap D(B^2))$ such that $x_k\to x$ w.p.1 as $k\to\infty$; due to Proposition 2.7, such a sequence exists. Consider the function

$$\psi_k(t)=(1+R)^{-1}(f-R(\cdot,0)x_k)(t); \tag{5.9}$$

it belongs to $C^0_{\mathcal{F}}([0,T],H)$ by Lemma 4.8 (i)-(iii) and Lemma 4.6, and in addition $\psi_k\to g$ in $L^p_{\mathcal{F}}(0,T,H)$ as $k\to\infty$ for each $p\in[1,\frac{1}{1-\alpha}\wedge 2[$, where $g=(1+R)^{-1}(f-R(\cdot,0)x)$.

As in the proof of Theorem 3.11, define $\psi_k(t)$ in $\mathbb{R}-[0,T]$ setting $\psi_k(t)=\psi_k(T)$ $\forall t>T$, $\psi_k(t)=\psi_k(0)$ $\forall t<0$, and take $\phi_k=\theta_k*\psi_k$, where θ_k is a mollifier.Then $\phi_k\in C^1_{\mathcal{F}}([0,T],H)$ and $\phi_k-\psi_k\to 0$ in $C^0([0,T],H)$ as $k\to\infty$ w.p.1. Next, set $\xi_k(t)=h_k^2R(h_k,A(0))\cdot$ $\cdot R(h_k,B)\phi_k(t)$,where $\{h_k\}$ is an increasing sequence of integers such that $\|\xi_k-\phi_k\|_{C^0([0,T],H)}<\frac{1}{k}$ w.p.1 (compare with Proposition 2.2). The functions ξ_k satisfy

$$\xi_k\in C^1_{\mathcal{F}}([0,T],H),\ \xi_k(t)\in D(A(0))\cap D(B^2)\ \ \forall t\in[0,T],\ \text{w.p.1},$$
$$B^2\ \xi_k\in C^1_{\mathcal{F}}([0,T],H), \tag{5.10}$$

as it can be easily verified. Finally, define

$$f_k=(1+R)\xi_k+R(\cdot,0)x_k.$$

Then $f_k\in C^0_{\mathcal{F}}([0,T],H)$, and $f_k\to f$ in $C^0([0,T],H)$ as $k\to\infty$ w.p.1: indeed, as R is bounded in $C^0_{\mathcal{F}}([0,T],H)$, by (5.9) we have as $k\to\infty$

$$f_k-f=(1+R)(\xi_k-\psi_k)+(1+R)\psi_k+R(\cdot,0)x_k-f=(1+R)(\xi_k-\psi_k)=$$
$$=(1+R)(\xi_k-\phi_k)+(1+R)(\phi_k-\psi_k)\to 0.$$

Consider now the function

$$u_k(t)=e^{W_tB}e^{tA(t)}x_k+\int_0^t e^{(W_t-W_s)B}e^{(t-s)A(t)}\xi_k(s)ds;$$

by Theorem 5.1 it is a strict solution of the stochastic problem

$$\begin{cases} du_k(t)=[A(t)u_k(t)+\frac{1}{2}B^2u_k(t)+f_k(t)]dt+Bu_k(t)dW_t \\ u_k(0)=x_k. \end{cases}$$

Moreover it is clear that $u_k \to u$ in $C^0([0,T],H)$ as $k\to\infty$ (u is given by (4.1)). Since also $f_k \to f$ in $C^0([0,T],H)$ and $x_k \to x$ in H w.p.1, by Egoroff's Theorem we deduce that the conditions of Definition 3.2 are satisfied; therefore u is a generalized solution of (S).

6. THE STOCHASTIC PROBLEM: UNIQUENESS

In order to prove that the strict,or generalized,solution of (S) is unique, we need some further lemmata. For each $n\in N$ and $t\in[0,T]$ define $J_n(t)=nA(t)R(n,A(t))$. Then we have:

LEMMA 6.1. *For each* $n\in\mathbb{N}$ *and* $t\in[0,T]$ *the following properties hold:*

(i) $J_n(t)\in\mathcal{L}(H)$;

(ii) $\rho(J_n(t))\supseteq\rho(A(t))$ and $R(\lambda,J_n(t))=\frac{1}{\lambda+n}[n-A(t)]R(\frac{\lambda n}{\lambda+n},A(t))=$

$$=\frac{\lambda n^2}{(\lambda+n)^2}R(\frac{\lambda n}{\lambda+n},A(t))+\frac{\lambda n}{(\lambda+n)^2}\quad\forall\lambda\in\rho(A(t));$$

(iii) $\|\frac{\partial}{\partial t}e^{\xi J_n(t)}\|_{\mathcal{L}(H)}\le\frac{c}{\xi^{1-\alpha}}\quad\forall\xi>0$;

(iv) $J_n(t)BJ_n(t)^{-1}x=[B+L_n(t)]x$, $\forall x\in D(B)$, $L_n(t)=$ $=nR(n,A(t))L(t)$;

(v) $e^{\xi[B+L_n(t)]}=J_n(t)e^{\xi B}J_n(t)^{-1}\quad\forall\xi\in\mathbb{R}$;

(vi) $\|e^{\xi(B+L_n(t))}-e^{\xi B}\|_{\mathcal{L}(H)} \le C|\xi|e^{c|\xi|} \quad \forall \xi\in\mathbb{R}$.

Proof. (i),(ii),(iii) are evident. Let us prove (iv): for each $x\in D(B)$ we have by Hypothesis III

$J_n(t)BJ_n(t)^{-1}x=nA(t)R(n,A(t))B\,\frac{n-A(t)}{n}A(t)^{-1}x=nR(n,A(t))\cdot$

$\cdot A(t)BA(t)^{-1}x-A(t)R(n,A(t))Bx=nR(n,A(t))Bx+nR(n,A(t))\cdot$

$\cdot L(t)x-A(t)R(n,A(t))Bx=Bx+nR(n,A(t))L(t)x$.

To prove (v), let us first verify that

$$R(\lambda,B+L_n(t))=J_n(t)R(\lambda,B)J_n(t)^{-1} \quad \forall\lambda\in\rho(B)\cap\rho(B+L_n(t)) \tag{6.1}$$

Indeed, for each $x\in H$ we have $y=R(\lambda,B+L_n(t))x\in D(B)$ and $\lambda y-[B+L_n(t)]y=x$. Hence

$$x = \lambda y-J_n(t)BJ_n(t)^{-1}y = J_n(t)(\lambda-B)J_n(t)^{-1}y$$

or

$$y = J_n(t)R(\lambda,B)J_n(t)^{-1}x.$$

Starting from (6.1), (v) is proved as in [9], proof of Proposition 1.

Finally, (vi) is proved as Proposition 2.9, since $\|L_n(t)\|_{\mathcal{L}(H)} \le C\|L(t)\|_{\mathcal{L}(H)}$.

For each $n\in N$, consider the stochastic problem

$$\begin{cases} du(t)=[J_n(t)u(t)+\frac{1}{2}B^2u(t)+f(t)]\,dt+Bu(t)\,dW_t \\ u(0)=x \end{cases} \qquad (S'_n)$$

with prescribed data $x\in L_{\mathcal{F}_0}(H)$, $f\in C^0_{\mathcal{F}}([0,T],H)$. Then, we have:

PROPOSITION 6.2. Let u be a strict solution of (S'_n).

<u>Then there exists</u> c(n) <u>such that</u>

$$\|u(t)\|_H \le c(n)\{\|x\|_H + \int_0^t \|f(s)\|_H ds\} \quad \forall t\in[0,T], \quad \text{w.p.1.}$$

<u>In particular, Problem</u> (S_n') <u>has at most one strict solution.</u>

<u>Proof</u>. Let $t\in]0,T]$. For each $s\in[0,t]$ define

$$v(s)=e^{(t-s)J_n(s)}e^{(W_t-W_s)B}u(s);$$

then taking into account Lemma 6.1, it is easy to verify that

$$\begin{cases} dv(s)=\{[-e^{(t-s)J_n(s)}(e^{(W_t-W_s)[B+L_n(s)]}-e^{(W_t-W_s)B})J_n(s)+ \\ \quad +[\frac{\partial}{\partial s}e^{\xi J_n(s)}]_{\xi=t-s}]u(s)+e^{(W_t-W_s)B}f(s)\}ds \\ v(0)=x \end{cases}$$

which implies

$$u(t)=x+\int_0^t\{-e^{(t-s)J_n(s)}[e^{(W_t-W_s)[B+L_n(s)]}-e^{(W_t-W_s)B}]J_n(s)+$$

$$+[\frac{\partial}{\partial s}e^{\xi J_n(s)}]_{\xi=t-s}]u(s)+e^{(W_t-W_s)B}f(s)\}ds.$$

Hence

$$\|u(t)\|_H \le \|x\|_H + C\int_0^t |W_t-W_s|\,\|J_n(s)\|_{\mathcal{L}(H)}\|u(s)\|_H ds+$$

$$+\ C\int_0^t \frac{1}{(t-s)^{1-\alpha}}\|u(s)\|_H ds + C\int_0^t \|f(s)\|_H ds,$$

and by a classical Gronwall-type argument (see e.g. Amann [2], Corollary 2.4) we get

$$\|u(t)\|_H \le C(n)\{\|x\|_H + \int_0^t \|f(s)\|_H ds\}.$$

<u>COROLLARY 6.3. Let</u> u <u>be a generalized solution of</u> (S_n').
<u>Then there exists</u> C(n) <u>such that</u>

$\|u(t)\|_H \leq C(n)\{\|x\|_H+\int_0^t\|f(s)\|_H ds\}.$

In particular, Problem (S_n') *has at most one generalized solution.*

PROPOSITION 6.4. Let $x\in L_{\mathcal{F}_0}(H), f\in C^0_{\mathcal{F}}([0,T],H)$. *Then Problem* (S_n') *has a generalized solution* u_n *given by*

$$u_n(t)=e^{W_tB}e^{tJ_n(t)}x+\int_0^t e^{(W_t-W_s)B}e^{(t-s)J_n}g_n(s)ds, \qquad (6.2)$$

$g_n(t)$ being the solution of the integral equation

$$g_n(t)+\int_0^t K_n(t,s)g_n(s)ds=f(t)-K_n(t,0)x \qquad \text{w.p.1.} \quad (6.3)$$

whose kernel $K_n(t,s)$ is defined by

$$K_n(t,s)=e^{(W_t-W_s)B}\Big[\frac{\partial}{\partial t}e^{\xi J_n(t)}\Big]_{\xi=t-s}-[J_n(t),e^{(W_t-W_s)B}]\ \cdot$$

$$\cdot e^{(t-s)J_n(t)},\quad 0\leq s<t\leq T. \qquad (6.4)$$

Proof. We proceed as in Section 5: first we prove that if $x\in L_{\mathcal{F}_0}(D(B^2))$ and f is such that the solution of (6.3) is suitably regular then (6.2) gives a strict solution of (S_n'); next, we approximate the general data x,f with more regular ones, and show that (6.2) is a generalized solution. We omit the proof because it is quite similar to that of Theorems 5.1 and 5.3, and even easier, since the role of A(t) is played by the bounded operator $J_n(t)$.

PROPOSIZION 6.5. Let u *be a strict, or generalized, solution of* (S_n'). *Then there exists* C *(independent of* n) *such that*

$$\|u(t)\|_H \leq C[\|x\|_H + \int_0^t \|f(s)\|_H ds] \quad \forall t \in [0,T], \quad \text{w.p.1.}$$

Proof. It follows by the representation formula (6.2) and from the fact that the operators $(1+K_n)^{-1}$, with $K_n(t,s)$ defined by (6.4), are bounded in $L^1_F(0,T,H)$ uniformly in $n \in \mathbb{N}$ (this is a consequence of Lemma 6.1 (iii)-(vi)).

Now we are able to prove the uniqueness theorem for the solution of (S).

THEOREM 6.6. Let u be a strict, or generalized, solution of (S). Then we have

$$\|u(t)\|_H \leq C\{\|x\|_H + \int_0^t \|f(s)\|_H ds\} \quad \forall t \in [0,T], \text{ w.p.1.}$$

In particular, Problem (S) has at most one strict, or generalized, solution.

Proof. If u is a strict solution of (P), then u is also a generalized solution of

$$\begin{cases} du(t) = [J_n(t)u(t) + \frac{1}{2}B^2 u(t) + f(t) + [A(t) - J_n(t)]u(t)]dt + Bu(t)dW_t \\ u(0) = x \end{cases}$$

Hence by Proposition (6.5) there exists c (independent of n) such that

$$\|u(t)\|_H \leq C\{\|x\|_H + \int_0^t \|f(s) + [A(s) - J_n(s)]u(s)\|_H ds\} \quad \forall t \in [0,T],$$

w.p.1.

As $n \to \infty$, the result follows by Lebesgue's Theorem, since $[A(s) - J_n(s)]u(s) \to 0$ for each $s \in [0,t]$.

By a standard argument, the estimate holds also for any generalized solution.

7. AN EXAMPLE

Take $H=L^2(0,1)$ and define

$$\begin{cases} D(B)=\{u\in L^2(0,1):\ gu'\in L^2(0,1)\}, \\ Bu=gu' \end{cases}$$

where $g\in C^2([0,1])$ with $g(0)=g(1)=g'(1)=0$; then it is well known that B generates a strongly continuous group and Hypothesis I holds.

Next, denote by $H^k(0,1)$ $(k\in\mathbb{N})$ the Sobolev space of functions $u\in L^2(0,1)$ whose distributional derivatives $u',u'',\ldots$ $\ldots u^{(k)}$ belong to $L^2(0,1)$, and define for each $t\in[0,T]$

$$\begin{cases} D(A(t))=\{u\in H^2(0,1):u(0)=0,\alpha(t)u(1)+\beta(t)u'(1)=0\} \\ A(t)u=u'' \end{cases}$$

where $\alpha(t),\beta(t)$ are real continuously differentiable functions, such that $\alpha\geq 0,\beta\geq 0,\alpha+\beta>0$ in $[0,T]$. It is also known that $A(t)$ generates an analytic semigroup, and Hypothesis II is satisfied with $\alpha=1/2$ (see Acquistapace-Terreni [1] in the case of $C([0,1])$ instead of $L^2(0,1)$).

Let us verify that Hypothesis III is fulfilled: clearly $D(A(t))\subseteq D(B^2)\subseteq D(B)$ for each $t\in[0,T]$; next, taking $\lambda_0(t)\equiv 0$, we have $D(B)\subseteq\{x\in L^2(0,1):BA(t)^{-1}\in D(A(t))\}$: indeed if $\phi\in D(B)$ and $\psi=A(t)^{-1}\phi$, we have $\psi\in H^2(0,1)$, so that

$$(B\psi)''=(g\psi')''=g''\psi'+2g'\psi''+g\psi'''=g''\psi'+2g'\psi''+B\phi\in L^2(0,1)$$

and addition

$$(B\psi)(0)=g(0)\psi'(0)=0,\ \alpha(t)(B\psi)(1)+\beta(t)(B\psi)'(1)=$$

$$=\alpha(t)g(1)\psi'(1)+\beta(t)[g'(1)\psi'(1)+g(1)\psi''(1)]=0.$$

In particular we get

$$A(t)BA(t)^{-1}\phi=(B\psi)''=g''\psi'+2g'\psi'+B\phi=g''\int_0^x\phi ds+2g'\phi+B\phi \quad \forall\phi\in D(B).$$

Define

$$[L(t)\phi](x) = g''(x)\int_0^x\phi(s)ds+2g'(x)\phi(x),$$

then $L(t)\equiv L$ and

$$A(t)BA(t)^{-1}\phi=[B+L]\phi \qquad \forall\phi\in D(B).$$

This shows that Hypothesis III holds.

Finally we observe that

$$[A(t)^{-1}f](x)=-\int_0^x f(s)(x-s)ds+ x\,\frac{\alpha(t)\int_0^1 f(s)(1-s)ds+\beta(t)\int_0^1 f(s)ds}{\alpha(t)+\beta(t)}, \quad \forall t\in[0,T], \ \forall x\in[0,1],$$

and consequently

$$[BA(t)^{-1}f](x)=g(x)[-\int_0^x f(s)ds+ \frac{\alpha(t)\int_0^1 f(s)(1-s)ds+\beta(t)\int_0^1 f(s)ds}{\alpha(t)+\beta(t)}] \quad \forall t\in[0,T], \ \forall x\in[0,1];$$

hence $\forall t,r\in[0,T]$

$$\|BA(t)^{-1}f-BA(r)^{-1}f\|_{L^2(0,1)} = \|g\|_{L^2(0,1)}\cdot \left| \frac{\alpha(t)\int_0^1 f(s)(1-s)ds+\beta(t)\int_0^1 f(s)ds}{\alpha(t)+\beta(t)} - \frac{\alpha(r)\int_0^1 f(s)(1-s)ds+\beta(r)\int_0^1 f(s)ds}{\alpha(r)+\beta(r)} \right|.$$

Thus Hypothesis IV is obviously fulfilled.

Therefore we can apply the theory in the previous sections to the stochastic problem

$$\begin{cases} du(t,x)=[(1+\frac{1}{2}g^2(x))u_{xx}(t,x)+\frac{1}{4}(g^2(x))'u_x(t,x)+f(t,x)]dt+ \\ \qquad\qquad +[g(x)u_x(t,x)]dW_t \\ u(0,x)=\phi(x) \\ u(t,0)=0 \\ \alpha(t)u(t,1)+\beta(t)u_x(t,1)=0 \end{cases} \tag{7.2}$$

where $f\in C^0_{\mathcal{F}}([0,T], L^2(0,1))$ and ϕ is a $\mathcal{F}_0$-measurable random variable with values in $L^2(0,1)$. By Theorems 5.3 and 6.6 we deduce:

THEOREM 7.1. *Let* g,α,β *real functions such that* $g\in C^2([0,1])$ *with* $g(0)=g(1)=g'(1)=0$, $\alpha,\beta\in C^1([0,T])$ *with* $\alpha\geq 0$, $\beta\geq 0$, $\alpha+\beta>0$ in $[0,T]$. *In addition, let* W_t *be a real Brownian motion, and* $\mathcal{F}_t$ *an increasing sequence of* σ*-algebras on the probability space* (Ω,ε,P), *non-anticipating with respect to* W_t *and such that* $\mathcal{F}_0\supseteq\varepsilon$ *and* $(\Omega,\mathcal{F}_0,P)$ *is a complete measure space* . *Then for each* $f\in C^0_{\mathcal{F}}([0,T],L^2(0,1)$ *and* $\phi\in L_{\mathcal{F}_0}(L^2(0,1))$, *Problem* (7.2) *has a unique generalized solution* $u\in C^0_{\mathcal{F}}([0,T],L^2(0,1))$.

APPENDIX

Here we want to prove the following result (see Remark 1.2):

PROPOSITION A.1. *Let Hypothesis* I,II *hold, and suppose that*:

(i)' $D(A(t))\subseteq D(B)$,

(ii)' *For each* $t\in[0,T]$ *there exist* $\lambda_0(t)\in\rho(A(t))$, $L(t)\in \mathcal{L}(H)$, $V(t)\subseteq D(B)$ *such that*:

(a) $\lambda_0 \in C([0,T],\mathbb{C}), L \in C([0,T],\mathcal{L}(H))$

(b) $V(t)$ <u>is a linear subspace of</u> $D(B)$, <u>dense in</u> $D(B)$ <u>with respect to the graph norm</u>;

(c) $V(t) \subseteq \{x \in H : BR(\lambda_0(t),A(t))x \in D(A(t))\}$

(d) $[\lambda_0(t)-A(t)]B\, R(\lambda_0(t),A(t))x = Bx+L(t)x \quad \forall x \in V(t)$

<u>Then</u> Hypothesis III <u>holds</u>.

<u>Proof</u>. We consider only the (unrestrictive) case $\lambda_0(t) \equiv 0$. For each $x \in V(t)$ and $\lambda \in \Sigma_{\theta_0}$ we have, as in the proof of Proposition 2.4:

$$[B,R(\lambda,A(t))]\,(1-\lambda A(t)^{-1})x = R(\lambda,A(t))L(t)x; \qquad (A.1)$$

define $W(t) = [1-\lambda A(t)^{-1}](V(t))$. As $V(t)$ is dense in $D(B)$ in the graph norm, the same is true for $W(t)$. To prove this, note that, obviously, $W(t) \subseteq D(B)$; next, if $y \in D(B)$ then, setting $x = -A(t)R(\lambda,A(t))y$, we have $x \in D(B)$ too, so by (b) there exists $\{x_n\}_{n\in\mathbb{N}} \subseteq V(t)$ such that as $n\to\infty$ $x_n \to x$ and $Bx_n \to Bx$ in H. Set $y_n = [A(t)-\lambda]A(t)^{-1}x_n$; then $\{y_n\}_{n\in\mathbb{N}} \subseteq W(t)$ and as $n\to\infty$ $y_n \to [A(t)-\lambda]A(t)^{-1}x = y$, $By_n = B(1+\lambda A(t)^{-1})x_n \to Bx+\lambda BA(t)^{-1}x = By$ (since $BA(t)^{-1} \in \mathcal{L}(H)$). This shows that $W(t)$ is dense in $D(B)$ with respect to the graph norm.

Next, by (A.1) we get

$$\begin{aligned}[B,R(\lambda,A(t))]y &= R(\lambda,A(t))L(t)\,(1-\lambda A(t)^{-1})^{-1}y = \\ &= -R(\lambda,A(t))L(t)A(t)R(\lambda,A(t))y \quad \forall y \in W(t)\end{aligned} \qquad (A.2)$$

Now let $x \in D(B)$: choose $\{y_n\} \subseteq W(t)$ such that $y_n \to x$ and $By_n \to Bx$. By (A.2)

$$BR(\lambda,A(t))y_n - R(\lambda,A(t))By_n = -R(\lambda,A(t))L(t)A(t)R(\lambda,A(t))y_n$$

$\forall n \in \mathbb{N}$,

and as $n\to\infty$ we get

$$[B,R(\lambda,A(t))]x = -R(\lambda,A(t))L(t)A(t)R(\lambda,A(t))x \quad \forall x\in D(B)$$
$$\forall t\in[0,T],\ \forall\lambda\in\Sigma_{\theta_0}. \tag{A.3}$$

Now we are ready to prove that (d) holds in the whole $D(B)$. Indeed, let $x\in D(B)$: then for each $n\in\mathbb{N}$ by (A.3) we have:

$$J_n(t)BA(t)^{-1}x = nA(t)R(n,A(t))BA(t)^{-1}x = -nBA(t)^{-1}x +$$
$$+n^2R(n,A(t))BA(t)^{-1}x = -nBA(t)^{-1}x + n^2[BR(n,A(t)) +$$
$$+R(n,A(t))L(t)A(t)R(n,A(t))]A(t)^{-1}x =$$
$$= nB[-1+nR(n,A(t))]A(t)^{-1}x + nR(n,A(t))L(t)nR(n,A(t))x =$$
$$= nBR(n,A(t))x + nR(n,A(t))L(t)nR(n,A(t))x = nR(n,A(t))Bx -$$
$$-nR(n,A(t))L(t)A(t)R(n,A(t))x + nR(n,A(t))L(t)nR(n,A(t))x =$$
$$= nR(n,A(t))Bx + nR(n,A(t))L(t)x,$$

which implies

$$J_n(t)BA(t)^{-1}x \to Bx + L(t)x \qquad \text{as } n\to\infty.$$

This proves that

$$BA(t)^{-1}x\in D(A(t)) \qquad \forall x\in D(B),$$

and

$$A(t)BA(t)^{-1}x = Bx + L(t)x \qquad \forall x\in D(B),\ \forall t\in[0,T],$$

so that Hypothesis III holds.

REFERENCES

[1] P. ACQUISTAPACE, B. TERRENI, Some existence and regularity results for abstract non-autonomous parabolic equations, pre-print Sc. Norm. Sup. Pisa (1982) to appear in J. Math. Anal. Appl..

[2] H. AMANN, Periodic solutions of semi-linear parabolic equations, Non-linear analysis -a collection of papers in honour of E.H. Rothe, ed. L. Cesari, R. Kannan, H.F. Weinberger, Academic Press, New York, 1978.

[3] A. BALAKRISHNAN, Stochastic bilinear partial differential equations, U.S. Italy Conference on variable structure systems, Oregon, 1974.

[4] A. CHOINOWSKA MICHALIK, Stochastic differential equations in Hilbert spaces and some of their applications, Thesis, Institute of Mathematics, Polish Academy of Sciences, 1971.

[5] R.F. CURTAIN, Stochastic evolution equations with general white noise disturbance, J. Math. Anal. Appl. 60 (1977) 570-595.

[6] R.F. CURTAIN, A.J. PRITCHARD, Infinite dimensional linear systems theory, Springer Verlag, Berlin, 1978.

[7] G. DA PRATO, Stochastic differential equations with non-continuous coefficients in Hilbert spaces, conference held at the meeting "Stochastic problems in Mechanics", Torino, 1981, pre-print Sc. Norm. Sup. Pisa (1981).

[8] G. DA PRATO, M. IANNELLI, L. TUBARO, Semilinear stochastic differential equations in Hilbert spaces, Boll. Un. Mat. Ital. (5) 16A (1979) 168-185.

[9] G. DA PRATO, M. IANNELLI, L. TUBARO, Some results on linear stochastic differential equations in Hilbert spaces, Stochastics 6 (1982) 105-116.

[10] D.A. DAWSON, Stochastic evolution equations, Math. Biosciences, 15 (1972), 287-316.

[11] A. FRIEDMAN, Stochastic differential equations and applications, vol. I, Academic Press, New York, 1975.

[12] A. ICHIKAWA, Linear Stochastic evolution equations in Hilbert spaces, J. Diff. Eq. 28 (1978) 266-283.

[13] T. KATO, H. TANABE, On the abstract evolution equa tions, Osaka Math. J. 14 (1962), 107-133.

[14] P. KOTELENEZ, A sub-martingale type inequality with applications to stochastic evolution equations, Stochastics 8(1982) 139-151.

[15] N.V. KRYLOV, B.L. ROZOVSKII, Cauchy problem for linear stochastic partial differential equations, Izvestia Ak. Nauk CCCP, Math. Series, (6) 41 (1977) 1329-1347.

[16] N.V. KRYLOV, B.L. ROZOVSKII, Stochastic evolution equations, Itogi Nauki i Tekhniki, Seryia Sovremennye Problemy Matematiki 14 (1979) 71-146.

[17] R.S. LIPSTER, A.N. SHIRYAYEV, Statistics of random processes. Theory and applications, Springer Verlag, Berlin, 1978.

[18] M. METIVIER, G. PISTONE, Une formule d'isometrie pour l'integrale stochastique hilbertienne et équa tions d'evolution linéaires stochastiques, Z. Wahrscheinlichkeitstheorie verw. Geb. 33 (1975) 1-18.

[19] E. PARDOUX, Equations aux dérivées partielles sto chastiques non linéaires monotones, Thèse, Univer sité Paris XI, 1975.

[20] E.PARDOUX, Stochastic partial differential equations and filtering of diffusion processes, Stochastics 3 (1979) 127-167.

[21] R.L. STRATONOVICH, A new representation for stochastic integrals, Vestnik Moskov Univ. Ser. I 1 (1964) 3-12; English translation: SIAM J.Control 4 (1968) 362-371.

[22] H. SUSSMANN, On the gap between deterministic and stochastic ordinary differential equations, Ann. of Prob. 6 (1978) 19-41.

[23] H. TANABE, On the equations of evolution in a Banach space, Osaka Math. J. 12 (1960) 363-376.

[24] E. WONG, M.ZAKAI, Riemann-Stieltjes approximation of stochastic integrals, Z. Wahrscheinlichkeitstheorie verw. Geb. 12 (1969) 87-97.

[25] K. YOSIDA, Functional analysis, Springer Verlag, Berlin 1968.

[26] M. ZAKAI, On the optimal filtering of diffusion processes, Z. Wahrscheinlichkeitstheorie verw. Geb. 11 (1969) 230-243.