ZYGMUND CLASSES WITH BOUNDARY CONDITIONS AS INTERPOLATION SPACES

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0. INTRODUCTION

This paper is concerned with the characterization of the real interpolation spaces $\mathcal{D}_A^{(\alpha,\omega)}$, $\mathcal{D}_A^{(\alpha,\omega)}$ [Lions-Mezaire, 1968] and $\mathcal{D}_A^{(\alpha,\omega)}$ [Bu Prato-Grisvard, 1979], where $A$ is an elliptic differential operator of order $2m$, with general boundary conditions, and $\mathcal{D}$ is the Banach space of continuous functions on a bounded open set $\mathcal{O}$. Following [Grisvard, 1969], we denote such spaces respectively by $\mathcal{D}_A^{(\alpha,\omega)}$ and $\mathcal{D}_A^{(\alpha,\omega)}$, where $\alpha = 1 - \nu$.

In an earlier paper [Acquistapace-Terreni, 1987] we studied the case $2m \notin \mathbb{N}$; the purpose here is to study the "critical" cases $2m \notin \mathbb{N}$.

All notations here are the same as in [Acquistapace-Terreni, 1987].

1. ASSUMPTIONS

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, $n \geq 1$, with $C^{2m}$ boundary, $m \geq 1$. We introduce the differential operators

\begin{align*}
(1.1) & \quad A(x,D) := \sum_{|\alpha| \leq 2m} a_\alpha(x)D^\alpha, \quad x \in \Omega \\
(1.2) & \quad B_j(x,D) := \sum_{|\beta| \leq m} a_j(x)D^\beta, \quad x \in \Omega, \quad j = 1, \ldots, m,
\end{align*}

under the following assumptions:

\begin{align*}
(1.3) & \quad a_\alpha \in C(\bar{\Omega}, \mathbb{C}), \quad |\alpha| \leq 2m; \quad b_j \in C(\bar{\Omega}, \mathbb{C}), \quad |\beta| \leq m, \quad j = 1, \ldots, m

(1.4) & \quad (\text{ellipticity}) \quad \text{There exist } \eta \in (0,2m), \nu > 0 \text{ such that}
\end{align*}

\begin{equation*}
v(|x|^{-\nu}, x \mapsto a_\alpha(x\zeta) - (-1)^{m-\nu} \zeta^{2m-\nu}) \in C_{\nu}(\mathbb{R}^n, \mathbb{C}), \quad \forall \xi \in \mathbb{R}^n, \forall \zeta \in \mathbb{C}.
\end{equation*}
(1.5) (root condition) If \( x \in \mathbb{R}^n \), \( \xi \in \mathbb{R}^n \), \( t \in \mathbb{R} \) and \( |\xi| \| t \| \geq 1 \), \( t \| v(x) \| = 0 \), then the polynomial
\[
\zeta = \sum_{|\xi| = 2m} a_\xi (x) (\xi v(x))^2 (-1)^{n+1} t^{2m}
\]
has exactly \( m \) roots \( \zeta_j^+(x, \xi, t) \) with positive imaginary part.

(1.6) (complementing condition) If \( x \in \mathbb{R}^n \), \( \xi \in \mathbb{R}^n \), \( t \in \mathbb{R} \) and \( |\xi| \| t \| \geq 1 \), \( t \| v(x) \| = 0 \), then the \( m \) polynomials
\[
\zeta = \sum_{|\xi| = m} b_\xi (x) (\xi v(x))^2
\]
are linearly independent modulo the polynomial
\[
\zeta = \prod_{j=1}^m (\xi - \zeta_j^+(x, \xi, t)).
\]

(1.7) \( 0 \leq m \leq n \leq 2m \) if \( 1 \leq j < 1 \leq m \).

**Remark 1.1.** Condition (1.7) replaces here the normality condition assumed in [Acquistapace-Terrani, 1987]. Indeed, it is easily seen that the transversality condition (1.9) of that paper, i.e.,
\[
\sum_{|\xi| = m} b_\xi (x) v(x)^2 \neq 0, \quad x \in \mathbb{R}^n, \quad j=1, \ldots, m
\]
is implied by the complementing condition (1.6); thus the only difference here with respect to [Acquistapace-Terrani, 1987] is that we just require the orders of the boundary operators to be less than \( 2m \), without forcing them to be different from one another.

We remark that this weakened form of the normality condition is sufficient to prove all results of [Acquistapace-Terrani, 1987]; actually, the proof of the main result there [Theorem 2.3] depends only on the basic elliptic existence theory and spectral estimates [Theorems 1.1-1.2 of that paper] which in turn still hold under these assumptions, as shown in [Geymonat-Grisvard, 1987, Theorem 4.1].

Under hypotheses (1.1), . . . , (1.7) the abstract operator \( A \), defined in the space \( E := C(\overline{\Omega}) \) by
\[
D_A := \{ u \in L^2(\Omega) : A^j(D_n u) \in L^2(\Omega), j=1, \ldots, m \}
\]
is the infinitesimal generator of an analytic semigroup in \( E \) [Stewart, 1980]; in particular, possibly replacing \( A(\cdot, D) \) by \( A(\cdot, D) - \omega I \) (\( \omega > 0 \)) we may assume that
\[
(1.9) \quad A(\lambda) \subset \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \}, \quad \Re (\lambda, A) \| v \| < \| A \| \quad \text{if} \quad \Re \lambda > 0.
\]

Hence the spaces \( D_A(\theta, \omega) \) and \( D_A(\theta) \) can be characterized [Grisvard, 1969] by:
\[
(1.10) \quad D_A(\theta, \omega) := \{ x \in E : D_A(\theta, \omega) := \sup_{\substack{\| s \| \leq \theta \| A \| \}} \| x \| E \leq \| x \| E \}
\]
\[
(1.11) \quad D_A(\theta) := \{ x \in E : \| D_A(\theta, \omega) \| \leq \sup_{\substack{\| s \| \leq \theta \| A \| \}} \| x \| E \leq \| x \| E \}.
\]

2. **Preliminaries and the Main Result.**

We list here some preliminary results which are necessary in order to state our main theorem. First of all we define the Zygmund classes: if \( \mathcal{E} \subset \mathbb{R}^n \) is an open set, we define for \( q \in \mathbb{N}^1 \):
\[
A^{q_0} := \{ x \in E^{q_0}(\mathcal{E}) : \| f \| E^{q_0}(\mathcal{E}) \}
\]
\[ a = \frac{1}{\lambda} \sup_{|a| \leq 1} \left\{ \int_0^1 \int_0^1 \frac{1}{|x-y|} \left( 1 - \frac{1}{2} |x-y| \right)^{q-1} \left( 1 - \frac{1}{2} |x-y| \right)^{q-1} \left( 1 - \frac{1}{2} |x-y| \right)^{q-1} \right\} \]

(2.2) \[ \lambda^q_0(\Omega) = \{ f \in C^q(\Omega) : \lim_{r \to 0} \sup_{x \in \Omega} \left\| f \right\|_{C^q(\Omega, \mathbb{R}^n)} = 0 \} \]

where the space \( C^q(\Omega) \) consists of functions whose derivatives of order \( |q| \leq q-1 \) are uniformly continuous and bounded in \( \Omega \).

It is well known that \( \lambda_0^q(\mathbb{R}^n) \) can be described in an alternative way: [Triebel, 1978, Theorem 2.7.2/2]:

(2.3) \[ \lambda_0^q(\mathbb{R}^n) = \{ f \in C^q(\mathbb{R}^n) : \sup_{|\alpha| \leq q-1} \left\| D^\alpha f \right\|_{L^\infty(\mathbb{R}^n)} = 0 \}, \]

where

(2.4) \[ D^\alpha f(x) := \sum_{j=0}^{q-1} (i\pi)^{-q+1-j} \frac{\alpha!}{\alpha_j!} \frac{\partial^j}{\partial x_j^j} f(x+\pi j). \]

Moreover we have the proper inclusions

\[ C_0^\infty(\mathbb{R}^n) \subset C_c^\infty(\mathbb{R}^n) \subset C_0^\infty(\mathbb{R}^n) \subset C_c^\infty(0,1) \]

provided \( \Omega \) is of class \( C^1 \).

It will be useful the following extension property:

**PROPOSITION 2.1.** If \( \Omega \) is bounded with \( \Omega \subset C_0^\infty(\mathbb{R}^n) \), then there exists an extension operator \( R : C^0(\Omega) \rightarrow C^0(\mathbb{R}^n) \) such that:

(1) \( R \) is \( f \),

(2) \( R \) has compact support in \( \mathbb{R}^n \),

(3) \[ \| Rf \|_{C^0(\mathbb{R}^n)} \leq C(0, c, q) \| f \|_{C^0(\Omega)}. \]

This result holds under much more general assumption [Johnson-Wallin, 1979, Theorem 5.1], but when \( \Omega \subset C_0^\infty(\mathbb{R}^n) \) it is possible to give an easier proof, which we omit for brevity.

We now turn to some interpolation properties:

**PROPOSITION 2.2.** If \( \Omega \) is bounded with \( \Omega \subset C_0^\infty(\mathbb{R}^n) \), then

\[ \lambda_0^q(\mathbb{R}^n) = \{ C_0^\infty(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n) \} \}

where \( \mathbb{R}^n \) is of class \( C^1 \).

**Proof.** The case \( \Omega \subset C_0^\infty(\mathbb{R}^n) \) is proved in [Triebel, 1978, Theorem 4.5.2/1].

Let \( f \in C_0^\infty(\mathbb{R}^n) \). By Proposition 2.1, we extend \( f \) to a function \( F \in C_0^\infty(\mathbb{R}^n) \).

As by [Triebel, 1978, Theorem 2.7.2/1],

\[ \lambda^q_0(\mathbb{R}^n) = \{ C_0^\infty(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n) \} \}

we have, by definition, \( F \subset U(0,1) \), where \( U(0,1) \rightarrow C_0^\infty(\mathbb{R}^n) \) satisfies

\[ \sup_{t \in [0,1]} \frac{t^{1/2} \left\{ \| u(t) \|_{C^0(\mathbb{R}^n)} + \| u(t) \|_{C^0(\mathbb{R}^n)} \right\}}{\| u(t) \|_{C^0(\mathbb{R}^n)}} \leq K \]

Hence \( u(t) = u(0) \) satisfies the same inequality with \( \mathbb{R}^n \) replaced by \( \mathbb{R}^n \) and \( u(0) = 0 \).

Thus \( f \in C_0^\infty(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n) \) \( \subset U(0,1) \). Conversely, let

\[ f \in C_0^\infty(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n) \] \( \subset U(0,1) \). Then let \( u(t) \), with \( u(t) \) satisfying

(2.5) \[ \sup_{t \in [0,1]} \frac{t^{1/2} \left\{ \| u(t) \|_{C^0(\mathbb{R}^n)} + \| u(t) \|_{C^0(\mathbb{R}^n)} \right\}}{\| u(t) \|_{C^0(\mathbb{R}^n)}} \leq K \]

Let now \( |a| = 1 \) and \( f \) is \( x, y, \frac{x+y}{2} \). If \( \Omega \) is convex, we write

\[ \int_0^1 \int_{\Omega} |D^2 f(x+y, \frac{x+y}{2})| \leq \int_0^1 \int_{\Omega} |D^2 f(x, \frac{x+y}{2})| \leq K \int_0^1 \int_{\Omega} |D^2 f(x, \frac{x+y}{2})| \leq K \]

and choosing \( t = \frac{|x-y|}{\min_n |a| |a|} \) we obtain \( \Omega \subset C_0^\infty(\mathbb{R}^n) \). If \( \Omega \) is not convex we...
have to construct two \( C^1 \) curves contained in \( \Omega \), joining \( \frac{x+y}{2} \) to \( x \) and \( y \), and whose lengths do not exceed \( M|x-y| \), where \( M \) depends only on \( \Omega \); this can be done since \( \Omega \) is of class \( C^1 \) at least, as shown in [Acquaspe-Trevisan, 1984, Lemma 1.16]. Then we can repeat the preceding argument, by integrating along such curves; and the result follows as above.

**Proposition 2.2.** Let \( \Omega \) be a bounded open set with \( \partial \Omega \in C^{k+1} \), \( k = 0,1, \ldots \).

Then

\[
\text{If } \begin{cases} \|\mathbf{g}\|_{k+1/2} < M, \\ \mathbf{g} \in C^{k+1}(\Omega) \end{cases} \text{, then } \partial \Omega \in C^{k+1}\left(\partial \Omega\right).
\]

This estimate is also well known.

Let us go back now to the situation described in Section 1. As \( \mathbb{R} = C^{2,\infty} \), the distance function

\[
d(x) := \inf(d(x, y) : y \in \partial \Omega), \quad x \in \partial \Omega
\]

belongs to \( C^{2,\infty}(\partial \Omega) \), when \( (\partial \Omega) := \{x \in \partial \Omega : d(x, \partial \Omega) \leq 1\} \) and \( r \) is a suitable positive number; in addition \( \partial \Omega = -v \) on \( \partial \Omega \), where \( v(x) \) is the unit outward normal vector at \( x \in \partial \Omega \) (see [Gilbarg-Trudinger, 1977, Appendix]), hence \( \mathbf{v} \in \mathbf{C}^{2,\infty}(\partial \Omega, \mathbb{R}^n) \).

We want now to define suitable subspaces of \( L^2(\Omega) \) determined by some kind of boundary conditions. We start with considering, for fixed \( j \), the boundary operator \( \overline{B}_j(x, \Omega) \) defined in (1.2). For each \( y \in \mathbb{R}^n \) with \( \|y\| = m_j \) we set

\[
N(y) := \text{number of non-zero components of } y = \text{cardinality of } \{z \in \mathbb{N}^3 : \|z\| = m_j, \beta + e_i = y \text{ for some } i \}
\]

Hence we have, denoting by \( \overline{B}_j \) the principal part of \( \overline{B}_j \):

\[
\overline{B}_j(x, \Omega) = \sum_{|y| = m_j} \overline{b}_j(y, x) \mathbf{v}(x)
\]

Set now for \( |\beta| = 1 \) and \( i = 1, \ldots, n 

Then we can rewrite \( \overline{B}_j(x, \Omega) \) as:

\[
\overline{B}_j(x, \Omega) = \sum_{|\beta| = 1} \overline{b}_j(\beta, x) \mathbf{v}(x)
\]

We now introduce the integral curves associated to the (real) vector fields \( \mathbf{v} \), \( \mathbf{d} \), \( \mathbf{c} \), \( \mathbf{e} \), \( \mathbf{d} \), \( \mathbf{c} \), namely

\[
\begin{align*}
\left\{ \frac{d}{dt} u(t, x) = \mathbf{v}(u(t, x)), \quad t > 0 \right\} \\
\left\{ u(0, x) = x \in \partial \Omega \right\}
\end{align*}
\]

\[
\left\{ \frac{d}{dt} x(t, x) - \text{Re } \mathbf{c}(t, x), \quad t > 0 \right\}
\]

\[
\left\{ x(0, x) = x \in \partial \Omega \right\}
\]

\[
\left\{ \frac{d}{dt} x(t, x) - \text{Im } \mathbf{c}(t, x), \quad t > 0 \right\}
\]

\[
\left\{ x(0, x) = x \in \partial \Omega \right\}
\]
and finally
\[ D_{\mathcal{A}}^{1/2}(\tau) = -\left( E &\mathcal{A}^{1/2}(\tau): f=0 \text{ on } \mathcal{A}\right), \quad D_{\mathcal{A}}^{1/2}(\tau) = -\left( E \mathcal{A}^{1/2}(\tau): f=0 \text{ on } \mathcal{A}\right), \]
\[ D_{\mathcal{A}}^{1/2}(\tau) = -\left( f \mathcal{A}^{1/2}(\tau): f=0 \text{ on } \mathcal{A}\right), \quad \sup_{\sigma \in \mathcal{A}} \sup_{\sigma \in \partial \mathcal{A}} \frac{1}{2} \left( \frac{\partial f}{\partial \nu}(X) + \frac{\partial f}{\partial \nu}(X) \right) \leq \frac{1}{2} \frac{\partial f}{\partial \nu}(X), \]
\[ D_{\mathcal{A}}^{1/2}(\tau) = -\left( f \mathcal{A}^{1/2}(\tau): f=0 \text{ on } \mathcal{A}\right), \quad \sup_{\sigma \in \mathcal{A}} \sup_{\sigma \in \partial \mathcal{A}} \frac{1}{2} \left( \frac{\partial f}{\partial \nu}(X) + \frac{\partial f}{\partial \nu}(X) \right) \leq \frac{1}{2} \frac{\partial f}{\partial \nu}(X). \]

**Remark 2.5.** In (2.13) we may replace the ratio \( \frac{\partial \bar{f}(x,y)}{\partial \nu}(X) \) by the simpler one \( \frac{\partial \bar{f}(x,y)}{\partial \nu}(X) \), indeed for \( f \in \mathcal{A}^{1/2}(\tau) \) the difference between such terms is \( o(1) \) as \( \sigma \to 0 \) (see Remark 5.3 below).

3. THE FIRST INCLUSION

Let \( \mathcal{A} \) be defined by (1.8) and suppose that (1.9) holds. We want to prove that

**Theorem 3.1.** If \( \theta \in (0,1] \) and \( q = 2\theta \in \mathbb{N} \), then
\[ \mathcal{A}^{1/2}(\tau) \subset \mathcal{A}(0,\tau). \]

**Proof.** According to (1.10) it is enough to show that
\[ \frac{d}{dx} \psi(x) = \mathcal{V}(\psi(x)), \quad \psi(x) = 0, \]
\[ \frac{d}{dx} \psi(x) = \mathcal{V}(\psi(x)), \quad \psi(x) = 0. \]
\[
\sup_{\max \{x, y\}} \frac{1}{|y-x|} \left| -\int_0^1 2f |x-uv(x)| - 3f |y-uv(Y)| \, dx \right| = 0,
\]
and finally
\[
D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\}, \quad D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\},
\]
\[
D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\}, \quad \sup_{\max \{x, y\}} \frac{1}{|y-x|} \left| -\int_0^1 2f |x-uv(x)| - 3f |y-uv(Y)| \, dx \right| = 0.
\]

\[
D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\}, \quad \lim_{p \to 0} \frac{1}{p} \left| -\int_0^1 2f |x-uv(x)| - 3f |y-uv(Y)| \, dx \right| = 0,
\]

\[
D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\}, \quad b_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\},
\]

\[
D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\}, \quad D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\}.
\]

\[
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\]

\[
D_A \left( \frac{1}{2}, w \right) = \left\{ f \in L^2(\Omega) : f = 0 \text{ on } \partial \Omega \right\}.
\]

**Remark 2.6.** In (2.13) we may replace the ratio \( \frac{\int \int f(x+y) - \int \int f(x)}{c} \) by the simpler one \( \frac{\int \int f(x+y) - \int \int f(x)}{c} \).

Indeed, for \( f \in L^2(\Omega) \), the difference between such terms is \( o(1) \) as \( c \to 0 \). (see Remark 5.3 below).

3. THE FIRST INCLUSION

Let \( A \) be defined by (1.8) and suppose that (1.9) holds. We want to prove the following

**Theorem 3.1.** If \( 0 \leq \theta \leq 1 \) and \( q := 2m \in \mathbb{N} \), then

\[
D_A(\theta, w) \subset D_A(\theta, w)
\]

**Proof.** According to (1.10) it is enough to show that

\[
\left( 3.1 \right) \quad \sup_{z \in \Omega} \frac{s}{A(z, A)} \, d \leq \frac{c_1}{s} \quad \text{for all} \quad z \in \Omega,
\]

As in [Acquistapace-Turrini, 1987], this will be done by constructing, for each fixed \( f \in L^2(\Omega) \), a function \( v : \Omega \to \mathbb{R} \) such that

\[
\left( 3.2 \right) \quad \int_{v(z)} \, d \leq \frac{c_2}{s} \quad \text{for all} \quad z \in \Omega,
\]

\[
\left( 3.3 \right) \quad \int_{v(z)} \, d \leq \frac{c_3}{s} \quad \text{for all} \quad z \in \Omega.
\]

Obviously (3.2) and (3.3) imply (3.1).

Let \( f \in L^2(\Omega) \) and consider an extension \( \tilde{f} \in L^2(\mathbb{R}^n) \), which exists by Proposition 2.1. Fix a real-valued function \( \phi \in C_0^\infty(\mathbb{R}^n) \) such that

\[
0 < \phi < \varepsilon \quad \text{outside } B(0, 1), \quad \int \varphi(x) \, dx = 1 \quad \text{and} \quad \phi \text{ is even in each variable},
\]

and set

\[
\hat{\phi}_M(z) := \frac{1}{M^m} \varphi(z), \quad z \in \mathbb{R}^n, \quad t \in \Omega.
\]

Define finally (compare with (2.4))

\[
v(t, \phi) = \int \left[ -\int_0^1 \left[ q(z, x) \right] \varphi(x) \, dx \right] \, \hat{\phi}_M(z) \, dx, \quad x \in \mathbb{R}^n, \quad t \in \Omega.
\]

then clearly

\[
v(t, \phi) = \hat{\phi}_M(x) = \int \left[ \sum_{n=0}^\infty \left( -1 \right)^n \left( \frac{q}{2^n} \right) \varphi(x+\varepsilon_n) \right] \, dx.
\]

Hence if \( |q| < 1 \) we have by (2.3) (since \( d \in L^2(\Omega) \))(\mathbb{R}^n));

\[
\left( 3.5 \right) \quad |d \cdot v(t, \phi)| \, \|d\|^2 \leq \int_{\mathbb{R}^n} \left| \sum_{n=0}^\infty \left( -1 \right)^n \left( \frac{q}{2^n} \right) \varphi(x+\varepsilon_n) \right| \, dx \leq \frac{c_4}{s} \quad \text{for all} \quad z \in \Omega,
\]

Next, if \( n = q-1 \) and \( \gamma \) is even and larger than 1, we get:
\[ |\partial^{q+1}v(t,x)| \leq \left( \sum_{n=1}^{\infty} \frac{1}{n^3} \right) \left( \sum_{n=1}^{\infty} \frac{1}{n^3} \right) |v(t,x)| \leq c \left( \frac{1}{n^3} \right) \]

Thus if \(|\mathbf{s}|>q\) and \(|\mathbf{s}|<q\) we obtain

\[ |\partial^{q}v(t,x)| \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

On the other hand if \(|\mathbf{s}|>q\) and \(|\mathbf{s}|<q\) is odd, by interpolation (3.6) yields

\[ |\partial^{q}v(t,x)| \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q+1/2} \]

By (3.6) and (3.7) we conclude that

\[ |\partial^{q}v(t,x)| \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

Finally if \(|\mathbf{s}|=q\) we do not have the boundedness of \(\partial^{q}v(t,x)\). But the weaker estimate (due to the fact that \(\partial^{q}v(t,x) \in L^r(\mathbb{R}^3)\)) is sufficient.

\[ |\partial^{q}v(t,x)| \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

Hence if we set

\[ w(s)(x) = w(s,x) := v(s^{-1/2m}x), \quad x \in \mathbb{R}^d, \quad s \geq 1 \]

we easily obtain:

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]

\[ \|w(s)\|_{L^1(\mathbb{R}^d)} \leq c \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \left( \frac{1}{n^3} \right)^{q-|\mathbf{s}|} \]
\[ I_j = \sum_{j=1}^{n} a_j \bigl[ I_j (s) \bigr] \bigr] \bigl( w(s) - r \bigr) \frac{1}{C^{m_j - 1}(G)} \leq c \left( \int_0^s \left( w(s) - r \right) \frac{1}{C^{m_j - 1}(G)} \right)^{1/2} \]

and it remains to consider the case \( m_j = q \), i.e., to estimate the quantity

\[ B_j(x, D)w(s, x) = B_j(x, D)w(s, x) + B_j(x, D)w(s, x) \quad s \geq 0, \quad s \geq 1, \]

where we have denoted by \( B_j \) the principal part of \( B_j \) and by \( B_j \) its lower order part. Obviously

\[ B_j(x, D)w(s, x) \leq c \| w(s) \| \leq c \| f \| \frac{1}{A^q(G)} \]

on the other hand, recalling (2.13) we can write:

\[ B_j(x, D)w(s, x) \leq B_j(x, D)w(s, x) - [D \gamma] w(s, x) \gamma] + \left( B_j(x, D)w(s, x) - [D \gamma] w(s, x) \gamma] \right) + \left( D \gamma] w(s, x) \gamma] \right) = T_1 + T_2 + T_3, \]

where \( \gamma \in [0, s] \). Now, taking into account (2.9) and (2.13) it can be checked that

\[ T_1 \leq c \| w(s) \| + \| w(s) \| \leq c \| w(s) \| \frac{1}{A^q(G)} \]

and, recalling (3.12), (3.10) and (3.11), the proof of (3.3) will be complete.

Let us prove (3.16) (note that this term appears only when \( m_j = q \)). If \( m_j = q \), by Propositions 2.3 and 2.2 we get:

\[ I_j (s) \leq c \left( \int_0^s \left( w(s) - r \right) \frac{1}{C^{m_j - 1}(G)} \right)^{1/2} \]

which implies, by (3.13), (3.14) and (3.15), that

\[ I_j \leq c \left( \int_0^s \left( w(s) - r \right) \frac{1}{C^{m_j - 1}(G)} \right)^{1/2} \]

and, recalling (3.12), (3.10) and (3.11), the proof of (3.3) will be complete.

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and, recalling (3.12), (3.10) and (3.11), the proof of (3.3) will be complete.
4. THE SECOND INCLUSION.

Let $A$ be defined again by (1.8) and assume (1.9). We want to prove:

**Theorem 4.1.** If $0 \leq 0.1$ and $q := 2n \in \mathbb{N}$, then

$$D_A(0,0) = \mathcal{H}(\mathcal{B}).$$

**Proof.** Let $f \in D_A(0,0)$. By the Reiteration Theorem [Triebel, 1978, Theorem 1.10.2] and by [Adams-Farey, 1987, Theorem 2.3] we have

$$D_A(0,0) = \left( \mathcal{H}(\mathcal{B}), \mathcal{H}(\mathcal{B}) \right)_{\frac{q}{2}} = \left( \mathcal{H}(\mathcal{B}), \mathcal{H}(\mathcal{B}) \right)_{\frac{q}{2}},$$

where $\mathcal{H}(\mathcal{B})$ is a norm. Then, by definition [Jous-Perret, 1964], there exists a function $u : [0,1] \to \mathcal{H}(\mathcal{B})$ such that:

$$\left\{ \begin{array}{l}
\sup_{t \leq 0.1} \left\| u(t) \right\|_{\mathcal{H}(\mathcal{B})} + \sup_{t \leq 0.1} \left\| u'(t) \right\|_{\mathcal{H}(\mathcal{B})} \\
\left\| u(0) \right\|_{\mathcal{H}(\mathcal{B})} = \left\{ f \right\}_{D_A(0,0)}
\end{array} \right.$$

Thus, in particular, $f \in \mathcal{H}(\mathcal{B})$. In addition, $u \in \mathcal{H}(\mathcal{B})$ if $f \in \mathcal{H}(\mathcal{B})$ as $t \to 0$.

We have to show that $\left\{ f \right\}_{D_A(0,0)}$ is finite. The following lemma is crucial for this purpose:

**Lemma 4.2.** Fix $|\gamma| = q - 1$. We have:

1. $|D^\gamma f(x) - D^\gamma f(y) - D^\gamma u(t,x) + D^\gamma u(t,y)| \\
\leq c_q \left( |D_A^\gamma(0,0)| \frac{c_{\mathcal{B}}}{2} \right)^{1/2} |x - y|^{1 - \epsilon} \quad \forall x, y \in \mathbb{R}^n$.

Let us show that $\left\{ f \right\}_{D_A(0,0)}$ is finite. Indeed, let $u \in \mathcal{H}(\mathcal{B})$ and $x \in \mathbb{R}^n$.

by Lemma 4.2, recalling that $B_2^s(x,0) = 0$, we get:

$$|D^\gamma f(x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq C_\gamma |D^\gamma u(t,x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq C_\gamma |D^\gamma u(t,x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq C_\gamma |D^\gamma u(t,x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq C_\gamma |D^\gamma u(t,x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq C_\gamma |D^\gamma u(t,x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq C_\gamma |D^\gamma u(t,x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq C_\gamma |D^\gamma u(t,x)|}$$

Thus, in particular, $f \in \mathcal{H}(\mathcal{B})$. In addition, $u \in \mathcal{H}(\mathcal{B})$ if $f \in \mathcal{H}(\mathcal{B})$ as $t \to 0$. We have to show that $\left\{ f \right\}_{D_A(0,0)}$ is finite. The following lemma is crucial for this purpose:

**Lemma 4.2.** Fix $|\gamma| = q - 1$. We have:

1. $|D^\gamma f(x) - D^\gamma f(y) - D^\gamma u(t,x) + D^\gamma u(t,y)| \\
\leq c_q \left( |D_A^\gamma(0,0)| \frac{c_{\mathcal{B}}}{2} \right)^{1/2} |x - y|^{1 - \epsilon} \quad \forall x, y \in \mathbb{R}^n$.

Let us show that $\left\{ f \right\}_{D_A(0,0)}$ is finite. Indeed, let $u \in \mathcal{H}(\mathcal{B})$ and $x \in \mathbb{R}^n$.

by Lemma 4.2, recalling that $B_2^s(x,0) = 0$, we get:

$$|D^\gamma f(x)| \leq \left( \left\| D_A^\gamma(0,0) \right\| \right)^{1/2} |D^\gamma u(t,x)| \leq c_q \left( |D_A^\gamma(0,0)| \frac{c_{\mathcal{B}}}{2} \right)^{1/2} |x - y|^{1 - \epsilon} \quad \forall x, y \in \mathbb{R}^n.$$
Now by Lemma 4.2(i)
\[ S_2 \leq c_q \frac{1}{\beta} \frac{1}{A(\theta, r)} \leq \frac{1}{A(\theta, r)} \leq \frac{1}{A(\theta, \infty)} \leq c \frac{1}{A(\theta, \infty)} , \]
and by Lemma 4.2(ii)
\[ S_2 \leq c_q \frac{1}{\beta} \frac{1}{A(\theta, r)} \leq c \frac{1}{A(\theta, \infty)} , \]
on the other hand
\[ S_2 \leq \int_0^{\infty} \| u(x) \|_{A(\theta, r)} \leq c \frac{1}{A(\theta, \infty)} , \]
and finally
\[ S_2 \leq c \frac{1}{A(\theta, \infty)} \leq c \frac{1}{A(\theta, \infty)} . \]

Hence choosing \( t = \left( \frac{c}{0} \right)^{2\nu} \) we get
\[ \left( f \right)_{A(\theta, r)} \leq c_q \frac{1}{A(\theta, \infty)} , \]
and the proof is complete.

5. IMPROVEMENTS AND REMARKS.

By Theorems 3.1 and 4.1 the first equality of Theorem 2.4 is established. In order to check the second one, just a few remarks are needed. Concerning the first inclusion, repeating the argument of Section 3 we see that the right-hand sides of the inequalities of Proposition 3.2 have to be multiplied by \( o(1) \) (as \( \delta \to \infty \)), due to the fact that \( f \) belongs to \( A(\theta, \infty) \) now. Consequently we get

\[ \lim_{\delta \to \infty} s^{\delta} |w(s)|^{1\beta} = 0 , \]

which replaces (3.1). In order to get the analogous of (3.2), i.e.

\[ \lim_{\delta \to \infty} s^{\delta} |w(s)|^{1\beta} = 0 , \]
it is readily seen that the main point is to show that if \( \nu_j \leq q \)
\[ \lim_{\delta \to \infty} s^{\delta} \frac{1}{A(\theta, r)} = 0 . \]
This is easily proved, similarly to (3.17), when \( \nu_j \leq q \). In the case \( \nu_j > q \) we split
\[ B_j (x) = B_j (x) + B_j (x) - B_j (x) - B_j (x) \]
but
\[ B_j (x) - B_j (x) ) = \frac{1}{2} \left( B_j (x) - B_j (x) - B_j (x) - B_j (x) \right) \]
whereas, using the notations of (3.19),
\[ \left| B_j (x) - B_j (x) - B_j (x) - B_j (x) \right| \leq \left| B_j (x) - B_j (x) - B_j (x) - B_j (x) \right| \leq \]
\[ \leq c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) \]
\[ \leq c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) + c_q \frac{1}{A(\theta, \infty)} o(1) \]
where in estimating the last term we have used the boundary condition satisfied by \( f \). Taking \( \delta_j = \delta_j \) we finally get (5.1), which together with (5.1) yields

\[ \lim_{\delta \to \infty} s^{\delta} |A(\theta, r)|^{1\beta} = 0 , \]

i.e., by (1.11), \( f \in D(\theta) \).

The second inclusion is easier: if \( f \in D(\theta) \), then, as \( D(\theta) \) is the closure of \( D \), in \( D(\theta) \), we take a sequence \( \{ u_n \} \) in \( D(\theta) \) such that \( u_n \to f \).
in $D_{r}(0,\infty)$; then

$$\begin{cases} u_n + f \text{ in } A^q_{0}(\Omega) \\
[u_n, q] \text{ in } A^q_{0}(\Omega) \\
\n\end{cases}$$

The first condition implies $f \in A^q_{0}(\Omega)$ since $D_{r} \subset A^q_{0}(\Omega)$ and $A^q_{0}(\Omega)$ is a closed subspace of $A^q(\Omega)$; the second one easily yields (since $B_{j}(\ast, D)u = 0$ for $j = 1, \ldots, m)$:

$$\lim_{\eta \to 0} B_{j}(\ast, D) f(x) = -\eta \int_{\Omega} (x, D)f(x)$$

provided $x \in \Omega$ and $m = q$. Thus $f \in A^q_{0}(\Omega)$ and Theorem 2.4 is completely proved.

REMARK 5.1. Theorem 2.4 still holds in the case of elliptic systems in a possibly unbounded open set which is uniformly regular of class $C^{2m}$ [Mann, 1984; Geymonat-Grisvard, 1967] [compare with [Acquistapace-Terreni, 1987, Remark 5.1]].

REMARK 5.2. Theorem 2.4 in the case $m = 1$ was proved, with different techniques, in [Acquistapace-Terreni, 1984].

REMARK 5.3. In the definition (2.13) of $\Lambda_{0}^{1}$ we may replace the integral curve $u(x,s)$ of (2.10) by the segment $x - sv(x)$, $s \in [0, c]$. Indeed, if $f \in A^q_{0}(\Omega)$ we have (denoting by $u(t, \ast)$ the function satisfying (2.5) and such that $u(0, \ast) = f$):

$$\frac{d^{2}f(u(x, s), x - sv(x))}{ds^2} = \frac{d^{2}f(x - sv(x))}{ds^2}$$

$$= \left[ - \frac{1}{c} \int_{0}^{c} \left\{ D^{2}u(s, x - sv(x)) \right\} ds \right] \ast +$$

$$\frac{1}{c} \int_{0}^{c} \left\{ (D^{2}u(t, x - sv(x))) \ast (D^{2}u(t, x - sv(x))) \right\} dt \ast$$

$$\leq 2 c^{\varepsilon} m^{1/2} \int_{\Omega} \left| (\mu_{0} + x - s \nu(x)) \right|^{1/2} \ast +$$

$$+ \frac{c}{\varepsilon} \int_{\Omega} \left| \left( \mu_{0} + x - s \nu(x) \right) \right|^{1/2} \ast + \int_{\Omega} \| \mu_{0} + x - s \nu(x) \| \ast$$

provided we choose $\varepsilon = \left( \frac{c}{2} \right)^{2/3}$.  

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