Chapter 2.3
Invariant objects

The main problem we must deal with now is how to generalize the theorems proved in the previous chapter. An analysis of the proofs (and a glance at the first part of this book) shows that the main tools used were the tautness of B^n and Schwarz's lemma. So our first concern should be how to get a sort of Schwarz's lemma on taut manifolds.

The right solution is suggested by the study of hyperbolic Riemann surfaces. There, we defined a distance, the Poincaré distance, contracted by holomorphic functions, and this allowed us to make use of Schwarz's lemma type arguments, with the results you surely know by heart. Therefore if we find a way to define on every taut manifold a distance contracted by holomorphic maps then we shall (hopefully) be able to work out the same arguments in new settings.

Well; in 1967 Kobayashi [1967a, b] introduced a pseudodistance on every complex manifold which is exactly what we need: it is contracted by holomorphic maps, it coincides with the Poincaré distance on hyperbolic Riemann surfaces (and with the Bergmann distance on B^n), and it is a true distance on taut manifolds. The impatient reader may now jump to the next chapter to see how to apply this wonderful new tool, but the far-sighted reader will better choose to study carefully this chapter, where we develop all the basic properties of the Kobayashi pseudodistance we shall need later on.

To be more specific, in this chapter we shall mainly do two things. First of all, we shall introduce and study the Kobayashi pseudodistance, its cousin the Carathéodory pseudodistance, and their relatives, the invariant metrics and volume forms. A special attention will be payed to manifolds where the Kobayashi pseudodistance is a true distance, showing in particular that this is the case for taut manifolds.

Our second concern will be the study of the boundary behavior of the Kobayashi distance and metric in strongly pseudoconvex domains. In fact, a lot of future work will be devoted to study the boundary behavior of several objects defined using the Kobayashi distance (remember Lemma 1.3.19, for instance), and thus we shall need very precise estimates. To give an idea of the strength of these tools, we shall prove a version of Fefferman's theorem: every biholomorphism of strongly pseudoconvex domains extends continuously to the boundary.

It should be noticed that this is a very utilitarian chapter: we shall not make any effort to present a complete exposition of the theory, limiting ourselves to the facts we shall need later on; for instance, the Carathéodory distance is somehow neglected, and we do not discuss extension theorems. A more comprehensive exposition (though not containing all the results presented here) can be found in Kobayashi [1970, 1976] or in Lang [1987].

2.3.1 *Invariant distances*

In this section we shall define the Kobayashi and the Carathéodory pseudodistances on complex manifolds, and we shall describe their main general properties. In particular, we shall devote a lot of (time and) space to the study of the so-called hyperbolic manifolds, where the Kobayashi pseudodistance is a true distance. By the way, we probably had better to specify what we mean by pseudodistance. A *pseudodistance* on a set X is a function $d: X \times X \to \mathbf{R}^+$ such that

(i) $d(x,y) = d(y,x)$ for every x, $y \in X$;
(ii) $d(x,z) \le d(x,y) + d(y,z)$ for every x, y, $z \in X$;
(iii) $d(x,x) = 0$ for every $x \in X$.

Clearly, if we replace (iii) by "$d(x,y) = 0$ iff $x = y$ for every x, $y \in X$" we get the usual notion of distance on a set.

So let X be a complex manifold. The *Carathéodory (pseudo)distance* c_X on X is defined by

$$\forall z, w \in X \qquad c_X(z,w) = \sup\bigl\{\omega\bigl(h(z), h(w)\bigr) \bigm| h \in \mathrm{Hol}(X, \Delta)\bigr\}, \tag{2.3.1}$$

where ω is the Poincaré distance on Δ. We shall see in a while that $c_X(z,w)$ is always finite; granted this, it is obvious that c_X is a pseudodistance on X. We shall denote by $B_c(z,r)$ the open *Carathéodory ball*, that is the open ball of center $z \in X$ and radius $r > 0$ for c_X.

The dual concept is the function $\delta_X: X \times X \to [0, +\infty]$ defined by

$$\delta_X(z,w) = \inf\bigl\{\omega(\zeta,\eta) \bigm| \exists \varphi \in \mathrm{Hol}(\Delta, X): \varphi(\zeta) = z,\ \varphi(\eta) = w\bigr\} \tag{2.3.2}$$

for all z, $w \in X$. Unfortunately, in general δ_X does not satisfy the triangular inequality; therefore to get a pseudodistance on X we need a more complicate definition.

An *analytic chain* $\alpha = \{\zeta_0, \ldots, \zeta_m; \eta_0, \ldots, \eta_m; \varphi_0, \ldots, \varphi_m\}$ connecting two points z_0 and w_0 in a complex manifold X is a sequence of points $\zeta_0, \ldots, \zeta_m, \eta_0, \ldots, \eta_m \in \Delta$ and holomorphic maps $\varphi_0, \ldots, \varphi_m \in \mathrm{Hol}(\Delta, X)$ such that $\varphi_0(\zeta_0) = z_0$, $\varphi_j(\eta_j) = \varphi_{j+1}(\zeta_{j+1})$ for $j = 0, \ldots, m-1$ and $\varphi_m(\eta_m) = w_0$. The *length* $\omega(\alpha)$ of the chain α is

$$\omega(\alpha) = \sum_{j=0}^{m} \omega(\zeta_j, \eta_j).$$

Then we can define the *Kobayashi (pseudo)distance* k_X on X by

$$\forall z, w \in X \qquad k_X(z,w) = \inf\bigl\{\omega(\alpha)\bigr\}, \tag{2.3.3}$$

where the infimum is taken with respect to all the analytic chains connecting z to w. Since X is connected, $k_X(z,w)$ is always finite, and it is clear that k_X is a pseudodistance on X. We shall denote by $B_k(z,r)$ the open *Kobayashi ball*, that is the open ball of center $z \in X$ and radius $r > 0$ for k_X.

Since Δ is homogeneous, it is not necessary to consider all the analytic chains connecting z to w in (2.3.3). For instance, we can limit to *linked analytic chains*, that is to analytic chains such that $\eta_j = \zeta_{j+1}$ for $j = 0, \ldots, m-1$, or to *fixed analytic chains*, that is to analytic chains such that $\zeta_j = 0$ for $j = 0, \ldots, m$. Furthermore, note that, by definition, for all z, $w \in X$ we have

$$k_X(z,w) = \inf\bigl\{\sum_{j=0}^{m} \delta_X(z_j, z_{j+1}) \bigm| z_0 = z,\ z_{m+1} = w,\ z_1, \ldots, z_m \in X,\ m \in \mathbf{N}\bigr\}. \tag{2.3.4}$$

The main property of Carathéodory and Kobayashi distances is:

Proposition 2.3.1: *Let $f: X \to Y$ be a holomorphic map between two complex manifolds. Then for all z, $w \in X$*
$$c_Y\big(f(z), f(w)\big) \le c_X(z, w)$$
and
$$k_Y\big(f(z), f(w)\big) \le k_X(z, w).$$

Proof: This clearly follows from the definitions, **q.e.d.**

Corollary 2.3.2: *Let X be a complex manifold. Then*
(i) *If $\gamma \in \mathrm{Aut}(X)$ then for every z, $w \in X$*
$$c_X\big(\gamma(z), \gamma(w)\big) = c_X(z, w) \qquad \textit{and} \qquad k_X\big(\gamma(z), \gamma(w)\big) = k_X(z, w);$$
(ii) *If Y is a submanifold of X then for every z, $w \in Y$*
$$c_X(z, w) \le c_Y(z, w) \qquad \textit{and} \qquad k_X(z, w) \le k_Y(z, w).$$

Now the scrupulous reader may ask why we singled out these two particular pseudodistances. The answer is that they are extremal in a very precise sense:

Proposition 2.3.3: *Let X be a complex manifold, and $d: X \times X \to \mathbf{R}^+$ a pseudodistance on X. Then*
(i) *if*
$$d\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) \le \omega(\zeta_1, \zeta_2)$$
for all ζ_1, $\zeta_2 \in \Delta$ and $\varphi \in \mathrm{Hol}(\Delta, X)$, then $d \le k_X$;
(ii) *if*
$$d(z_1, z_2) \ge \omega\big(f(z_1), f(z_2)\big)$$
for all z_1, $z_2 \in X$ and $f \in \mathrm{Hol}(X, \Delta)$, then $d \ge c_X$.

Proof: (i) If $\alpha = \{\zeta_0, \ldots, \zeta_m; \eta_0, \ldots, \eta_m; \varphi_0, \ldots, \varphi_m\}$ is any analytic chain connecting two points z, $w \in X$ we have
$$d(z, w) \le \sum_{j=0}^{m} d\big(\varphi_j(\zeta_j), \varphi_j(\eta_j)\big) \le \sum_{j=0}^{m} \omega(\zeta_j, \eta_j) = \omega(\alpha),$$
and so $d \le k_X$.

(ii) Obvious, **q.e.d.**

We already have a distance satisfying Proposition 2.3.1, namely the Poincaré distance on Δ. This is not casual:

Proposition 2.3.4: (i) $k_\Delta = \omega = c_\Delta$.
(ii) *For any complex manifold X we have $c_X \le k_X$. In particular, c_X is always finite.*

Proof: It is clear that $k_\Delta \le \omega$; on the other hand, Proposition 2.3.3.(i) and the Schwarz-Pick lemma yield $\omega \le k_\Delta$, and the first equality in (i) is proved. In particular, we have

$$\omega\bigl(h(z), h(w)\bigr) \le k_X(z, w)$$

for all z, $w \in X$ and $h \in \mathrm{Hol}(X, \Delta)$, and (ii) follows. Finally, (ii) implies

$$\omega \le c_\Delta \le k_\Delta = \omega,$$

and we are done, **q.e.d.**

So k_X and c_X are generalizations of the Poincaré distance, and Proposition 2.3.1 is the ultimate generalization of Schwarz's lemma, as promised.

The Carathéodory and Kobayashi pseudodistances are wonderful theoretical tools, but they are very hard to compute explicitely in given examples (and indeed we shall spend a whole section to estimate them using euclidean objects). An important exception is the following case:

Proposition 2.3.5: *Let $\|\cdot\|\colon \mathbf{C}^n \to \mathbf{R}^+$ be a norm on $\mathbf{C}^n$, and B the unit ball for this norm. Then for all $z \in B$*

$$c_B(0, z) = k_B(0, z) = \omega(0, \|z\|).$$

Proof: Take $z \in B$, $z \ne 0$, and define $\varphi\colon \Delta \to B$ by $\varphi(\zeta) = \zeta z/\|z\|$. Then Propositions 2.3.1 and 2.3.4 yield

$$c_B(0, z) \le k_B(0, z) \le \omega(0, \|z\|).$$

On the other hand, for every $z \in \mathbf{C}^n$ there exists a linear form $\lambda_z\colon \mathbf{C}^n \to \mathbf{C}$ such that $\lambda_z(z) = \|z\|$ and $\lambda_z(w) \le \|w\|$ for all $w \in \mathbf{C}^n$. Therefore $\lambda_z|_B$ sends B into Δ and, if $z \in B$,

$$\omega(0, \|z\|) \le c_B(0, z),$$

q.e.d.

In particular, as already reflected by the notations, in B^n we find nothing new:

Corollary 2.3.6: *On B^n, the Bergmann, Carathéodory and Kobayashi distances coincide.*

Proof: By Proposition 2.3.5 and (2.2.19) they coincide at the origin. Since B^n is homogeneous, they coincide everywhere, **q.e.d.**

The *unit polydisk* Δ^n of $\mathbf{C}^n$ is the unit ball for the norm

$$\|z\| = \max\{|z_1|, \ldots, |z_n|\};$$

as a set, $\Delta^n = \Delta \times \cdots \times \Delta$ n times. Using Proposition 2.3.5 we can compute k_{Δ^n} and c_{Δ^n}:

Corollary 2.3.7: *For any z, $w \in \Delta^n$*

$$k_{\Delta^n}(z,w) = c_{\Delta^n}(z,w) = \omega\big(0, \|\gamma_z(w)\|\big) = \max_{j=1,\ldots,n}\big\{\omega(z_j,w_j)\big\},$$

where

$$\gamma_z(w) = \left(\frac{w_1 - z_1}{1 - \overline{z_1}w_1}, \ldots, \frac{w_n - z_n}{1 - \overline{z_n}w_n}\right).$$

Proof: γ_z is an automorphism of Δ^n such that $\gamma_z(z) = 0$. The assertion then follows from Proposition 2.3.5, **q.e.d.**

In general, the relationship between the Carathéodory and Kobayashi distances of two manifolds and the respective distances on the cartesian product is described in

Proposition 2.3.8: *Let X and Y be two complex manifolds, z_1, $z_2 \in X$ and w_1, $w_2 \in Y$. Then*

$$c_X(z_1,z_2) + c_Y(w_1,w_2) \geq c_{X\times Y}\big((z_1,w_1),(z_2,w_2)\big) \geq \max\big\{c_X(z_1,z_2), c_Y(w_1,w_2)\big\},$$

and

$$k_X(z_1,z_2) + k_Y(w_1,w_2) \geq k_{X\times Y}\big((z_1,w_1),(z_2,w_2)\big) \geq \max\big\{k_X(z_1,z_2), k_Y(w_1,w_2)\big\}.$$

Proof: The inequalities on the right follow applying Proposition 2.3.1 to the canonical projections $(z,w) \mapsto z$ and $(z,w) \mapsto w$. On the other hand, Proposition 2.3.1 applied to the maps $z \mapsto (z,w_1)$ and $w \mapsto (z_2,w)$ yields

$$\begin{aligned} c_X(z_1,z_2) + c_Y(w_1,w_2) &\geq c_{X\times Y}\big((z_1,w_1),(z_2,w_1)\big) + c_{X\times Y}\big((z_2,w_1),(z_2,w_2)\big), \\ k_X(z_1,z_2) + k_Y(w_1,w_2) &\geq k_{X\times Y}\big((z_1,w_1),(z_2,w_1)\big) + k_{X\times Y}\big((z_2,w_1),(z_2,w_2)\big), \end{aligned}$$

and the inequalities on the left follow from the triangular inequality, **q.e.d.**

A natural question is whether the distance topology induced by k_X or c_X coincides with the manifold topology of X. One direction is settled by

Proposition 2.3.9: *Let X be a complex manifold; then c_X and k_X are continuous. In particular, the manifold topology is finer than the distance topology induced either by k_X or by c_X.*

Proof: Since for all z_0, w_0, z, $w \in X$

$$|c_X(z_0,w_0) - c_X(z,w)| \leq c_X(z_0,z) + c_X(w_0,w),$$

and analogously for k_X, it suffices to show that for any $z_0 \in X$ the functions $z \mapsto c_X(z_0,z)$ and $z \mapsto k_X(z_0,z)$ are continuous at z_0.

Let $U \subset X$ be a coordinate neighbourhood of z_0 biholomorphic to some B^n. By Proposition 2.3.5, $c_U(z_0,z)$ and $k_U(z_0,z)$ are continuous. Therefore, by Corollary 2.3.2.(ii), $c_X(z_0,z)$ and $k_X(z_0,z)$ are continuous at z_0, and we are done, **q.e.d.**

In general, however, k_X and c_X do not induce the manifold topology. In fact, it is evident that if X is compact then $c_X \equiv 0$, or that $c_{\mathbf{C}^n} \equiv k_{\mathbf{C}^n} \equiv 0$, whereas to induce the manifold topology k_X or c_X should be at least true distances.

Now, c_X is a distance iff for every pair of distinct points z, $w \in X$ there is a *bounded* holomorphic function $h\colon X \to \mathbf{C}$ such that $h(z) \neq h(w)$, which is quite a restrictive assumption (it is almost equivalent to requiring X to be a relatively compact domain of a Stein manifold, if you know what that means). Therefore, from now on we shall focus on the Kobayashi distance, and we are led to the following definition: a complex manifold X is *hyperbolic* iff k_X is a true distance (the reason for this name will become apparent after Corollary 2.3.12). Then

Proposition 2.3.10: *Let X be a complex manifold. Then X is hyperbolic iff k_X induces the manifold topology on X.*

Proof: One direction is clear. Conversely assume that X is hyperbolic; we have to show that for every $z \in X$ and every neighbourhood U of z in X there is a Kobayashi ball $B_k(z,r)$ contained in U. Clearly, we can assume U relatively compact in X.

Assume, by contradiction, that this is not the case. Then there is a sequence $\{z_\nu\} \subset X$ such that $z_\nu \notin U$ for all ν and $k_X(z,z_\nu) < 1/\nu$. This means that there is a linked analytic chain $\{\zeta_0,\ldots,\zeta_{m_\nu+1};\varphi_0,\ldots,\varphi_{m_\nu}\}$ such that

$$\sum_{j=0}^{m_\nu} \omega(\zeta_j,\zeta_{j+1}) < 1/\nu.$$

Let σ_j^ν be the geodesic arc for the Poincaré metric on Δ joining ζ_j and ζ_{j+1}. Then the arcs $\varphi_j \circ \sigma_j^\nu$ in X connect to form a continuous curve σ^ν from z to z_ν such that $k_X(z,w) < 1/\nu$ for every w in the range of σ^ν. Now, since $z \in U$ and $z_\nu \notin U$, there is $w_\nu \in \partial U$ in the range of σ^ν; in particular,

$$\lim_{\nu\to\infty} k_X(z,y_\nu) = 0.$$

But ∂U is compact, and $k_X(z,\cdot)$ is continuous and never zero on ∂U (for X is hyperbolic); hence

$$\inf_{y\in\partial U} k_X(z,y) > 0,$$

contradiction, **q.e.d.**

At this point, an anxious reader may begin to tremble, thinking at a hypothetic confusion between hyperbolic Riemann surfaces and, well, hyperbolic Riemann surfaces. But don't be frightened: the names were chosen with a bit of foreseeing. Indeed, if A and B are subsets of a complex manifold X, define the Kobayashi distance from A to B by

$$k_X(A,B) = \inf\{k_X(z,w) \mid z \in A,\ w \in B\},$$

as usual; then

Proposition 2.3.11: *Let $\pi\colon \widetilde{X} \to X$ be a covering map of complex manifolds. Take z_0, $w_0 \in X$; then*

$$k_X(z_0, w_0) = k_{\widetilde{X}}\big(\tilde{z}_0, \pi^{-1}(w_0)\big),$$

where $\tilde{z}_0$ is any element of $\pi^{-1}(z_0)$.

Proof: It is clear that

$$k_X(z_0, w_0) \le k_{\widetilde{X}}\big(\tilde{z}_0, \pi^{-1}(w_0)\big).$$

Assume, by contradiction, that there is $\varepsilon > 0$ such that

$$k_X(z_0, w_0) + \varepsilon \le k_{\widetilde{X}}\big(\tilde{z}_0, \pi^{-1}(w_0)\big),$$

and choose a linked analytic chain $\alpha = \{\zeta_0, \ldots, \zeta_{m+1}; \varphi_0, \ldots, \varphi_m\}$ connecting z_0 to w_0 such that

$$\sum_{j=0}^{m} \omega(\zeta_j, \zeta_{j+1}) < k_X(z_0, w_0) + \varepsilon.$$

We can lift α to a linked analytic chain $\tilde{\alpha} = \{\zeta_0, \ldots, \zeta_{m+1}; \tilde{\varphi}_0, \ldots, \tilde{\varphi}_m\}$ connecting the given point $\tilde{z}_0$ to a point $\tilde{w}_0 \in \pi^{-1}(w_0)$ so that $\pi \circ \tilde{\varphi}_j = \varphi_j$ for $j = 0, \ldots, m$. Then

$$k_{\widetilde{X}}(\tilde{z}_0, \tilde{w}_0) \le \sum_{j=0}^{m} \omega(\zeta_j, \zeta_{j+1}),$$

contradiction, **q.e.d.**

Corollary 2.3.12: *Let X be a Riemann surface. Then*
(i) *if X is hyperbolic, then the Poincaré and the Kobayashi distances coincide;*
(ii) *if X is not hyperbolic, then $k_X \equiv 0$.*

Proof: (i) follows from Propositions 2.3.4, 2.3.11 and (1.1.27). To prove (ii), by Proposition 2.3.11 it suffices to show that $k_{\mathbf{C}} \equiv k_{\widehat{\mathbf{C}}} \equiv 0$. It is easy to check that $k_{\mathbf{C}} \equiv 0$; therefore $k_{\widehat{\mathbf{C}}}|_{\mathbf{C}\times\mathbf{C}} \equiv 0$ and, by Proposition 2.3.9, $k_{\widehat{\mathbf{C}}} \equiv 0$, **q.e.d.**

This is the reason of the name "hyperbolic". Note that, in particular, there are compact hyperbolic manifolds (the compact hyperbolic Riemann surfaces, for instance), whereas on compact manifolds the Carathéodory distance is identically zero.

It is now time to give examples of hyperbolic manifolds. Besides the hyperbolic Riemann surfaces, a first set of examples is provided by the homogeneous unit balls of norms in $\mathbf{C}^n$, thanks to Proposition 2.3.5. Other examples can be constructed using the following

Proposition 2.3.13: (i) *A submanifold of a hyperbolic manifold is hyperbolic;*
(ii) *the product of two hyperbolic manifolds is hyperbolic;*
(iii) *if $\pi\colon \widetilde{X} \to X$ is a covering map of complex manifolds, then $\widetilde{X}$ is hyperbolic iff X is hyperbolic.*

Proof: (i) This follows from Corollary 2.3.2.(ii).

(ii) This follows from Proposition 2.3.8.

(iii) Assume $\widetilde{X}$ is hyperbolic. If $k_X(z_0, w_0) = 0$ for two points z_0, $w_0 \in X$, then for any $\tilde{z}_0 \in \pi^{-1}(z_0)$ there is a sequence $\{\tilde{w}_\nu\} \subset \pi^{-1}(w_0)$ such that $k_{\widetilde{X}}(\tilde{z}_0, \tilde{w}_\nu) \to 0$ as $\nu \to +\infty$ (by Proposition 2.3.11). Then $\tilde{w}_\nu \to \tilde{z}_0$ (Proposition 2.3.10) and so $\tilde{z}_0 \in \pi^{-1}(w_0)$, that is $z_0 = w_0$.

Conversely, assume X hyperbolic. Suppose $\tilde{z}_0$, $\tilde{w}_0 \in \widetilde{X}$ are so that $k_{\widetilde{X}}(\tilde{z}_0, \tilde{w}_0) = 0$; then $k_X\big(\pi(\tilde{z}_0), \pi(\tilde{w}_0)\big) = 0$ and so $\pi(\tilde{z}_0) = \pi(\tilde{w}_0) = z_0$. Let $\widetilde{U}$ be a neighbourhood of $\tilde{z}_0$ such that $\pi|_{\widetilde{U}}$ is a biholomorphism between $\widetilde{U}$ and $B_k(z_0, \varepsilon)$, for $\varepsilon > 0$ small enough; in particular, $\tilde{w}_0 \notin \widetilde{U}$. Since $k_{\widetilde{X}}(\tilde{z}_0, \tilde{w}_0) = 0$, there is a linked analytic chain $\{\zeta_0, \ldots, \zeta_{m+1}; \varphi_0, \ldots, \varphi_m\}$ connecting $\tilde{z}_0$ to $\tilde{w}_0$ such that

$$\sum_{j=0}^{m} \omega(\zeta_j, \zeta_{j+1}) < \varepsilon.$$

Let σ_j be the geodesic arc for the Poincaré metric on Δ joining ζ_j to ζ_{j+1}. Then the arcs $\varphi_j \circ \sigma_j$ in $\widetilde{X}$ connect to form a continuous curve σ from $\tilde{z}_0$ to $\tilde{w}_0$. Now the maps $\pi \circ \varphi_j \in \mathrm{Hol}(\Delta, X)$ are distance-decreasing; therefore every point of the curve $\pi \circ \sigma$ should belong to $B_k(z_0, \varepsilon)$. But then σ is contained in $\widetilde{U}$, and so $\tilde{z}_0 = \tilde{w}_0$, **q.e.d.**

But the main source of examples is

Theorem 2.3.14: *Every relatively taut manifold is hyperbolic.*

Proof: Assume that the complex manifold X is not hyperbolic, and take z_0, w_0 two distinct points of X such that $k_X(z_0, w_0) = 0$. Choose a coordinate neighbourhood U of z_0 relatively compact in X, and biholomorphic to B^n for some n, such that $w_0 \notin \overline{U}$, and choose also another neighbourhood $V \subset\subset U$ of z_0.

We claim that for any $\nu \in \mathbf{N}$ there is $\varphi_\nu \in \mathrm{Hol}(\Delta, X)$ such that $\varphi_\nu(0) \in \overline{V}$ but $\varphi_\nu(\Delta_{1/\nu}) \not\subset U$. In fact, assume by contradiction that $\nu \in \mathbf{N}$ is such that $\varphi(0) \in \overline{V}$ implies $\varphi(\Delta_{1/\nu}) \subset U$ for any $\varphi \in \mathrm{Hol}(\Delta, X)$. Choose a constant $c > 0$ such that $\omega(0, \zeta) \geq c\, k_{\Delta_{1/\nu}}(0, \zeta)$ for all $\zeta \in \Delta_{1/(2\nu)}$, and let $\varepsilon = c\, k_U(z_0, \partial V) > 0$. Since we have $k_X(z_0, w_0) = 0$, we can find a fixed analytic chain $\{\zeta_1, \ldots, \zeta_m; \varphi_1, \ldots, \varphi_m\}$ connecting z_0 to w_0 such that

$$\sum_{j=1}^{m} \omega(0, \zeta_j) < \varepsilon.$$

Let $m_0 \leq m$ be the first integer such that $\{\varphi_{m_0}(t\zeta_{m_0}) \mid t \in (0,1)\} \not\subset V$. Adding enough points of the form $t\zeta_j$ with $t \in (0,1)$ and $j = 1, \ldots, m_0$ we can assume $\zeta_j \in \Delta_{1/(2\nu)}$,

$\varphi_j(\zeta_j) \in V$ for $j = 1, \ldots, m_0 - 1$, $\zeta_{m_0} \in \Delta_{1/(2\nu)}$ and $\varphi_{m_0}(\zeta_{m_0}) \in \partial V$. Then

$$\begin{aligned}\sum_{j=1}^{m} \omega(0, \zeta_j) &\geq \sum_{j=1}^{m_0} \omega(0, \zeta_j) \geq c \sum_{j=1}^{m_0} k_{\Delta_{1/\nu}}(0, \zeta_j) \geq c \sum_{j=1}^{m_0} k_U\big(\varphi_j(0), \varphi_j(\zeta_j)\big) \\ &\geq c\, k_U\big(z_0, \varphi_{m_0}(\zeta_{m_0})\big) \geq \varepsilon,\end{aligned}$$

contradiction.

So we have a sequence $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, X)$ such that $\varphi_\nu(0) \in \overline{V}$ but $\varphi_\nu(\Delta_{1/\nu}) \not\subset U$; take $\zeta_\nu \in \Delta_{1/\nu}$ such that $\varphi_\nu(\zeta_\nu) \notin U$. Then if X were relatively taut in $\overline{X}$, $\{\varphi_\nu\}$ would have a subsequence $\{\varphi_{\nu_k}\}$ converging to $\varphi \in \mathrm{Hol}(X, \overline{X})$; in particular, $\{\varphi_{\nu_k}(\zeta_{\nu_k})\}$ would converge to $\varphi(0) \in \overline{V}$, impossible, **q.e.d.**

So taut manifolds are hyperbolic, as anticipated, but hyperbolic manifolds are much more common: for instance, every submanifold of a bounded domain of $\mathbf{C}^n$, or every manifold covered by a bounded domain, or covering a bounded domain, is hyperbolic.

However, this way of producing examples of hyperbolic manifolds is slightly different from what we did for hyperbolic Riemann surfaces. There, we first proved that the Kobayashi (formerly Poincaré) distance is complete, and then we showed that a hyperbolic Riemann surface is taut (Montel's theorem). Our next aim is to repeat this argument in general, thus proving that every hyperbolic manifold with complete Kobayashi distance is taut.

We say that a hyperbolic manifold X is *complete hyperbolic* if k_X is a complete distance. First of all, we want to show that k_X is complete iff every closed Kobayashi ball is compact (cf. Proposition 1.1.39). We need a notation: if A is a subset of X and $r > 0$, set

$$B_k(A, r) = \bigcup_{z \in A} B_k(z, r).$$

Lemma 2.3.15: *Let X be a complex manifold, and choose $z_0 \in X$ and r_1, $r_2 > 0$. Then*

$$B_k\big(B_k(z_0, r_1), r_2\big) = B_k(z_0, r_1 + r_2).$$

Proof: The inclusion $B_k\big(B_k(z_0, r_1), r_2\big) \subset B_k(z_0, r_1 + r_2)$ follows immediately from the triangular inequality. For the converse, let $z \in B_k(z_0, r_1 + r_2)$, and set $3\varepsilon = r_1 + r_2 - k_X(z_0, z)$. Then there is a linked analytic chain $\{\zeta_0, \ldots, \zeta_{m+1}; \varphi_0, \ldots, \varphi_m\}$ connecting z_0 to z such that

$$\sum_{j=0}^{m} \omega(\zeta_j, \zeta_{j+1}) < r_1 + r_2 - 2\varepsilon.$$

Let $\mu \leq m$ be the largest integer such that

$$\sum_{j=0}^{\mu-1} \omega(\zeta_j, \zeta_{j+1}) < r_1 - \varepsilon.$$

Let η_μ be the point on the geodesic arc for the Poincaré distance connecting ζ_μ to $\zeta_{\mu+1}$ such that

$$\sum_{j=0}^{\mu-1} \omega(\zeta_j, \zeta_{j+1}) + \omega(\zeta_\mu, \eta_\mu) = r_1 - \varepsilon.$$

If we set $w = \varphi_\mu(\eta_\mu)$, then $k_X(z_0, w) < r_1$ and $k_X(w, z) < r_2$, so that

$$z \in B_k(w, r_2) \subset B_k\big(B_k(z_0, r_1), r_2\big),$$

q.e.d.

Lemma 2.3.16: *Let X be a hyperbolic manifold, $z_0 \in X$ and $r > 0$. Then $\overline{B_k(z_0, r)}$ is compact if there exists $\rho > 0$ such that $\overline{B_k(z, \rho)}$ is compact for all $z \in B_k(z_0, r)$.*

Proof: Since X is locally compact and hyperbolic, there is $0 < s < r$ such that $\overline{B_k(z_0, s)}$ is compact. So it suffices to show that if $\overline{B_k(z_0, s)}$ is compact then also $\overline{B_k(z_0, s + \rho/2)}$ is compact. Let $\{z_\nu\}$ be a sequence in $\overline{B_k(z_0, s + \rho/2)}$, and $\{w_\nu\}$ a sequence in $\overline{B_k(z_0, s)}$ such that $k_X(z_\nu, w_\nu) < 3\rho/4$ for all $\nu \in \mathbf{N}$ (by Lemma 2.3.15). Up to a subsequence, we can assume that $\{w_\nu\}$ converges to $w_0 \in \overline{B_k(z_0, s)}$. Then $z_\nu \in \overline{B_k(w_0, \rho)}$ for all large ν; hence, by assumption, $\{z_\nu\}$ admits a converging subsequence, **q.e.d.**

Then:

Proposition 2.3.17: *Let X be a hyperbolic manifold. Then k_X is complete iff every closed Kobayashi ball is compact.*

Proof: One direction is obvious. Conversely, assume k_X complete; by Lemma 2.3.16, it suffices to show that there is $\rho > 0$ such that $\overline{B_k(z_0, \rho)}$ is compact for every $z_0 \in X$. Assume the contrary. Then there exists $z_1 \in X$ such that $\overline{B_k(z_1, 1/2)}$ is noncompact. By Lemma 2.3.16, there is $z_2 \in B_k(z_1, 1/2)$ such that $\overline{B_k\big(z_2, (1/2)^2\big)}$ is noncompact. In this way we obtain a Cauchy sequence $\{z_\nu\}$ such that $z_\nu \in B_k(z_{\nu-1}, 1/2^{\nu-1})$ and $\overline{B_k(z_\nu, 1/2^\nu)}$ is noncompact. Let w_0 be the limit of $\{z_\nu\}$; since X is locally compact, there is $\varepsilon > 0$ such that $\overline{B_k(w_0, \varepsilon)}$ is compact. But for ν sufficiently large $\overline{B_k(z_\nu, 1/2^\nu)}$ is contained in $\overline{B_k(w_0, \varepsilon)}$, and hence is compact, contradiction, **q.e.d.**

And so we can prove:

Theorem 2.3.18: *Every complete hyperbolic manifold is taut.*

Proof: Let X be a complete hyperbolic manifold, and let $\{\varphi_\nu\}$ be a sequence in $\mathrm{Hol}(\Delta, X)$ which is not compactly divergent; we should extract a subsequence converging uniformly on compact subsets.

Up to a subsequence, there are a compact set $K_0 \subset \Delta$ and a compact set $K_1 \subset X$ such that $\varphi_\nu(K_0) \cap K_1 \neq \emptyset$ for all $\nu \in \mathbf{N}$. Let K be another compact subset of Δ; we claim that there is a Kobayashi ball B in X such that $\varphi_\nu(K) \subset B$ for all $\nu \in \mathbf{N}$. Without loss of generality, we can assume that K is connected and contains K_0. Choose $\varepsilon > 0$, and cover K by a finite number of Poincaré disks of radius $\varepsilon/2$ and centers in $\zeta_1, \dots, \zeta_m \in K$. It is clear that the diameter of $\varphi_\nu(K)$ for the Kobayashi distance of X cannot exceed $m\varepsilon$ for all $\nu \in \mathbf{N}$. Now fix a point $w \in K_1$, and let ρ be the diameter of K_1 for k_X. Then, being $\varphi_\nu(K) \cap K_1 \neq \emptyset$, $\varphi_\nu(K)$ is contained in the Kobayashi ball of center w_0 and radius $\rho + m\varepsilon$, which is independent of ν, as claimed.

So $\{\varphi_\nu\}$ is an equicontinuous family such that $\bigcup_\nu \varphi_\nu(K)$ is bounded and hence compact (by Proposition 2.3.17) for every compact subset K of Δ; by the Ascoli-Arzelà theorem, $\{\varphi_\nu\}$ admits a subsequence converging uniformly on compact subsets, **q.e.d.**

This is the source of examples of taut manifolds we often mentioned in chapter 2.1. The following propositions will give a large list of complete hyperbolic (and hence taut) manifolds; moreover, later on we shall prove that convex and strongly pseudoconvex domains of $\mathbf{C}^n$ are complete hyperbolic.

Proposition 2.3.19: *Every homogeneous hyperbolic manifold is complete hyperbolic. In particular, every bounded homogeneous domain of $\mathbf{C}^n$ is complete hyperbolic.*

Proof: Let X be a homogeneous hyperbolic manifold, and take $z_0 \in X$ and $\rho > 0$ such that $\overline{B_k(z_0, \rho)}$ is compact. But then, by homogeneity, $\overline{B_k(z, \rho)}$ is compact for every $z \in X$, and so, by Lemma 2.3.16, X is complete hyperbolic, **q.e.d.**

Proposition 2.3.20: (i) *A closed submanifold of a complete hyperbolic manifold is complete hyperbolic;*
(ii) *the product of two complete hyperbolic manifolds is complete hyperbolic;*
(iii) *if $\pi\colon \widetilde{X} \to X$ is a covering map of complex manifolds, then $\widetilde{X}$ is complete hyperbolic iff X is complete hyperbolic.*

Proof: (i) This follows from Corollary 2.3.2.(ii).

(ii) This follows from Proposition 2.3.8.

(iii) Assume first $\widetilde{X}$ complete hyperbolic. Take $z_0 \in X$ and choose $\tilde{z}_0 \in \pi^{-1}(z_0)$. Then, by Proposition 2.3.11, for every $r > 0$ there is $\delta > 0$ such that

$$\overline{B_k(z_0, r)} \subset \pi\Big(\overline{B_k(\tilde{z}_0, r + \delta)}\Big),$$

and so X is complete hyperbolic.

Conversely, assume X complete hyperbolic, and let $\{\tilde{z}_\nu\}$ be a Cauchy sequence in $\widetilde{X}$. Since π is distance decreasing, $\{\pi(\tilde{z}_\nu)\}$ is a Cauchy sequence in X, and thus converges to $z_0 \in X$. Choose $\varepsilon > 0$ so small that $U = B_k(z_0, 2\varepsilon)$ is an admissible neighbourhood of z_0, i.e., π induces a homeomorphism of each connected component of $\pi^{-1}(U)$ onto U. Clearly, there is $\nu_0 \in \mathbf{N}$ such that $\pi(\tilde{z}_\nu) \in U$ for $\nu \geq \nu_0$. Let $\widetilde{U}_\nu$ be the connected component of $\pi^{-1}(U)$ containing $\tilde{z}_\nu$; we claim that $B_k(\tilde{z}_\nu, \varepsilon) \subset \widetilde{U}_\nu$ for all $\nu \geq \nu_0$. Indeed, take $\tilde{w} \in B_k(\tilde{z}_\nu, \varepsilon)$, and choose a linked analytic chain $\{\zeta_0, \dots, \zeta_{m+1}; \varphi_0, \dots, \varphi_m\}$ connecting $\tilde{z}_\nu$ to $\tilde{w}$ such that

$$\sum_{j=0}^{m} \omega(\zeta_j, \zeta_{j+1}) < \varepsilon.$$

If σ_j is the geodesic arc for the Poincaré metric joining ζ_j to ζ_{j+1}, the curves $\varphi_j \circ \sigma_j$ connect forming a continuous curve $\tilde{\sigma}$ from $\tilde{z}_\nu$ to $\tilde{w}$; set $\sigma = \pi \circ \tilde{\sigma}$. It is clear that the image of σ is contained in U; therefore the image of $\tilde{\sigma}$ must be contained in $\widetilde{U}_\nu$, and so $\tilde{w} \in \widetilde{U}_\nu$, as claimed.

In particular, then, $\widetilde{U}_\nu$ does not depend on ν for ν large enough, for $\{\tilde{z}_\nu\}$ is a Cauchy sequence; let $\widetilde{U}$ denote this uniquely determined connected component of $\pi^{-1}(U)$, and let $\tilde{z}_0 \in \widetilde{U}$ be the unique point of $\widetilde{U}$ such that $\pi(\tilde{z}_0) = z_0$. Then it is clear that $\tilde{z}_\nu \to \tilde{z}_0$, **q.e.d.**

Finally, it should be mentioned that not every taut manifold is complete hyperbolic; an example is in Rosay [1982].

2.3.2 *Invariant metrics*

Both the Poincaré and the Bergmann distances were the integrated form of a Riemannian metric; in this section we shall see that something similar is true for the Kobayashi distance.

Let X be a complex manifold, and TX its tangent bundle; then the *Kobayashi pseudometric* $\kappa_X\colon TX \to \mathbf{R}^+$ is defined by

$$\kappa_X(z; v) = \inf\bigl\{|\xi| \bigm| \exists \varphi \in \mathrm{Hol}(\Delta, X) : \varphi(0) = z,\ d\varphi_0(\xi) = v\bigr\}$$

for every $z \in X$ and $v \in T_z X$; note that if $\xi \in \mathbf{C}$ then $|\xi|$ is the length in the Poincaré metric of ξ considered as tangent vector to Δ at 0.

Analogously, the *Carathéodory pseudometric* $\gamma_X\colon TX \to \mathbf{R}^+$ is defined by

$$\gamma_X(z; v) = \sup\bigl\{|df_z(v)| \bigm| f \in \mathrm{Hol}(X, \Delta), f(z) = 0\bigr\},$$

for every $z \in X$ and $v \in T_z X$; as we shall see in a moment, γ_X is always finite.

Roughly speaking, a metric is something to measure length of curves with. Then the suspicious reader may suspect that κ_X and γ_X deserved the name of (pseudo)metrics because they enjoy the following property:

$$\gamma_X(z; \lambda v) = |\lambda|\, \gamma_X(z; v) \qquad \text{and} \qquad \kappa_X(z; \lambda v) = |\lambda|\, \kappa_X(z; v) \tag{2.3.5}$$

for every $z \in X$, $v \in T_z X$ and $\lambda \in \mathbf{C}$, which is the least requirement for a metric. A large part of this section will be devoted to prove that it is actually possible measure length of curves using the Kobayashi metric; but before that we need a few general facts.

The Kobayashi and Carathéodory pseudometrics enjoy properties very similar to the one enjoyed by the corresponding distances, with very similar proofs. We list the statements, leaving the details to the willing reader:

Proposition 2.3.21: (i) *Let $f\colon X \to Y$ be a holomorphic map between two complex manifolds. Then for all $z \in X$ and $v \in T_zX$*

$$\gamma_Y\bigl(f(z); df_z(v)\bigr) \le \gamma_X(z;v) \qquad \text{and} \qquad \kappa_Y\bigl(f(z); df_z(v)\bigr) \le \kappa_X(z;v).$$

(ii) *If γ is an automorphism of a complex manifold X then for every $z \in X$ and $v \in T_zX$*

$$\gamma_X\bigl(\gamma(z); d\gamma_z(v)\bigr) = \gamma_X(z;v) \qquad \text{and} \qquad \kappa_X\bigl(\gamma(z); d\gamma_z(v)\bigr) = \kappa_X(z;v);$$

(iii) *If Y is a submanifold of X then for every $z \in Y$ and $v \in T_zY$*

$$\gamma_X(z;v) \le \gamma_Y(z;v) \qquad \text{and} \qquad \kappa_X(z;v) \le \kappa_Y(z;v).$$

Proposition 2.3.22: (i) *κ_Δ and γ_Δ coincide with the Poincaré metric;*
(ii) *For every complex manifold X we have $\gamma_X \le \kappa_X$. In particular, γ_X is always finite.*

Proposition 2.3.23: *Let X be a complex manifold, and $\eta\colon TX \to \mathbf{R}^+$ a function such that $\eta(z;\lambda v) = |\lambda|\eta(z;v)$ for all $z \in X$, $v \in T_zX$ and $\lambda \in \mathbf{C}$. Then:*
(i) *if $\eta\bigl(\varphi(\zeta); d\varphi_\zeta(\xi)\bigr) \le \kappa_\Delta(\zeta;\xi)$ for all $\varphi \in \mathrm{Hol}(\Delta, X)$, $\zeta \in \Delta$ and $\xi \in \mathbf{C}$, then $\eta \le \kappa_X$;*
(ii) *if $\eta(z;v) \ge \kappa_\Delta\bigl(f(z); df_z(v)\bigr)$ for all $f \in \mathrm{Hol}(X,\Delta)$, $z \in X$ and $v \in T_zX$, then $\eta \ge \gamma_X$.*

Proposition 2.3.24: *Let $\|\cdot\|\colon \mathbf{C}^n \to \mathbf{R}^+$ be a norm on $\mathbf{C}^n$, and B the unit ball for this norm. Then for all $v \in T_0B \cong \mathbf{C}^n$ we have*

$$\gamma_B(0;v) = \kappa_B(0;v) = \|v\|.$$

Corollary 2.3.25: *On B^n, the Bergmann, Carathéodory and Kobayashi metrics coincide.*

Corollary 2.3.26: *For every $v \in \mathbf{C}^n$ we have*

$$\kappa_{\Delta^n}(0;v) = \max_{j=1,\dots,n}\{|v_j|\}.$$

However, there is a new statement:

Proposition 2.3.27: *Let X and Y be two complex manifolds. Then we have*

$$\kappa_{X\times Y}\bigl((z,w);(u,v)\bigr) = \max\{\kappa_X(z;u), \kappa_Y(w;v)\}$$

for every $(z,w) \in X \times Y$ and $(u,v) \in T_{(z,w)}(X\times Y) = T_zX \oplus T_wY$.

Proof: It is easy to see, using the canonical projections, that

$$\kappa_{X\times Y}\bigl((z,w);(u,v)\bigr) \ge \max\{\kappa_X(z;u), \kappa_Y(w;v)\}. \tag{2.3.6}$$

Assume, by contradiction, that (2.3.6) is not an equality. Then we can find $\varphi \in \mathrm{Hol}(\Delta, X)$, $\psi \in \mathrm{Hol}(\Delta, Y)$ and ξ, $\eta \in \mathbf{C}$ such that $\varphi(0) = z$, $\psi(0) = w$, $d\phi_0(\xi) = u$, $d\psi_0(\eta) = v$ and

$$\kappa_{X\times Y}\bigl((z,w);(u,v)\bigr) > \max\{|\xi|, |\eta|\} > \max\{\kappa_X(z;u), \kappa_Y(w;v)\}.$$

On the other hand, applying Proposition 2.3.21.(i) to the map $f \in \mathrm{Hol}(\Delta^2, X \times Y)$ given by $f(\zeta_1,\zeta_2) = \bigl(\varphi(\zeta_1), \psi(\zeta_2)\bigr)$ we find

$$\kappa_{X\times Y}\bigl((z,w);(u,v)\bigr) \le \kappa_{\Delta^2}\bigl((0,0);(\xi,\eta)\bigr) = \max\{|\xi|, |\eta|\},$$

by Corollary 2.3.26, contradiction, **q.e.d.**

Now, the distance associated to a Riemannian metric is obtained as infimum of length of curves. Therefore if we want a similar relation between the Kobayashi pseudodistance and the Kobayashi pseudometric we must first of all give a meaning to the writing

$$\int_a^b \kappa_X\bigl(\sigma(t);\dot\sigma(t)\bigr)\,dt, \tag{2.3.7}$$

where $\sigma\colon[a,b]\to X$ is a piecewise C^1 curve in X.

To do so, we need a technical lemma.

Lemma 2.3.28: *Let X be a complex manifold of dimension n, and $\varphi\in\mathrm{Hol}(\Delta,X)$ such that $\varphi'(0)\ne 0$. Then for every $r<1$ there exist a neighbourhood U_r of $\overline{\Delta_r}\times\{0\}$ in Δ^n and a map $f_r\in\mathrm{Hol}(U_r,X)$ such that $f_r|_{\overline{\Delta_r}\times\{0\}}=\varphi|_{\overline{\Delta_r}}$ and f_r is a biholomorphism in a neighbourhood of* 0.

Proof: The proof of this lemma for a generic manifold is quite complicated, and requires techniques out of the scope of this book (see Royden [1974] or Siu [1976]). Fortunately, we shall need the results of this section only for domains of $\mathbf{C}^n$, where an easy proof is available.

So let D be a domain in $\mathbf{C}^n$, set $v_0=\varphi'(0)$ and let V denote the orthogonal complement of v_0 in $\mathbf{C}^n$. Define $g\colon\Delta\times V\to\mathbf{C}^n$ by

$$\forall\zeta\in\Delta\ \forall w\in V \qquad\qquad g(\zeta,w)=\varphi(\zeta)+w.$$

Clearly, g is holomorphic and $g|_{\Delta\times\{0\}}=\varphi$; moreover, since $dg_{(0,0)}(\xi,w)=\xi v_0+w$, g is a biholomorphism in a neighbourhood of the origin. Now, since $\overline{\Delta_r}\times\{0\}$ is compact and $g(\overline{\Delta_r}\times\{0\})\subset\subset D$, there is a neighbourhood U_r of $\overline{\Delta_r}\times\{0\}$ in Δ^n such that $g(U_r)\subset D$, and $f_r=g|_{U_r}$ is as we need, **q.e.d.**

We are now able to prove a regularity theorem for the Kobayashi pseudometric:

Theorem 2.3.29: *Let X be a complex manifold. Then κ_X is upper semicontinuous.*

Proof: Choose $z_0\in X$, $v_0\in T_{z_0}X$ and $\varepsilon>0$; we must show that there is a neighbourhood $\widetilde V$ of $(z_0;v_0)$ in TX such that

$$\forall(z;v)\in\widetilde V \qquad\qquad \kappa_X(z;v)<\kappa_X(z_0;v_0)+\varepsilon.$$

By definition, there are $\varphi\in\mathrm{Hol}(\Delta,X)$ and $\xi\in\mathbf{C}$ such that $\varphi(0)=z_0$, $d\varphi_0(\xi)=v_0$ and $|\xi|<\kappa_X(z_0;v_0)+\varepsilon/2$. Choose $r_0<1$ such that $|\xi|/r_0$ is still less than $\kappa_X(z_0;v_0)+\varepsilon/2$, and let $U\subset\Delta^n$ and $f\in\mathrm{Hol}(U,X)$ be given by Lemma 2.3.28 applied to φ and r_0; we can clearly take $U=\Delta_{r_0}\times\Delta_\rho^{n-1}$ for a suitable $\rho>0$.

Now, f is a biholomorphism in a neighbourhood of 0, $f(0)=z_0$ and $df_0(\xi e_1)=v_0$, where $e_1=(1,0,\dots,0)$ as usual. Therefore we can find a neighbourhood $\widetilde U$ of $(0;\xi e_1)$ in $TU\cong U\times\mathbf{C}^n$ such that $(f;df)$ is a biholomorphism between $\widetilde U$ and a neighbourhood $\widetilde V$

of $(z_0; v_0)$ in TX. Moreover, since by Proposition 2.3.24 κ_U is continuous, we can also assume that

$$\forall (\zeta; v) \in \widetilde{U} \qquad \kappa_U(\zeta; v) \le \kappa_U(0; \xi e_1) + \varepsilon/2.$$

So take $(z; v) \in \widetilde{V}$ and $(\zeta; v) \in \widetilde{U}$ so that $z = f(\zeta)$ and $v = df_\zeta(v)$. Then

$$\kappa_X(z; v) \le \kappa_U(\zeta; v) \le \kappa_U(0; \xi e_1) + \varepsilon/2 = |\xi|/r_0 + \varepsilon/2 < \kappa_X(z_0; v_0) + \varepsilon,$$

q.e.d.

So (2.3.7) is well-defined; at least, κ_X is integrable. We can also show that (2.3.7) is always finite:

Lemma 2.3.30: *Let X be a complex manifold, fix any hermitian metric h on TX, and let $\|\cdot\|_h : TX \to \mathbf{R}^+$ denote the associated norm. Then for every compact subset K of X there is a constant $c_K > 0$ such that*

$$\forall z \in K \; \forall v \in T_z X \qquad \kappa_X(z; v) \le c_K \|v\|_h.$$

Proof: Fix $z_0 \in X$, and choose a coordinate neighbourhood U of z_0 biholomorphic to B^n for some n. Clearly it suffices to show that c_K exists for $K \subset\subset U$. Then the assertion follows from (2.2.16), Propositions 2.3.25, 2.3.21 and remarking that two hermitian metrics on a compact subset of a complex manifold are always equivalent, **q.e.d.**

Later on we shall need an analogous fact for the Kobayashi distance:

Lemma 2.3.31: *Let X be a complex manifold, and fix a point $z_0 \in X$, a coordinate neighbourhood U of z_0 and a biholomorphism $\psi: U \to B^n$, where n is the (complex) dimension of X. Then for every compact subset K of U there is a constant $c'_K > 0$ such that*

$$\forall z, \, w \in K \qquad k_X(z, w) \le c'_K \|\psi(z) - \psi(w)\|.$$

Proof: Indeed, $k_X(z, w) \le k_{B^n}\big(\psi(z), \psi(w)\big)$ for every z, $w \in U$, and the assertion follows from the explicit form of the Kobayashi distance on B^n, **q.e.d.**

Now, let $\sigma: [a, b] \to X$ be a piecewise C^1 curve in a complex manifold X. Then the *Kobayashi length* $\ell_k(\sigma)$ of σ is given by

$$\ell_k(\sigma) = \int_a^b \kappa_X\big(\sigma(t); \dot\sigma(t)\big)\, dt.$$

By Theorem 2.3.29, $\ell_k(\sigma)$ is well-defined and, by Lemma 2.3.30, it is always finite. Furthermore it does not depend on the parametrization of σ, thanks to (2.3.5). So we can define a pseudodistance $k^i_X : X \times X \to \mathbf{R}^+$ on X, the *integrated form* of κ_X, by

$$\forall z, \, w \in X \qquad k^i_X(z, w) = \inf\{\ell_k(\sigma)\},$$

where the infimum is taken with respect to the set of all piecewise C^1 curves connecting z to w.

k^i_X is constructed starting from κ_X exactly as the distance associated to a Riemannian metric; therefore the often announced main result of this section is

Theorem 2.3.32: *Let X be a complex manifold. Then k_X is the integrated form of κ_X.*

Proof: First of all, we show that $k_X^i \le k_X$. Since k_X^i is a pseudodistance, it suffices to show that $k_X^i \le \delta_X$. Take z_0, $w_0 \in X$. If $\delta_X(z_0, w_0) = +\infty$, there is nothing to prove; otherwise, fix $\varepsilon > 0$ and choose $\varphi \in \mathrm{Hol}(\Delta, X)$ with $\varphi(0) = z_0$ and $\varphi(t_0) = w_0$ for a suitable $t_0 \in [0, 1)$ such that $\omega(0, t_0) < \delta_X(z_0, w_0) + \varepsilon$. Let $\sigma(t) = \varphi(t)$. Then

$$k_X^i(z_0, w_0) \le \int_0^{t_0} \kappa_X\bigl(\sigma(t); \dot\sigma(t)\bigr)\, dt \le \int_0^{t_0} \kappa_\Delta(t; 1)\, dt = \omega(0, t_0) < \delta_X(z_0, w_0) + \varepsilon,$$

and $k_X^i \le k_X$, for ε is arbitrary.

It remains to show that $k_X(z_0, w_0) \le \ell_k(\sigma)$ for every piecewise C^1 curve $\sigma\colon [a, b] \to X$ connecting z_0 to w_0. Let $f\colon [a, b] \to \mathbf{R}^+$ be defined by $f(t) = k_X\bigl(z_0, \sigma(t)\bigr)$. Using Lemma 2.3.31 it is easy to see that f is locally Lipschitz, and so it is differentiable almost everywhere. In particular,

$$k_X(z_0, w_0) = f(b) - f(a) \le \int_a^b |f'(t)|\, dt;$$

hence it suffices to prove that if f is differentiable in $t_0 \in (a, b)$ then

$$|f'(t_0)| \le \kappa_X\bigl(\sigma(t_0); \dot\sigma(t_0)\bigr).$$

Fix $\varepsilon > 0$, and choose $\varphi \in \mathrm{Hol}(\Delta, X)$ and $\xi \in \mathbf{C}$ such that $\varphi(0) = \sigma(t_0)$, $d\varphi_0(\xi) = \dot\sigma(t_0)$ and $|\xi| < \kappa_X\bigl(\sigma(t_0); \dot\sigma(t_0)\bigr) + \varepsilon$. Then if $h \in \mathbf{R}$ is small enough

$$\begin{aligned}|f(t_0 + h) - f(t_0)| &\le k_X\bigl(\sigma(t_0 + h), \sigma(t_0)\bigr) \le k_X\bigl(\sigma(t_0 + h), \varphi(h\xi)\bigr) + k_X\bigl(\varphi(h\xi), \varphi(0)\bigr)\\ &\le k_X\bigl(\sigma(t_0 + h), \varphi(h\xi)\bigr) + \omega(0, h\xi).\end{aligned}$$

Now, since $\varphi(0) = \sigma(t_0)$ and $d\varphi_0(\xi) = \dot\sigma(t_0)$, Lemma 2.3.31 implies that

$$k_X\bigl(\sigma(t_0 + h), \varphi(h\xi)\bigr) = o(|h|).$$

Therefore

$$|f'(t_0)| \le \lim_{h\to 0} \frac{\omega(0, h\xi)}{|h|} = |\xi| < \kappa_X\bigl(\sigma(t_0); \dot\sigma(t_0)\bigr) + \varepsilon,$$

and we are done, **q.e.d.**

Therefore from now on to compute the Kobayashi distance we can measure length of curves, a fact that will be quite crucial a couple of times in this chapter. For the moment, we limit ourselves to some immediate applications. For instance, we can prove a sort of converse of Lemma 2.3.30, showing how to characterize hyperbolic manifolds by means of the Kobayashi metric:

Proposition 2.3.33: *Let X be a complex manifold, fix any hermitian metric h on TX and let $\|\cdot\|_h\colon TX \to \mathbf{R}^+$ denote the associated norm. Then X is hyperbolic iff for every compact subset K of X there is a constant $c''_K > 0$ such that*

$$\forall z \in K\ \forall v \in T_zX \qquad \kappa_X(z;v) \ge c''_K \|v\|_h. \tag{2.3.8}$$

Proof: Assume (2.3.8) holds, and take two distinct points z_0, $w_0 \in X$. Then there is a compact neighbourhood K of z_0 such that $w_0 \notin K$. In particular, every curve connecting z_0 to w_0 must leave K, and so, by Theorem 2.3.32,

$$k_X(z_0, w_0) \ge c''_K\, d_h(z_0, \partial K) > 0,$$

where d_h is the distance induced by the hermitian metric h, and X is hyperbolic.

Conversely, assume X hyperbolic. Take $z_0 \in X$, fix a coordinate neighbourhood U of z_0 biholomorphic to B^n for some n, and choose $\varepsilon > 0$ so that $B_k(z_0, 2\varepsilon) \subset\subset U$. In particular, for every $\varphi \in \mathrm{Hol}(\Delta, X)$ such that $\varphi(0) \in B_k(z_0,\varepsilon)$ we have $\varphi\bigl(\Delta_{\tanh\varepsilon}\bigr) \subset B_k(z_0, 2\varepsilon)$.

Take $z \in B_k(z_0,\varepsilon)$, $v \in T_zX$, $\varphi \in \mathrm{Hol}(\Delta, X)$ and $\xi \in \mathbf{C}$ so that $\varphi(0) = z$ and $d\varphi_0(\xi) = v$. Then, setting $\psi(\zeta) = \varphi\bigl((\tanh\varepsilon)\zeta\bigr)$, we have $\psi \in \mathrm{Hol}(\Delta, U)$, $\psi(0) = z$ and $d\psi_0(\xi) = (\tanh\varepsilon)v$. In other words, for every $z \in B_k(z_0,\varepsilon)$ and $v \in T_zX$ we have

$$(\tanh\varepsilon)\, \kappa_U(z;v) \le \kappa_X(z;v).$$

But (2.3.8) is clearly true for B^n (and thus for U), and hence the assertion follows as in Lemma 2.3.30, **q.e.d.**

In particular, then, if X is hyperbolic then $\kappa_X(z;v) > 0$ for every $z \in X$ and $v \in T_zX$ with $v \ne 0$; in this case we shall speak of Kobayashi *metric,* instead of pseudometric.

We end this section showing how tautness reflects on the Kobayashi metric:

Proposition 2.3.34: *Let X be a taut manifold. Then κ_X is continuous.*

Proof: By Theorem 2.3.29, it suffices to show that κ_X is lower semicontinuous.

First of all, we claim that for every $z \in X$ and $v \in T_zX$ there are $\varphi \in \mathrm{Hol}(\Delta, X)$ and $\xi \in \mathbf{C}$ such that $\varphi(0) = z$, $d\varphi_0(\xi) = v$ and $|\xi| = \kappa_X(z;v)$. Indeed, take sequences $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, X)$ and $\{\xi_\nu\} \subset \mathbf{C}$ so that $\varphi_\nu(0) = z$, $d(\varphi_\nu)_0(\xi_\nu) = v$ for all $\nu \in \mathbf{N}$, and $|\xi_\nu| \to \kappa_X(z;v)$. Up to a subsequence, we can assume $\varphi_\nu \to \varphi \in \mathrm{Hol}(\Delta, X)$ and $\xi_\nu \to \xi \in \mathbf{C}$; φ and ξ are clearly as desired.

Now assume, by contradiction, that κ_X is not lower semicontinuous. Then there are $z_0 \in X$, $v_0 \in T_{z_0}X$, $\varepsilon > 0$ and a sequence $\{(z_\nu; v_\nu)\} \subset TX$ converging to $(z_0; v_0)$ such that $\kappa_X(z_\nu; v_\nu) \le \kappa_X(z_0; v_0) - \varepsilon$ for all $\nu \in \mathbf{N}$. Choose $\varphi_\nu \in \mathrm{Hol}(\Delta, X)$ and $\xi_\nu \in \mathbf{C}$ such that $\varphi_\nu(0) = z_\nu$, $d(\varphi_\nu)_0(\xi_\nu) = v_\nu$ and $|\xi_\nu| = \kappa_X(z_\nu; v_\nu)$. Then $\{\xi_\nu\} \subset \mathbf{C}$ is bounded and, up to a subsequence, we can assume $\varphi_\nu \to \varphi \in \mathrm{Hol}(\Delta, X)$ and $\xi_\nu \to \xi \in \mathbf{C}$. Hence $\varphi(0) = z_0$, $d\varphi_0(\xi) = v_0$ and

$$|\xi| \le \kappa_X(z_0; v_0) - \varepsilon < \kappa_X(z_0; v_0),$$

impossible, **q.e.d.**

2.3.3 *Invariant volume forms*

There is still another construction related to the invariant distances: the invariant pseudovolume forms. Using them we shall be able to construct invariant measures on complex manifolds, an useful tool in studying holomorphic maps between compact hyperbolic manifolds.

We begin recalling a few facts of linear algebra. Let V be a complex vector space of dimension n; we shall denote by V^* its dual space, and by $\overline{V}^*$ its antidual space, i.e., the space of all anti-linear forms on V, that is of functions $f\colon V \to \mathbf{C}$ such that

$$\forall u, v \in V \ \forall \lambda, \mu \in \mathbf{C} \qquad f(\lambda u + \mu v) = \overline{\lambda} f(u) + \overline{\mu} f(v).$$

$\bigwedge^n V^*$ will denote the n-th exterior power of V^*, and $\bigwedge^{n,n} V^* = (\bigwedge^n V^*) \wedge (\bigwedge^n \overline{V}^*)$ the space of (n,n)-forms on V.

If $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a basis of V, we shall denote by $\mathcal{V}^* = \{v_1^*, \ldots, v_n^*\}$ the dual basis of V^*, and by $\overline{\mathcal{V}}^* = \{\bar{v}_1^*, \ldots, \bar{v}_n^*\}$ the antidual basis of $\overline{V}^*$ (still defined by $\bar{v}_h^*(v_k) = \delta_{hk}$, where δ_{hk} is the Kronecker delta). If $\mathcal{U} = \{u_1, \ldots, u_n\}$ is another basis of V and $A \in \mathbf{GL}(n, \mathbf{C})$ is the transition matrix, that is the invertible matrix $A = (a_{hk})$ such that

$$\forall h = 1, \ldots, n \qquad\qquad u_h = \sum_{\mu=1}^{n} a_{h\mu} v_\mu,$$

then we shall write $\mathcal{U} = A\mathcal{V}$. It is easy to check that, using these notations, $\mathcal{U}^* = {}^t A^{-1} \mathcal{V}^*$, and $\overline{\mathcal{U}}^* = {}^t \overline{A}^{-1} \overline{\mathcal{V}}^*$.

Now, $\bigwedge^{n,n} V^*$ has complex dimension 1; therefore if $\mathcal{V}$ is a basis of V then the (n,n)-form $v_1^* \wedge \bar{v}_1^* \wedge \cdots \wedge v_n^* \wedge \bar{v}_n^*$ is a generator of $\bigwedge^{n,n} V^*$. If $\mathcal{U}$ is another basis of V, and $A \in \mathbf{GL}(n, \mathbf{C})$ is the transition matrix, then

$$u_1^* \wedge \bar{u}_1^* \wedge \cdots \wedge u_n^* \wedge \bar{u}_n^* = |\det A|^{-2} v_1^* \wedge \bar{v}_1^* \wedge \cdots \wedge v_n^* \wedge \bar{v}_n^*. \tag{2.3.9}$$

This equation has an important consequence. Set $\eta_{\mathcal{V}} = 2^{-n}\, i^n\, v_1^* \wedge \bar{v}_1^* \wedge \cdots \wedge v_n^* \wedge \bar{v}_n^*$ — the factor i^n makes $\eta_{\mathcal{V}}$ *real*, i.e., such that $\overline{\eta}_{\mathcal{V}} = \eta_{\mathcal{V}}$; the factor 2^{-n} is due to (2.3.11) —; then $\eta_{\mathcal{V}}$ is a generator of the real one-dimensional vector space $\bigwedge_{\mathbf{R}}^{n,n} V^*$ of real (n,n)-forms on V. Now, stating that $\eta_{\mathcal{V}}$ is positive, we fix an ordering on $\bigwedge_{\mathbf{R}}^{n,n} V^*$; well, by (2.3.9) this ordering is *natural*, i.e., does not depend on the chosen basis of V. In particular, we have a well-defined notion of infimum and supremum of a family of real (n,n)-forms — allowing the result to be $\pm\infty$, of course.

There is another consequence of (2.3.9) worth mentioning. Let $T\colon V \to \overline{V}^*$ be a linear operator. If we fix a basis $\mathcal{V}$ of V, and the antidual basis $\overline{\mathcal{V}}^*$ on $\overline{V}^*$, we can compute the determinant $\det_{\mathcal{V}} T$ of T with respect to these two basis. If $\mathcal{U} = A\mathcal{V}$ is another basis of V, it is easy to check that

$$\det\nolimits_{\mathcal{U}} T = |\det A|^2 \det\nolimits_{\mathcal{V}} T.$$

In particular, by (2.3.9), this implies that the (n,n)-form $\Theta_T = (\det_{\mathcal{V}} T)\eta_{\mathcal{V}}$ is well-defined, being independent of the chosen basis of V. Θ_T is the *volume form* associated to T.

Linear operators from V in $\overline{V}^*$ arise naturally from hermitian products $h: V \times V \to \mathbf{C}$, setting $T_h(v) = h(v, \cdot)$; in this case the (n,n)-form $\Theta_{T_h} = \Theta_h$ will be called the *volume form* associated to h. Note that, by definition of hermitian product, $T_h(u)(v) = \overline{T_h(v)(u)}$ for all u, $v \in V$; this implies that $\det_{\mathcal{V}} T_h \in \mathbf{R}$ for every basis $\mathcal{V}$ of V, and so Θ_h belongs to $\bigwedge_{\mathbf{R}}^{n,n} V^*$. Furthermore, if h is positive semidefinite it immediately follows that Θ_h is non-negative.

This was the local picture; the next step is to globalize it. Let X be a complex manifold, and h a hermitian metric on it. If we denote by T^*X the cotangent bundle and by $\overline{T}^*X$ the anticotangent bundle of X, then h gives rise to a bundle map $T_h: TX \to \overline{T}^*X$. By the previous arguments, then, h defines a *volume form* Θ_h on X, that is a positive (n,n)-form on X or, in other words, a positive section of the bundle $\bigwedge_{\mathbf{R}}^{n,n} T^*X$, which has fiber $\bigwedge_{\mathbf{R}}^{n,n}(T_z^*X)$ at the point $z \in X$. If $\{z_1, \ldots, z_n\}$ is a local chart centered about $z_0 \in X$, then $\Theta_h(z_0)$ is given by

$$\Theta_h(z_0) = \det\bigl(h_{i\bar{j}}(z_0)\bigr)\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n, \tag{2.3.10}$$

where $(h_{i\bar{j}})$ is the positive definite hermitian matrix representing h in these local coordinates.

For instance, if X is $\mathbf{C}^n$ and h is the euclidean metric we find

$$\Theta_h(z_0) \equiv \Theta = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n, \tag{2.3.11}$$

for every $z_0 \in \mathbf{C}^n$, where we set $z_j = x_j + iy_j$ for $j = 1, \ldots, n$ (this formula is the rationale under the factor 2^{-n} in the previous definition of Θ_h).

Another example: if X is B^n and h is the Bergmann metric then for every $z \in B^n$ we have

$$\Theta_h(z) = \frac{1}{(1 - \|z\|^2)^{n+1}}\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n = \frac{1}{(1 - \|z\|^2)^{n+1}}\Theta. \tag{2.3.12}$$

Now we are almost ready to define the invariant pseudovolume forms. Let X be a complex manifold of (complex) dimension n, $z_0 \in X$, and take $f \in \mathrm{Hol}(X, B^n)$ with $f(z_0) = 0$. Then $(f^*\Theta)(z_0)$ belongs to $\bigwedge_{\mathbf{R}}^{n,n}(T_{z_0}^*X)$; moreover, if $\{z_1, \ldots, z_n\}$ is a local chart about z_0, then

$$(f^*\Theta)(z_0) = |\det\, df_{z_0}|^2 \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n, \tag{2.3.13}$$

where the determinant is computed with respect to the canonical basis of $\mathbf{C}^n \cong T_0B^n$ and to the basis $\{dz_1, \ldots, dz_n\}$ of $T_{z_0}^*X$.

Analogously, take $f \in \mathrm{Hol}(B^n, X)$ with $f(0) = z_0$ and df_0 invertible; then the (n,n)-form $\bigl((f^{-1})^*\Theta\bigr)(z_0)$ is a well-defined element of $\bigwedge_{\mathbf{R}}^{n,n}(T_{z_0}^*X)$, given locally by

$$\bigl((f^{-1})^*\Theta\bigr)(z_0) = |\det\, df_0|^{-2}\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \tag{2.3.14}$$

Since $\bigwedge_{\mathbf{R}}^{n,n} X$ has a natural ordering (which is essentially induced by the canonical orientation of X), we can finally define the *Carathéodory pseudovolume form* $\tilde{\gamma}_X$ by

$$\forall z_0 \in X \quad \tilde{\gamma}_X(z_0) = \sup\bigl\{(f^*\Theta)(z_0) \bigm| f \in \mathrm{Hol}(X, B^n),\, f(z_0) = 0\bigr\},$$

and the *Kobayashi pseudovolume form* $\tilde{\kappa}_X$ by

$$\tilde{\kappa}_X(z_0) = \inf\bigl\{\bigl((f^{-1})^*\Theta\bigr)(z_0) \bigm| f \in \mathrm{Hol}(B^n, X),\, f(0) = z_0,\, df_0 \text{ invertible}\bigr\}$$

for all $z_0 \in X$. $\tilde{\gamma}_X$ and $\tilde{\kappa}_X$ are (not necessarily continuous) non-negative (n,n)-forms on X, thanks to (2.3.13) and (2.3.14), provided that $\tilde{\gamma}_X$ is finite. But this is proved in the usual way:

Proposition 2.3.35: *Let $f\colon X \to Y$ be a holomorphic map between two complex manifolds of the same dimension. Then $f^*\tilde{\gamma}_Y \le \tilde{\gamma}_X$ and $f^*\tilde{\kappa}_Y \le \tilde{\kappa}_X$.*

Proposition 2.3.36: (i) *$\tilde{\gamma}_{B^n}$ and $\tilde{\kappa}_{B^n}$ coincide with the volume form of the Bergmann metric;*
(ii) *For any complex manifold X we have $\tilde{\gamma}_X \le \tilde{\kappa}_X$. In particular, $\tilde{\gamma}_X$ is always finite.*

Proof: Let h be the Bergmann metric on B^n. Since both $\tilde{\kappa}_{B^n}$ (by Proposition 2.3.35) and Θ_h (by Corollary 2.2.3) are invariant under $\mathrm{Aut}(B^n)$, to show that $\tilde{\kappa}_{B^n} = \Theta_h$ it suffices to prove that $\tilde{\kappa}_{B^n}(0) = \Theta_h(0) = \Theta$. Clearly, $\tilde{\kappa}_{B^n}(0) \le \Theta$. On the other hand, if $f \in \mathrm{Hol}(B^n, B^n)$ is such that $f(0) = 0$ and $\det df_0 \ne 0$, we have $|\det df_0|^{-2} \ge 1$ by Theorem 2.1.21.(ii), and so $\tilde{\kappa}_{B^n}(0) \ge \Theta$, by (2.3.14).

In particular, if X is any complex manifold, for every $z_0 \in X$ and $f \in \mathrm{Hol}(X, B^n)$ with $f(z_0) = 0$ we have

$$(f^*\Theta)_{z_0} \le \tilde{\kappa}_X(z_0),$$

and (ii) follows. Finally, (ii) implies

$$\Theta_h \le \tilde{\gamma}_{B^n} \le \tilde{\kappa}_{B^n} = \Theta_h,$$

and we are done, **q.e.d.**

Having a non-negative (n,n)-form on a complex manifold X, the first thing we would like to do is to integrate it so to get a measure on X. As usual, we need some regularity information:

Proposition 2.3.37: *Let X be a complex manifold. Then*
(i) *$\tilde{\kappa}_X$ is upper semicontinuous;*
(ii) *if X is taut, $\tilde{\kappa}_X$ is continuous.*

Proof: (i) Take $z_0 \in X$ and fix a coordinate neighbourhood U of z_0 with local coordinates $\{z_1, \dots, z_n\}$ centered about z_0, and set

$$\forall z \in U \qquad \eta(z) = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \tag{2.3.15}$$

Clearly, there is a function $K: U \to \mathbf{R}^+$ such that $\tilde{\kappa}_X(z) = K(z)\,\eta(z)$ for all $z \in U$; we must show that for every $\varepsilon > 0$ there is a neighbourhood $V \subset U$ of z_0 such that

$$\forall z \in V \qquad\qquad K(z) < K(z_0) + \varepsilon.$$

Fix $\varepsilon > 0$. By definition, there is a map $f \in \mathrm{Hol}(B^n, X)$ with $f(0) = z_0$ which is a biholomorphism in a neighbourhood of 0 and such that $|\det df_0|^{-2} < K(z_0) + \varepsilon/2$. Then there is a neighbourhood $\widetilde{V}$ of 0 such that $f|_{\widetilde{V}}$ is a biholomorphism between $\widetilde{V}$ and $V = f(\widetilde{V})$ and moreover

$$\forall w \in \widetilde{V} \qquad |\det d(f \circ \gamma_w)_0|^{-2} = \frac{|\det df_w|^{-2}}{(1 - \|w\|^2)^{n+1}} < K(z_0) + \varepsilon,$$

where γ_w is the automorphism of B^n given by (2.2.1), and we used Corollary 2.2.3.

Then, since $f \circ \gamma_w(0) = f(w)$, if for every $z \in V$ we take $w \in \widetilde{V}$ so that $f(w) = z$ we get

$$K(z) \le |\det d(f \circ \gamma_w)_0|^{-2} < K(z_0) + \varepsilon,$$

and we are done.

(ii) By part (i), it suffices to show that $\tilde{\kappa}_X$ is lower semicontinuous. Assume, by contradiction, there is a $z_0 \in X$ where $\tilde{\kappa}_X$ is not lower semicontinuous. Then, retaining the notations introduced in the proof of part (i), there are $\varepsilon > 0$ and a sequence $\{z_\nu\} \subset X$ converging to z_0 such that $K(z_\nu) \le K(z_0) - \varepsilon$ for all $\nu \in \mathbf{N}$. Choose $f_\nu \in \mathrm{Hol}(B^n, X)$ such that $f_\nu(0) = z_\nu$ and $|\det d(f_\nu)_0|^{-2} < K(z_\nu) + \varepsilon/2$ for all $\nu \in \mathbf{N}$. Up to a subsequence, we can assume $f_\nu \to f \in \mathrm{Hol}(B^n, X)$, for X is taut; hence $f(0) = z_0$ and

$$|\det df_0|^{-2} \le K(z_0) - \varepsilon/2 < K(z_0),$$

impossible, **q.e.d.**

In particular, then, if X is a complex manifold, $\tilde{\kappa}_X$ defines by integration a nonnegative Borel measure on X (note that $\tilde{\kappa}_X$ is locally bounded: it suffices to compare it with the volume form of a coordinate neighbourhood biholomorphic to B^n) contracted by holomorphic maps. In particular, the *Kobayashi volume* $\mathrm{vol}_k(X)$ of a complex manifold X is given by

$$\mathrm{vol}_k(X) = \int_X \tilde{\kappa}_X \in [0, +\infty].$$

Clearly, a natural problem now is when $\tilde{\kappa}_X$ is everywhere positive. The clever reader will immediately suspect that this is the case when X is hyperbolic; our next aim is to confirm this brilliant suspect.

A complex manifold X is *measure hyperbolic* if $\tilde{\kappa}_X > 0$; *strongly measure hyperbolic* if there is a hermitian metric h on X such that for every $z_0 \in X$ there are a constant $c > 0$ and a neighbourhood U of z_0 such that

$$\tilde{\kappa}_X(z) \ge c\,\Theta_h(z)$$

for all $z \in U$.

Clearly, a strongly measure hyperbolic manifold is measure hyperbolic; our idea is that every hyperbolic manifold is strongly measure hyperbolic. We need two lemmas:

Lemma 2.3.38: *Let X be a hyperbolic manifold, $U \subset X$ an open set and H a compact subset of U. Then there is $r \in (0,1)$ such that, if we set $rB^n = \{z \in \mathbf{C}^n \mid \|z\| < r\}$, then for every $n \in \mathbf{N}$ and for every $f \in \mathrm{Hol}(B^n, X)$ such that $f(0) \in H$ we have $f(rB^n) \subset U$.*

Proof: Let $a = k_X(H, X \setminus U) > 0$. If $z \in B^n$ is such that $k_{B^n}(0,z) < a$ we have

$$k_X\big(f(0), f(z)\big) \le k_{B^n}(0,z) < a,$$

and so $f(z) \in U$. Therefore $r = \tanh a$ will do, **q.e.d.**

Lemma 2.3.39: *Let X be a hyperbolic manifold, $U \subset X$ an open set and H a compact subset of U. Then there exists $c > 0$ such that*

$$\tilde{\kappa}_U|_H \le c\, \tilde{\kappa}_X|_H. \tag{2.3.16}$$

Proof: Clearly, we can assume U is a coordinate neighbourhood of a point $z_0 \in X$, and $H = \{z_0\}$. Let $\{z_1, \dots, z_n\}$ be local coordinates in U centered about z_0, and for every $z \in U$ define $\eta(z) \in \bigwedge_{\mathbf{R}}^{n,n}(T_z^* X)$ as in (2.3.15), and K_X, $K_U \colon U \to \mathbf{R}^+$ by $\tilde{\kappa}_X(z) = K_X(z)\,\eta(z)$ and $\tilde{\kappa}_U(z) = K_U(z)\,\eta(z)$.

Choose $\varepsilon > 0$. Then there is $f \in \mathrm{Hol}(B^n, X)$ such that $f(0) = z_0$, and

$$|\det df_0|^{-2} < K_X(z_0) + \varepsilon.$$

Let $r > 0$ be given by Lemma 2.3.38, and define $g \in \mathrm{Hol}(B^n, X)$ by $g(z) = f(rz)$. Then $g(B^n) \subset U$ and

$$K_U(z_0) \le |\det dg_0|^{-2} = \frac{1}{r^{2n}} |\det df_0|^{-2} < \frac{1}{r^{2n}}\big(K_X(z_0) + \varepsilon\big).$$

Since $\varepsilon > 0$ is arbitrary, (2.3.16) follows, **q.e.d.**

This is what we need for

Theorem 2.3.40: *Every hyperbolic manifold X is strongly measure hyperbolic.*

Proof: Fix a hermitian metric h on X, and let $z_0 \in X$. Choose two neighbourhoods $V \subset\subset U$ of z_0 such that U is biholomorphic to B^n, where n is the complex dimension of X. By Lemma 2.3.39 there is $c > 0$ such that

$$\tilde{\kappa}_X|_V \ge c^{-1} \tilde{\kappa}_U|_V.$$

Now, by Proposition 2.3.36 $\tilde{\kappa}_U$ is the volume form of a hermitian metric on U; therefore it is clear that there is $c_1 > 0$ such that

$$\tilde{\kappa}_U|_V \ge c_1 \Theta_h|_V,$$

and the assertion follows, **q.e.d.**

We end this section with two applications of these methods. The first one is the announced theorem on holomorphic maps between compact hyperbolic manifolds:

Theorem 2.3.41: *Let X and X' be two compact hyperbolic manifolds of the same dimension. Then*
(i) *if* $\mathrm{vol}_k(X) < \mathrm{vol}_k(X')$ *then every holomorphic map* $f\colon X \to X'$ *is everywhere degenerate;*
(ii) *if* $\mathrm{vol}_k(X) < 2\,\mathrm{vol}_k(X')$ *then every holomorphic map* $f\colon X \to X'$ *is either everywhere degenerate or biholomorphic.*

Proof: Since X and X' are compact and hyperbolic, they are complete hyperbolic (and thus taut), and their Kobayashi volume is finite and positive (by Proposition 2.3.37.(ii) and Theorem 2.3.40). Furthermore, being $\tilde\kappa_X$ and $\tilde\kappa_{X'}$ continuous and never zero, the topological degree of f is given by

$$\deg f = \frac{1}{\mathrm{vol}_k(X')} \int\limits_X f^*\tilde\kappa_{X'}. \tag{2.3.17}$$

Therefore $f^*\tilde\kappa_{X'} \le \tilde\kappa_X$ implies

$$\deg f \le \frac{\mathrm{vol}_k(X)}{\mathrm{vol}_k(X')}; \tag{2.3.18}$$

note that $\deg f \ge 0$ by (2.3.17), because $f^*\tilde\kappa_{X'} \ge 0$.

Now, $\deg f = 0$ iff $f^*\tilde\kappa_{X'} \equiv 0$, that is, since X' is strongly measure hyperbolic, iff f is everywhere degenerate. In particular, (2.3.18) implies (i). If $\mathrm{vol}_k(X) < 2\,\mathrm{vol}_k(X')$, then either $\deg f = 0$ or $\deg f = 1$. In the first case, f is everywhere degenerate. In the second case,

$$\mathrm{vol}_k(X) = \int\limits_X f^*\tilde\kappa_{X'}$$

together with $f^*\tilde\kappa_{X'} \le \tilde\kappa_X$ imply

$$f^*\tilde\kappa_{X'} = \tilde\kappa_X. \tag{2.3.19}$$

It remains to show that (2.3.19) implies f is a biholomorphism.

First of all, f is injective. In fact, if there are $z \ne w \in X$ such that $f(z) = f(w)$, then we can find two disjoint open subsets U and V of X such that $f(U) = f(V)$. But then if, by a slight abuse of notation, we denote by $\tilde\kappa_X$ the measure induced by the Kobayashi volume form, we have

$$\tilde\kappa_X(U\cup V) = \tilde\kappa_X(U) + \tilde\kappa_X(V) \ge \tilde\kappa_{X'}\bigl(f(U)\bigr) + \tilde\kappa_X(V) > \tilde\kappa_{X'}\bigl(f(U)\bigr) = \tilde\kappa_{X'}\bigl(f(U\cup V)\bigr),$$

contradiction.

Finally, f is also surjective. Indeed, suppose not. Then, since X is compact, $X'\setminus f(X)$ is open, and so

$$\mathrm{vol}_k(X') = \tilde\kappa_{X'}\bigl(f(X)\bigr) + \tilde\kappa_{X'}\bigl(X'\setminus f(X)\bigr) > \tilde\kappa_{X'}\bigl(f(X)\bigr) = \mathrm{vol}_k(X),$$

impossible. Hence f is bijective, and thus, by Osgood's theorem, a biholomorphism, **q.e.d.**

The second application is a characterization of B^n we shall need in chapter 2.6. We begin with a general lemma:

Lemma 2.3.42: *Let X, Y be complex manifolds of dimension n, and let $\{f_\nu\} \subset \mathrm{Hol}(X,Y)$ be such that there are $z_0 \in X$ and $w_0 \in Y$ so that $f_\nu(z_0) = w_0$ for all $\nu \in \mathbf{N}$. Furthermore, suppose that there is $g \in \mathrm{Hol}(Y,X)$ such that $\{g \circ f_\nu\}$ converges to $\varphi \in \mathrm{Aut}(X)$. Then g is a biholomorphism.*

Proof: By Osgood's theorem, it suffices to show that g is bijective. First of all, g is surjective. In fact, fix $z \in X$, and choose a neighbourhood U of $z' = \varphi^{-1}(z)$ relatively compact in X. Since $g \circ f_\nu \to \varphi$ uniformly on U, for ν large enough $g \circ f_\nu|_U$ is a biholomorphism with its image. Then we can apply Lemma 2.1.19, concluding that $z = \varphi(z') \in g \circ f_\nu(U)$ for ν large enough, and so $z \in g(Y)$.

It remains to show that g is injective. Fix $\tilde{w} \in Y$; we claim that there exists a compact set $K \subset X$ such that $\tilde{w} \in f_\nu(K)$ for all ν large enough. To prove this, let σ be a curve in Y joining w_0 to $\tilde{w}$, and set $\tau = g \circ \sigma$, $\tilde{z}_0 = g(w_0) = \varphi(z_0)$ and $\tilde{z} = g(\tilde{w})$. Thus τ connects $\tilde{z}_0$ to $\tilde{z}$ in X. Let W be an open neighbourhood of τ (we are identifying a curve and its image) which is relatively compact in X, and set $V = \varphi^{-1}(W)$; note that $z_0 \in V$. Then $K = \overline{V}$ will do.

Indeed, suppose not. Since $g \circ f_\nu \to \varphi$ uniformly on a neighbourhood U of $\overline{V}$, we can choose ν_0 so large that if $\nu \geq \nu_0$ then $g \circ f_\nu$ is a biholomorphism of U with $g\big(f_\nu(U)\big)$ and, moreover, τ is a compact subset of $g\big(f_\nu(V)\big)$ (this is possible for $\varphi^{-1} \circ \tau$ is a compact subset of V). In particular, $g|_{f_\nu(U)}$ is a biholomorphism between $f_\nu(U)$ and $g\big(f_\nu(U)\big)$.

By our assumption, there is $\nu \geq \nu_0$ such that $\tilde{w} \notin f_\nu(V)$. However, $f_\nu(z_0) = w_0$; therefore $\sigma \cap f_\nu(V)$ does not have compact closure in $f_\nu(V)$. But the biholomorphic map $g|_{f_\nu(U)}$ sends $\sigma \cap f_\nu(V)$ into a subset of τ, which does have compact closure in $g\big(f_\nu(V)\big)$, contradiction.

Now we can prove that g is injective. Take $w_1, w_2 \in Y$ with $w_1 \neq w_2$ and assume, by contradiction, that $g(w_1) = g(w_2)$. There is a compact subset K of X such that $w_1, w_2 \in f_\nu(K)$ for all ν large enough; choose $z_1^\nu, z_2^\nu \in K$ so that $f_\nu(z_j^\nu) = w_j$ for $j = 1, 2$; clearly $z_1^\nu \neq z_2^\nu$. Up to a subsequence, we can assume $z_1^\nu \to z_1 \in K$ and $z_2^\nu \to z_2 \in K$ as $\nu \to +\infty$. Since $g \circ f_\nu(z_1^\nu) = g \circ f_\nu(z_2^\nu)$ for all ν, and $g \circ f_\nu \to \varphi$, it follows that $z_1 = z_2$. But now, $g \circ f_\nu$ must be injective in a neighbourhood of $z_1 = z_2$ for ν large enough; it follows that
$$g(w_1) = g \circ f_\nu(z_1^\nu) \neq g \circ f_\nu(z_2^\nu) = g(w_2)$$
for any ν sufficiently large, contradiction, **q.e.d.**

Then

Theorem 2.3.43: *Let X be a complex manifold of dimension n. Suppose that for some $z_0 \in X$ we have $\tilde{\kappa}_X(z_0) = \tilde{\gamma}_X(z_0) \neq 0$. Then X is biholomorphic to B^n.*

Proof: By definition, there exist sequences $\{f_\nu\} \subset \mathrm{Hol}(B^n, X)$ and $\{g_\nu\} \subset \mathrm{Hol}(X, B^n)$ such that $f_\nu(0) = z_0$, $g_\nu(z_0) = 0$ for all $\nu \in \mathbf{N}$ and
$$\big((f_\nu^{-1})^*\Theta\big)_0 \to \tilde{\kappa}_X(z_0),$$
$$(g_\nu^*\Theta)_{z_0} \to \tilde{\gamma}_X(z_0).$$

Since B^n is taut, up to a subsequence we can assume that $\{g_\nu\}$ converges to a holomorphic map $g: X \to B^n$; clearly, $(g^*\Theta)_{z_0} = \tilde{\gamma}_X(z_0)$. Then the sequence $\{g \circ f_\nu\}$ is such that $g\bigl(f_\nu(0)\bigr) = 0$ for all $\nu \in \mathbf{N}$ and $|\det d(g \circ f_\nu)_0| \to 1$ as $\nu \to +\infty$. Up to a subsequence, we can assume that $\{g \circ f_\nu\}$ converges to $\varphi \in \mathrm{Hol}(B^n, B^n)$ such that $\varphi(0) = 0$ and $|\det d\varphi_0| = 1$. But then Theorem 2.1.21 implies that $\varphi \in \mathrm{Aut}(B^n)$ and hence, by Lemma 2.3.42, g is a biholomorphism, **q.e.d.**

2.3.4 *The Kobayashi distance in convex domains*

In the following chapters we shall be often concerned with the study of convex domains of $\mathbf{C}^n$, mainly because in these domains the Kobayashi distance is particularly well behaved, as we shall see in detail in chapter 2.6. In this short section we present three propositions, giving a first idea of the characteristic features of the Kobayashi distance in convex domains.

First of all, in convex domains the definition of Kobayashi distance can be considerably simplified:

Proposition 2.3.44: *Let $D \subset\subset \mathbf{C}^n$ be a bounded convex domain. Then $\delta_D = k_D$.*

Proof: First of all, note that $\delta_D(z,w) < +\infty$ for all z, $w \in D$. Indeed, let

$$\Omega = \{\lambda \in \mathbf{C} \mid (1-\lambda)z + \lambda w \in D\}.$$

Since D is convex, Ω is a convex subset of $\mathbf{C}$ containing 0 and 1. Let $\phi: \Delta \to \Omega$ be a biholomorphism such that $\phi(0) = 0$; then the map $\varphi: \Delta \to D$ given by

$$\varphi(\zeta) = \bigl(1 - \phi(\zeta)\bigr)z + \phi(\zeta)w$$

is such that z, $w \in \varphi(\Delta)$.

Now, if z, $w \in D$ are distinct, then $\delta_D(z,w) \geq k_D(z,w) > 0$, by Proposition 2.3.14; hence it suffices to show that δ_D satisfies the triangular inequality — cf. (2.3.4). Take z_1, z_2, $z_3 \in D$ and fix $\varepsilon > 0$. Then there are φ_1, $\varphi_2 \in \mathrm{Hol}(\Delta, D)$ and ζ_1, $\zeta_2 \in \Delta$ such that $\varphi_1(0) = z_1$, $\varphi_1(\zeta_1) = \varphi_2(\zeta_1) = z_2$, $\varphi_2(\zeta_2) = z_3$ and

$$\begin{aligned} \omega(0, \zeta_1) &< \delta_D(z_1, z_2) + \varepsilon, \\ \omega(\zeta_1, \zeta_2) &< \delta_D(z_2, z_3) + \varepsilon. \end{aligned}$$

Moreover, we can assume that ζ_1 and ζ_2 are real, and that $\zeta_2 > \zeta_1 > 0$. Furthermore, up to replacing φ_j by φ_j^r defined by $\varphi_j^r(\zeta) = \varphi_j(r\zeta)$ for r close enough to 1, we can also assume that φ_j is defined and continuous on $\overline{\Delta}$ (and this for $j = 1, 2$).

Let $\lambda: \mathbf{C} \setminus \{\zeta_1, \zeta_1^{-1}\} \to \mathbf{C}$ be given by

$$\lambda(\zeta) = \frac{(\zeta - \zeta_2)(\zeta - \zeta_2^{-1})}{(\zeta - \zeta_1)(\zeta - \zeta_1^{-1})}.$$

λ is meromorphic in $\mathbf{C}$ and in a neighbourhood of $\overline{\Delta}$ the only pole is the simple pole at ζ_1. Moreover, $\lambda(0) = 1$, $\lambda(\zeta_2) = 0$ and $\lambda(\partial\Delta) \subset [0,1]$. Then define $\psi: \overline{\Delta} \to \mathbf{C}^n$ by

$$\psi(\zeta) = \lambda(\zeta)\varphi_1(\zeta) + \bigl(1 - \lambda(\zeta)\bigr)\varphi_2(\zeta).$$

ψ is holomorphic on Δ — for $\varphi_1(\zeta_1) = \varphi_2(\zeta_1)$ —, $\psi(0) = z_1$, $\psi(\zeta_2) = z_3$ and $\psi(\partial\Delta) \subset \overline{D}$; hence $\psi(\Delta) \subset D$. Indeed, otherwise there would be $\zeta_0 \in \Delta$ such that $\psi(\zeta_0) = x_0 \in \partial D$. Let $\lambda: \mathbf{C}^n \to \mathbf{C}$ be the weak peak function for D at x_0 defined in the proof of Corollary 2.1.11. Then we would have $|\lambda\circ\psi| \le 1$ on ∂D and $|\lambda\circ\psi(\zeta_0)| = 1$; thus, by the maximum principle, $|\lambda \circ \psi| \equiv 1$, i.e., $\psi(\Delta) \subset \partial D$, whereas $\psi(0) \in D$, contradiction.

In particular, then,

$$\delta_D(z_1, z_3) \le \omega(0, \zeta_2) = \omega(0, \zeta_1) + \omega(\zeta_1, \zeta_2) \le \delta_D(z_1, z_2) + \delta_D(z_2, z_3) + 2\varepsilon,$$

and the assertion follows, since ε is arbitrary, **q.e.d.**

Next, a fact already mentioned:

Proposition 2.3.45: *Let $D \subset\subset \mathbf{C}^n$ be a bounded convex domain. Then D is complete hyperbolic.*

Proof: We can assume $0 \in D$. It clearly suffices to show that all the closed Kobayashi balls $\overline{B_k(0,r)}$ of center 0 are compact. Let $\{z_\nu\} \subset \overline{B_k(0,r)}$; we must find a subsequence converging to a point of D. Clearly, we may suppose that $z_\nu \to w_0 \in \overline{D}$ as $\nu \to +\infty$, for D is bounded. Assume, by contradiction, that $w_0 \in \partial D$; since D is convex, there is a linear functional $\lambda: \mathbf{C}^n \to \mathbf{C}$ such that $\operatorname{Re}\lambda(z) < \operatorname{Re}\lambda(w_0)$ for all $z \in D$; in particular, $\lambda(w_0) \ne 0$ (for $0 \in D$).

Set $H = \{\zeta \in \mathbf{C} \mid \operatorname{Re}\lambda(\zeta w_0) < \operatorname{Re}\lambda(w_0)\}$; clearly H is a half-plane of $\mathbf{C}$, and the linear map $\pi: \mathbf{C}^n \to \mathbf{C}$ given by $\pi(z) = \lambda(z)/\lambda(w_0)$ sends D into H. In particular

$$r \ge k_D(0, z_\nu) \ge k_H\bigl(0, \pi(z_\nu)\bigr).$$

Since H is complete hyperbolic, by Proposition 2.3.17 the closed Kobayashi balls in H are compact; therefore, up to a subsequence $\{\pi(z_\nu)\}$ tends to a point of H. On the other hand, $\pi(z_\nu) \to \pi(w_0) = w_0 \in \partial H$, and this is a contradiction, **q.e.d.**

Finally, the convexity is reflected by the shape of Kobayashi balls:

Proposition 2.3.46: *Let $D \subset\subset \mathbf{C}^n$ be a bounded convex domain. Then for all $\lambda \in [0,1]$ and z_0, z_1, $z_2 \in D$ we have*

$$k_D\bigl(z_0, \lambda z_1 + (1-\lambda)z_2\bigr) \le \max\{k_D(z_0, z_1), k_D(z_0, z_2)\}. \tag{2.3.20}$$

In particular, the closed Kobayashi balls of D are compact and convex.

Proof: Choose z_0, z_1, $z_2 \in D$ with, for instance, $k_D(z_0, z_2) \le k_D(z_0, z_1)$, and fix $\varepsilon > 0$. By Proposition 2.3.44, there are φ_1, $\varphi_2: \Delta \to D$ and ζ_1, $\zeta_2 \in \Delta$ such that $\varphi_j(0) = z_0$,

$\varphi_j(\zeta_j) = z_j$ and $\omega(0, \zeta_j) < k_D(z_0, z_j) + \varepsilon$, for $j = 1, 2$; moreover, we may assume both ζ_1 and ζ_2 real and $\zeta_1 \geq \zeta_2 > 0$. Define $\psi: \Delta \to D$ by

$$\psi(\zeta) = \varphi_2\left(\frac{\zeta_2}{\zeta_1}\zeta\right),$$

and $\phi_\lambda: \Delta \to \mathbf{C}^n$ by

$$\phi_\lambda(\zeta) = \lambda\varphi_1(\zeta) + (1 - \lambda)\psi(\zeta),$$

for $\lambda \in [0, 1]$. Since D is convex, every ϕ_λ maps Δ into D; hence

$$k_D\big(z_0, \lambda z_1 + (1 - \lambda)z_2\big) = k_D\big(z_0, \phi_\lambda(\zeta_1)\big) \leq \omega(0, \zeta_1) < k_D(z_0, z_1) + \varepsilon,$$

and (2.3.20) follows. The convexity of the closed Kobayashi balls is then immediate, and the compactness follows from Proposition 2.3.45 together with Proposition 2.3.17, **q.e.d.**

2.3.5 *Boundary behavior of the Kobayashi distance*

As already mentioned, the Kobayashi distance is quite difficult to compute; so, for the applications, it becomes important to find a way of approximating it using something more explicit. Near interior points of a hyperbolic manifold X, this is easily accomplished by means of Lemma 2.3.31, Theorem 2.3.32 and Proposition 2.3.33, showing that k_X is locally equivalent to the distance induced by any hermitian metric on X. On the other hand, if $D \subset \mathbf{C}^n$ is a hyperbolic domain, at present we have no way of estimating the behavior of k_D near ∂D.

The aim of this section is to give tools to handle this problem when D is strongly pseudoconvex. In B^n, the Bergmann distance $k_{B^n}(0, z)$ is of the same order of $-\frac{1}{2}\log(1 - \|z\|)$ as $z \to \partial B^n$; therefore $k_{B^n}(0, z)$ diverges exactly as $-\frac{1}{2}\log d(z, \partial B^n)$, where $d(z, \partial B^n)$ denotes the euclidean distance of z from ∂B^n. The idea is that this happens in every strongly pseudoconvex domain D: $k_D(z_0, z)$ blows up exactly as $-\frac{1}{2}\log d(z, \partial D)$, for any base point $z_0 \in D$. Actually, we shall not stop here: we shall study the behavior of $k_D(z, w)$ when both z and w go to the boundary, again comparing it with the euclidean distance from the boundary. We shall get quite powerful and precise estimates and, to give you an idea of how to work with them, we shall end the section with a proof of the continuous version of Fefferman's theorem resting on these estimates. In the next section using different arguments we shall study the boundary behavior of the Kobayashi metric, and we shall come back again to boundary estimates in the next chapter, for weakly convex domains.

Complex analysis focused on strongly pseudoconvex domains because on them it is possible to solve quite accurately the $\overline{\partial}$-equation. Accordingly, we begin our job quoting two standard facts of complex analysis, which are the only external results we shall need about strongly pseudoconvex domains.

We shall denote by $L^2_{(0,1)}(D)$ the space of $(0, 1)$-forms on a domain $D \subset \mathbf{C}^n$ with square-integrable coefficients, and by $L^\infty_{(0,1)}(D)$ the space of $(0, 1)$-forms on D with bounded coefficients. Note that if D is bounded then $L^\infty_{(0,1)}(D)$ is contained in $L^2_{(0,1)}(D)$. Then the first fact is the existence of a solution of the $\overline{\partial}$-equation on strongly pseudoconvex domains:

Theorem 2.3.47: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain with smooth boundary. Let η be a $\overline{\partial}$-closed smooth $(0,1)$-form in $L^2_{(0,1)}(D)$. Then there is a unique smooth function $u = S\eta \in L^2(D)$ such that $\overline{\partial} u = \eta$ and u is orthogonal in $L^2(D)$ to the holomorphic functions on D. Moreover, S is a bounded linear operator, that is there exists $C > 0$ depending only on D such that*

$$\|u\|_2 \le C\|\eta\|_{L^2_{(0,1)}(D)}. \tag{2.3.21}$$

And the second fact is the continuous dependence of the solution on parameters:

Theorem 2.3.48: *Let M be a compact subset of $\mathbf{R}^N$, and $D \subset\subset \mathbf{C}^n$ a strongly pseudoconvex domain with smooth boundary. Let $\eta\colon M \to L^\infty_{(0,1)}(D)$ be a continuous map such that $\eta_x = \eta(x)$ is smooth and $\overline{\partial}$-closed for every $x \in M$. Set $u_x = S\eta_x$. Then $u\colon M \times D \to \mathbf{C}$ given by $u(x,z) = u_x(z)$ is continuous on $M \times D$.*

Proofs of these theorems can be found in Krantz [1982], for instance.

We now set up some notations about strongly pseudoconvex domains. Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain (that we recall is always bounded and with C^2 boundary), and let $\rho\colon \mathbf{C}^n \to \mathbf{R}$ be a defining function for D; we shall always take ρ so that for every $x \in \partial D$ the Levi form $L_{\rho,x}$ of ρ at x is positive definite on $\mathbf{C}^n$. In particular, ρ is *strictly plurisubharmonic* in a neighbourhood U of ∂D, i.e., $L_{\rho,z}$ is positive definite for all $z \in U$. Furthermore, since ∂D is compact, there are c_1, $c_2 > 0$ such that for all $v \in \mathbf{C}^n$ and $x_0 \in \partial D$

$$c_1\|v\|^2 \le L_{\rho,x_0}(v,v) \le c_2\|v\|^2. \tag{2.3.22}$$

By the way, it is easy to check that for every $x \in \partial D$ the positive definite hermitian form $L_{D,x}$ on $T^{\mathbf{C}}_x(\partial D)$ given by $L_{D,x} = \|\mathrm{grad}\rho(x)\|^{-1}L_{\rho,x}$ is independent of ρ; $L_{D,x}$ is the *Levi form* of D at $x \in \partial D$.

The expression

$$p_x(z) = \sum_{j=1}^n \frac{\partial\rho}{\partial z_j}(x)(z_j - x_j) + \frac{1}{2}\sum_{h,k=1}^n \frac{\partial^2\rho}{\partial z_h\partial z_k}(x)(z_h - x_h)(z_k - x_k)$$

is the *Levi polynomial* of ρ at $x \in \partial D$. The expansion of ρ about $x_0 \in \partial D$ can be written

$$\rho(z) = 2\,\mathrm{Re}\big(p_{x_0}(z)\big) + L_{\rho,x_0}(z - x_0, z - x_0) + o(\|z - x_0\|^2). \tag{2.3.23}$$

Since $\rho(z) < 0$ in $D \cap U$ and L_{ρ,x_0} is positive definite, there is a neighbourhood V_{x_0} of x_0 such that $\mathrm{Re}(p_{x_0}) < 0$ in $V_{x_0} \cap D$. Moreover, since ∂D is compact, we can assume that V_{x_0} is of *uniform size*, that is that there is a fixed neighbourhood V of the origin such that $V_{x_0} = x_0 + V$ for all $x_0 \in \partial D$.

The notion of strongly pseudoconvex domain is stable under perturbation. In fact, if D is given as before by means of a function ρ strictly plurisubharmonic in a neighbourhood U of ∂D and ψ is *any* C^2 real-valued function compactly supported in U, then for any $\varepsilon > 0$ sufficiently small the function $\rho - \varepsilon\psi$ is strictly plurisubharmonic in U and

$$\widetilde{D} = \{z \in \mathbf{C}^n \mid (\rho - \varepsilon\psi)(z) < 0\}$$

is strongly pseudoconvex. In particular, $\overline{D}$ has a fundamental system of neighbourhoods composed by strongly pseudoconvex C^∞ domains.

In chapter 2.1 we saw how to use peak function to prove tautness. We constructed a peak function at any point of the boundary of a strongly pseudoconvex domain; now we shall use the material introduced so far to prove that we can link together peak functions in a continuous way:

Theorem 2.3.49: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex C^2 domain. Then there exist a neighbourhood D' of $\overline{D}$ and a continuous function $\Psi\colon \partial D \times D' \to \mathbf{C}$ such that:*
(i) $\Psi_{x_0} = \Psi(x_0, \cdot)$ *is holomorphic in D' for any $x_0 \in \partial D$;*
(ii) Ψ_{x_0} *is a peak function for D at x_0 for each $x_0 \in \partial D$.*

Proof: Let V be a neighbourhood of the origin such that $\mathrm{Re}\, p_{x_0} < 0$ in $D \cap V_{x_0}$ for every $x_0 \in \partial D$, where $V_{x_0} = x_0 + V$. Let $A(x_0) = \{z \in \mathbf{C}^n \mid p_{x_0}(z) = 0\}$. We claim that there exist two euclidean balls $B_2 \subset\subset B_1 \subset\subset V$ centered at the origin and a strongly pseudoconvex neighbourhood $\widehat{D}$ of $\overline{D}$ such that for all $x_0 \in \partial D$

$$\big(B_1(x_0) \setminus B_2(x_0)\big) \cap \widehat{D} \cap A(x_0) = \emptyset,$$

where $B_j(x_0) = x_0 + B_j$ for $j = 1, 2$. In fact, let $\varepsilon > 0$ be smaller than the eigenvalues of L_{ρ,x_0} for all $x_0 \in \partial D$, where ρ is a defining function for D strictly plurisubharmonic in a neighbourhood U of ∂D — cf. (2.3.22). Choose a ball B_1 so that

$$A(x_0) \cap B_1(z_0) \cap \{z \in U \mid \rho(z) - \varepsilon\|z - x_0\|^2 = 0\} = \{x_0\}.$$

Then take any ball $B_2 \subset\subset B_1$, of radius r, say, and put

$$\widehat{D} = \{z \in \mathbf{C}^n \mid \rho(z) < \varepsilon r^2\}.$$

Finally, let $\widetilde{D}$ be a strongly pseudoconvex smooth domain such that $D \subset\subset \widetilde{D} \subset\subset \widehat{D}$. We shall solve $\overline{\partial}$ on $\widetilde{D}$.

Let $\chi\colon \mathbf{C}^n \to [0,1]$ be a smooth function with support contained in B_1 and identically 1 on B_2, and put $\chi_{x_0}(z) = \chi(z - x_0)$. Then χ_{x_0}/p_{x_0} is well defined on D for every $x_0 \in \partial D$, and $\mathrm{Re}(\chi_{x_0}/p_{x_0}) \le 0$ on D. Moreover, the (0,1)-form $\eta_{x_0} = \overline{\partial}(\chi_{x_0}/p_{x_0})$ has bounded smooth coefficients on $\widetilde{D}$, is $\overline{\partial}$-closed and depends continuously on $x_0 \in \partial D$. Therefore Theorem 2.3.48 can be applied with $M = \partial D$, and the solutions $u_{x_0} = S\eta_{x_0}$ yield a continuous function $u\colon \partial D \times \widetilde{D} \to \mathbf{C}$. Moreover, slightly shrinking $\widetilde{D}$ if necessary (but still with $D \subset\subset \widetilde{D}$) we may assume that u is bounded on $\partial D \times \widetilde{D}$ by a constant $k > 0$, say. In particular, the functions $\chi_{x_0}/p_{x_0} - u_{x_0} - k$ are meromorphic on $\widetilde{D}$ and have negative real part there.

The function $h(w) = (w+1)/(w-1)$ maps the left half-plane onto Δ, sending ∞ into 1. Thus if we set

$$\Psi_{x_0} = h\big(\chi_{x_0}/p_{x_0} - u_{x_0} - k\big),$$

Ψ defined by $\Psi(x_0, \cdot) = \Psi_{x_0}$ is exactly as we want, provided that it is defined in a neighbourhood of $\overline{D}$.

Ψ_{x_0} is clearly holomorphic on $D \cup \big(\widetilde{D} \setminus B_2(x_0)\big)$. On $B_2(x_0)$,

$$\Psi_{x_0} = \frac{1 - (u_{x_0} + k - 1)p_{x_0}}{1 - (u_{x_0} + k + 1)p_{x_0}}.$$

This function is holomorphic when the denominator is not zero. Since $\|u_{x_0}\|_{L^\infty(\widetilde{D})} < k$ and the Levi polynomials are equicontinuous on ∂D, there is an even smaller strongly pseudoconvex smooth neighbourhood D' of $\overline{D}$ contained in $\widetilde{D}$ where all the functions Ψ_{x_0} are holomorphic, and we are done, **q.e.d.**

This is all we need for our investigation of the boundary behavior of the Kobayashi distance. The idea is to compare $k_D(z_0, z)$, where z_0 is a given point of a strongly pseudoconvex domain D and $z \in D$ is near ∂D, with $d(z, \partial D)$, the euclidean distance from the boundary. The sort of results we should expect is exemplified in

Lemma 2.3.50: *Let B_r be the euclidean ball of radius r in $\mathbf{C}^n$ centered at the origin. Then for every $z \in B_r$*

$$\tfrac{1}{2}\log r - \tfrac{1}{2}\log d(z, \partial B_r) \le c_{B_r}(0, z) = k_{B_r}(0, z) \le \tfrac{1}{2}\log 2r - \tfrac{1}{2}\log d(z, \partial B_r).$$

Proof: We have

$$k_{B_r}(0, z) = c_{B_r}(0, z) = \omega\left(0, \frac{\|z\|}{r}\right),$$

and

$$d(z, \partial B_r) = r - \|z\|.$$

Then, setting $t = \|z\|/r$, we get

$$\begin{aligned}\tfrac{1}{2}\log r - \tfrac{1}{2}\log d(z, \partial B_r) &= \tfrac{1}{2}\log\frac{1}{1-t} \le \tfrac{1}{2}\log\frac{1+t}{1-t} = \omega(0, t) \le \tfrac{1}{2}\log\frac{2}{1-t} \\ &= \tfrac{1}{2}\log 2r - \tfrac{1}{2}\log d(z, \partial B_r),\end{aligned}$$

q.e.d.

And indeed this is the general case. We need a definition. Let M be a (not necessarily smooth; C^2 is enough) compact hypersurface of $\mathbf{R}^N$, and fix an unit normal vector field $\mathbf{n}$ on M (since M is orientable, there are only two choices: $\mathbf{n}$ and $-\mathbf{n}$). We shall say that M has a *tubular neighbourhood* U_ε *of radius* ε if the segments $\{x + t\mathbf{n}_x \mid t \in (-\varepsilon, \varepsilon)\}$ are pairwise disjoint, and we set

$$U_\varepsilon = \bigcup_{x \in M} \{x + t\mathbf{n}_x \mid t \in (-\varepsilon, \varepsilon)\}.$$

Note that if M has a tubular neighbourhood of radius ε, then $d(x + t\mathbf{n}_x, M) = |t|$ for every $t \in (-\varepsilon, \varepsilon)$ and $x \in M$; in particular, $U_\varepsilon = \bigcup_{x \in M} B(x, \varepsilon)$. A proof of the existence of a tubular neighbourhood of radius sufficiently small for any compact hypersurface of $\mathbf{R}^N$ can be found, e.g., in Spivak [1979].

And now, we begin with the estimates. The upper estimate does not even depend on the strong pseudoconvexity:

Theorem 2.3.51: *Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain, and $z_0 \in D$. Then there is a constant $c_1 \in \mathbf{R}$ depending only on D and z_0 such that for all $z \in D$*

$$c_D(z_0, z) \le k_D(z_0, z) \le c_1 - \tfrac{1}{2} \log d(z, \partial D). \tag{2.3.24}$$

Proof: Since D is a C^2 domain, ∂D admits tubular neighbourhoods U_ε of radius $\varepsilon < 1$ small enough. Put

$$c_1 = \sup\bigl\{k_D(z_0, w) \bigm| w \in D \setminus U_{\varepsilon/4}\bigr\} + \max\bigl\{0, \tfrac{1}{2}\log \operatorname{diam}(D)\bigr\},$$

where $\operatorname{diam}(D)$ is the euclidean diameter of D.

There are two cases:

(i) $z \in U_{\varepsilon/4} \cap D$. Let $x \in \partial D$ be such that $\|x - z\| = d(z, \partial D)$. Since $U_{\varepsilon/2}$ is a tubular neighbourhood of ∂D, there exists $\lambda \in \mathbf{R}$ such that $w = \lambda(x - z) \in \partial U_{\varepsilon/2} \cap D$ and the euclidean ball B of center w and radius $\varepsilon/2$ is contained in $U_\varepsilon \cap D$ and tangent to ∂D in x. Therefore Lemma 2.3.50 yields

$$\begin{aligned} c_D(z_0, z) \le k_D(z_0, z) &\le k_D(z_0, w) + k_D(w, z) \le k_D(z_0, w) + k_B(w, z) \\ &\le k_D(z_0, w) + \tfrac{1}{2}\log\varepsilon - \tfrac{1}{2}\log d(z, \partial B) \\ &\le c_1 - \tfrac{1}{2}\log d(z, \partial D), \end{aligned}$$

because $w \notin U_{\varepsilon/4}$ (and $\varepsilon < 1$).

(ii) $z \in D \setminus U_{\varepsilon/4}$. Then

$$c_D(z_0, z) \le k_D(z_0, z) \le c_1 - \tfrac{1}{2}\log \operatorname{diam}(D) \le c_1 - \tfrac{1}{2}\log d(z, \partial D),$$

because $d(z, \partial D) \le \operatorname{diam}(D)$, and we are done, **q.e.d.**

Now we pass to the more interesting lower estimate:

Theorem 2.3.52: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain, and $z_0 \in D$. Then there is a constant $c_2 \in \mathbf{R}$ depending only on D and z_0 such that for all $z \in D$*

$$c_2 - \tfrac{1}{2}\log d(z, \partial D) \le c_D(z_0, z) \le k_D(z_0, z). \tag{2.3.25}$$

Proof: Let $D' \supset\supset D$ and $\Psi\colon \partial D \times D' \to \mathbf{C}$ be given by Theorem 2.3.49, and define $\phi\colon \partial D \times \Delta \to \mathbf{C}$ by

$$\phi(x, \zeta) = \frac{1 - \overline{\Psi(x, z_0)}}{1 - \Psi(x, z_0)} \cdot \frac{\zeta - \Psi(x, z_0)}{1 - \overline{\Psi(x, z_0)}\zeta}. \tag{2.3.26}$$

Then the map $\Phi(x, z) = \Phi_x(z) = \phi\bigl(x, \Psi(x, z)\bigr)$ is defined on a neighbourhood $\partial D \times D_0$ of $\partial D \times \overline{D}$ (with $D_0 \subset\subset D'$) and satisfies

(a) Φ is continuous and Φ_x is a holomorphic peak function for D at x for any $x \in \partial D$;
(b) for every $x \in \partial D$ we have $\Phi_x(z_0) = 0$.

Now set $U_\varepsilon = \bigcup_{x\in\partial D} P(x,\varepsilon)$, where $P(x,\varepsilon)$ is the polydisk of center x and polyradius $(\varepsilon,\dots,\varepsilon)$. The family $\{U_\varepsilon\}$ is a basis for the neighbourhoods of ∂D; hence there exists $\varepsilon>0$ such that $U_\varepsilon \subset\subset D_0$ and U_ε is contained in a tubular neighbourhood of ∂D. Then for any $x\in\partial D$ and $z\in P(x,\varepsilon/2)$ the Cauchy estimates yield

$$\begin{aligned}|1-\Phi_x(z)| = |\Phi_x(x)-\Phi_x(z)| &\le \left\|\frac{\partial\Phi_x}{\partial z}\right\|_{P(x,\varepsilon/2)}\|z-x\|\\ &\le \frac{2\sqrt{n}}{\varepsilon}\|\Phi\|_{\partial D\times U_\varepsilon}\|z-x\| = M\|z-x\|,\end{aligned}$$

where M is independent of z and x. Put $c_2 = -\frac{1}{2}\log M$; note that $c_2 \le \frac{1}{2}\log(\varepsilon/2)$, for $\|\Phi\|_{\partial D\times U_\varepsilon}\ge 1$. Then we again have two cases:

(i) $z\in D\cap U_{\varepsilon/2}$. Choose $x\in\partial D$ so that $d(z,\partial D) = \|z-x\|$. Since $\Phi_x(D)\subset\Delta$ and $\Phi_x(z_0)=0$, we have

$$k_D(z_0,z)\ge c_D(z_0,z)\ge\omega\big(\Phi_x(z_0),\Phi_x(z)\big)\ge\tfrac{1}{2}\log\frac{1}{1-|\Phi_x(z)|}.$$

Now,

$$1-|\Phi_x(z)|\le|1-\Phi_x(z)|\le M\|z-x\| = M\,d(z,\partial D);$$

therefore

$$k_D(z_0,z)\ge c_D(z_0,z)\ge -\tfrac{1}{2}\log M-\tfrac{1}{2}\log d(z,\partial D)\ge c_2-\tfrac{1}{2}\log d(z,\partial D).$$

(ii) $z\in D\setminus U_{\varepsilon/2}$. Then $d(z,\partial D)\ge\varepsilon/2$; hence

$$k_D(z_0,z)\ge c_D(z_0,z)\ge 0\ge\tfrac{1}{2}\log(\varepsilon/2)-\tfrac{1}{2}\log d(z,\partial D)\ge c_2-\tfrac{1}{2}\log d(z,\partial D),$$

and we are done, **q.e.d.**

A first consequence is the promised:

Corollary 2.3.53: *Every strongly pseudoconvex domain D is complete hyperbolic.*

Proof: Take $z_0\in D$, $r>0$ and let $z\in B_k(z_0,r)$. Then (2.3.25) yields

$$d(z,\partial D)\ge\exp\big(2(c_2-r)\big),$$

where c_2 depends only on z_0. Then $B_k(z_0,r)$ is relatively compact in D, and the assertion follows, **q.e.d.**

Now we want to study the behavior of $k_D(z_1,z_2)$ when both z_1 and z_2 tend to the boundary of D. A deeper examination of the proof of Theorem 2.3.52 yields the following result:

Theorem 2.3.54: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain, and $\delta > 0$. Then there exist ε_1, $\varepsilon_0 \in (0,\delta)$ with $\varepsilon_0 < \varepsilon_1$ and a constant $c \in \mathbf{R}$ such that for all $x_0 \in \partial D$ and $z \in D \cap B(x_0,\varepsilon_0)$ we have*

$$k_D\bigl(z, D \setminus B(x_0, 2\varepsilon_1)\bigr) \geq -\tfrac{1}{2}\log d(z,\partial D) + c. \tag{2.3.27}$$

Proof: Let $D' \supset\supset D$ and $\Psi\colon \partial D \times D' \to \mathbf{C}$ be given by Theorem 2.3.49, and set again $U_\varepsilon = \bigcup_{x\in\partial D} P(x,\varepsilon)$. Now choose $\varepsilon_1 \in (0,\delta)$ such that $U_{2\varepsilon_1}$ is contained in a tubular neighbourhood of ∂D and moreover $U_{2\varepsilon_1} \subset\subset D'$. Put

$$V_{\varepsilon_1} = \bigl\{(x,z_0) \in \partial D \times \overline{D} \mid \|z_0 - x\| \geq \varepsilon_1\bigr\};$$

since V_{ε_1} is compact and $|\Psi(x,z_0)| < 1$ for all $(x,z_0) \in V_{\varepsilon_1}$, there is $\eta < 1$ such that $|\Psi(x,z_0)| < \eta < 1$ for all $(x,z_0) \in V_{\varepsilon_1}$.

Define $\phi\colon V_{\varepsilon_1} \times \Delta \to \mathbf{C}$ by

$$\phi(x,z_0,\zeta) = \frac{1-\overline{\Psi(x,z_0)}}{1-\Psi(x,z_0)} \cdot \frac{\zeta - \Psi(x,z_0)}{1-\overline{\Psi(x,z_0)}\zeta},$$

and fix $\gamma \in (\eta, 1)$. If we take a neighbourhood $D_0 \subset\subset D'$ of $\overline{D}$ such that $|\Psi(x,z)| < \gamma/\eta$ for all $x \in \partial D$ and $z \in \overline{D}_0$, then the map $\Phi(x,z_0,z) = \Phi_{x,z_0}(z) = \phi\bigl(x,z_0,\Psi(x,z)\bigr)$ is defined on $V_{\varepsilon_1} \times D_0$; moreover for every $(x,z_0,z) \in V_{\varepsilon_1} \times D_0$ we have

$$|\Phi(x,z_0,z)|^2 \leq 1 + \frac{1}{\eta(1-\gamma)^2} < +\infty.$$

Now choose $\varepsilon_0 \in (0,\varepsilon_1/2)$ so that $U_{2\varepsilon_0} \subset\subset D_0$. Then for every $(x,z_0) \in V_{\varepsilon_1}$ and $z \in B(x,\varepsilon_0) \subset P(x,\varepsilon_0)$ we have

$$\begin{aligned} |1 - \Phi_{x,z_0}(z)| = |\Phi_{x,z_0}(x) - \Phi_{x,z_0}(z)| &\leq \left\| \frac{\partial \Phi_{x,z_0}}{\partial z} \right\|_{P(x,\varepsilon_0)} \|z - x\| \\ &\leq \frac{\sqrt{n}}{\varepsilon} \|\Phi\|_{V_{\varepsilon_1}\times D_0} \|z - x\|. \end{aligned} \tag{2.3.28}$$

Set $c = -\frac{1}{2}\log(\sqrt{n}\|\Phi\|_{V_{\varepsilon_1}\times D_0}/\varepsilon)$. Take $x \in \partial D$, $z \in B(x,\varepsilon_0) \cap D$ and $z_0 \in D \setminus B(x, 2\varepsilon_1)$. Then there is $y \in B(x, 2\varepsilon_0) \cap \partial D$ such that $\|z - y\| = d(z,\partial D)$; moreover,

$$\|y - z_0\| \geq \|x - z_0\| - \|y - x\| > 2\varepsilon_1 - 2\varepsilon_0 \geq \varepsilon_1,$$

that is $(y,z_0) \in V_{\varepsilon_1}$. Then

$$\begin{aligned} k_D(z,z_0) \geq c_D(z,z_0) &\geq \omega\bigl(\Phi_{y,z_0}(z), \Phi_{y,z_0}(z_0)\bigr) \\ &\geq \tfrac{1}{2}\log\frac{1}{1 - |\Phi_{y,z_0}(z)|} \geq c - \tfrac{1}{2}\log\|z - y\| = c - \tfrac{1}{2}\log d(z,\partial D), \end{aligned}$$

by (2.3.28), **q.e.d.**

Then the first step is done:

Corollary 2.3.55: *Let $D \subset\subset \mathbf{C}^n$ be a bounded strongly pseudoconvex domain of $\mathbf{C}^n$, and choose two points x_1, $x_2 \in \partial D$ with $x_1 \neq x_2$. Then there exist $\varepsilon_0 > 0$ and $K \in \mathbf{R}$ such that for any $z_1 \in D \cap B(x_1, \varepsilon_0)$ and $z_2 \in D \cap B(x_2, \varepsilon_0)$ we have*

$$k_D(z_1, z_2) \geq -\tfrac{1}{2} \log d(z_1, \partial D) - \tfrac{1}{2} \log d(z_2, \partial D) + K. \tag{2.3.29}$$

Proof: Let ε_0, $\varepsilon_1 \in (0, \delta)$ be given by Theorem 2.3.54, where $\delta > 0$ is so small that $B(x_1, 2\delta) \cap B(x_2, 2\delta) = \emptyset$. Take $z_j \in B(x_j, \varepsilon_0)$ for $j = 1$, 2, and let σ be any curve from z_1 to z_2. Then part of the image of σ should be outside both $B(x_1, 2\varepsilon_1)$ and $B(x_2, 2\varepsilon_1)$; therefore (2.3.27) yields

$$\ell_k(\sigma) \geq -\tfrac{1}{2} \log d(z_1, \partial D) - \tfrac{1}{2} \log d(z_2, \partial D) + O(1),$$

and (2.3.29) follows from Theorem 2.3.32, **q.e.d.**

The last step is the description of what happens to $k_D(z_1, z_2)$ when z_1 and z_2 approach the same point of the boundary:

Theorem 2.3.56: *Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain and $x_0 \in \partial D$. Then there exist $\varepsilon > 0$ and $C \in \mathbf{R}$ such that for all z_1, $z_2 \in D \cap B(x_0, \varepsilon)$ we have*

$$k_D(z_1, z_2) \leq -\tfrac{1}{2} \sum_{j=1}^{2} \log d(z_j, \partial D) + \tfrac{1}{2} \sum_{j=1}^{2} \log\bigl(d(z_j, \partial D) + \|z_1 - z_2\|\bigr) + C. \tag{2.3.30}$$

Proof: For every $x \in \partial D$ denote by $\mathbf{n}_x$ the outer unit normal vector to ∂D at x. Choose $\varepsilon > 0$ so small that $\partial D \cap B(x_0, 4\varepsilon)$ is connected and

(i) $\|\mathbf{n}_x - \mathbf{n}_{x_0}\| < 1/8$ for all $x \in \partial D \cap B(x_0, \varepsilon)$;
(ii) for every $\delta \in [0, 2\varepsilon]$, $z \in D \cap B(x_0, \varepsilon)$ and $x \in \partial D \cap B(x_0, 4\varepsilon)$ we have $z - \delta \mathbf{n}_x \in D$ and

$$d(z - \delta \mathbf{n}_x, \partial D) > 3\delta/4.$$

Set $U = B(x_0, \varepsilon)$. Let z_1, $z_2 \in U \cap D$, and choose x_1, $x_2 \in \partial D$ so that $\|z_j - x_j\| = d(z_j, \partial D)$ for $j = 1$, 2. Set $z_j' = z_j - \|z_1 - z_2\| \mathbf{n}_{x_j}$; then

$$k_D(z_1, z_2) \leq k_D(z_1', z_2') + \sum_{j=1}^{2} k_D(z_j, z_j').$$

We begin bounding from above the first term. Since $\|z_1 - z_2\| < 2\varepsilon$, by (ii) we have $d(z_j', \partial D) > 3\|z_1 - z_2\|/4$, and by (i) we have $\|z_1' - z_2'\| < 5\|z_1 - z_2\|/4$. Define the open set Ω in $\mathbf{C}$ by

$$\Omega = \bigl\{\zeta \in \mathbf{C} \bigm| \min\{|\zeta|, |\zeta - 1|\} < 3/5\bigr\},$$

and $\varphi \colon \Omega \to \mathbf{C}^n$ by

$$\varphi(\zeta) = z_1' + \zeta(z_2' - z_1').$$

Then $\varphi(\Omega) \subset D$, $\varphi(0) = z_1'$ and $\varphi(1) = z_2'$; hence

$$k_D(z_1', z_2') \leq k_\Omega(0, 1).$$

To end the proof we must bound from above $k_D(z_j, z_j')$. Let $\varphi_j \in \mathrm{Hol}(\mathbf{C}, \mathbf{C}^n)$ be defined by

$$\varphi_j(\zeta) = x_j - \zeta \mathbf{n}_{x_j};$$

then $\varphi_j(0) = x_j$, $\varphi_j\big(d(z_j, \partial D)\big) = z_j$ and $\varphi_j\big(d(z_j, \partial D) + \|z_1 - z_2\|\big) = z_j'$. Set for every $\alpha > 0$

$$\Omega_\alpha = \big\{\zeta = \xi + i\eta \in \mathbf{C} \bigm| |\zeta| < 4\varepsilon,\, \xi > \alpha|\eta|^2\big\}.$$

If α is large enough, then $\varphi_j(\Omega_\alpha) \subset D \cap U$. For convenience, fix a domain $\Omega_\alpha' \subset \Omega_\alpha$, symmetric with respect to the real axis, obtained by smoothing $\partial\Omega_\alpha$ in a small neighbourhood of its two angular points. We have

$$k_D(z_j, z_j') \leq k_{\Omega_\alpha'}\big(d(z_j, \partial D), d(z_j, \partial D) + \|z_1 - z_2\|\big).$$

So it remains to show that if a and b are real numbers satisfying $0 < a < b < 3\varepsilon$, then

$$k_{\Omega_\alpha'}(a, b) \leq \tfrac{1}{2}\big(\log b - \log a\big) + O(1).$$

Let τ be a biholomorphism of Ω_α' with Δ such that $\tau(0) = 1$ and τ is real on the real axis. Since $\partial\Omega_\alpha'$ is of class C^1, τ extends to a diffeomorphism between $\overline{\Omega_\alpha'}$ and $\overline{\Delta}$ (by Theorem 1.1.28). Therefore there are $K > 1$ and $\theta \in (-1, 1)$ such that for every $c \in (0, 3\varepsilon)$

$$\max\{\theta, 1 - Kc\} \leq \tau(c) \leq 1 - c/K.$$

Then

$$k_{\Omega_\alpha'}(a, b) = \omega\big(\tau(a), \tau(b)\big) = \omega\big(0, \tau(a)\big) - \omega\big(0, \tau(b)\big) \leq \frac{1}{2}\left[\log \frac{2}{a/K} - \log \frac{1+\theta}{Kb}\right],$$

q.e.d.

We end the section proving the promised version of Fefferman's theorem, showing that every biholomorphism between two strongly pseudoconvex domains extends to a homeomorphism of the closures.

We need *Hopf's lemma:*

Proposition 2.3.57: *Let $U \subset \mathbf{R}^N$ be a C^2 domain. Let $f\colon \overline{U} \to \mathbf{R}$ be subharmonic in U, continuous in $\overline{U}$ and suppose that f has a local maximum at $x_0 \in \partial U$. Let $\mathbf{n} = \mathbf{n}_{x_0}$ be the outer unit normal to ∂U at x_0; then*

$$\liminf_{t \to 0^+} \frac{f(x_0) - f(x_0 - t\mathbf{n})}{t} > 0.$$

Figure 2.1

In particular, $\partial f/\partial \mathbf{n}(x_0) > 0$ *when it exists.*

Proof: Let $\varepsilon > 0$ be such that there exists a ball B of radius ε internally tangent to ∂U at x_0 so that $f(x_0) > f(x)$ for all $x \in B$. Up to a translation, we can assume that the center of B is the origin. Let B_1 be a ball centered at x_0 of radius $\varepsilon_1 < \varepsilon$, and let $B' = B \cap B_1$. Then $\partial B'$ is the union of $S' = \partial B' \cap \overline{B}$ and $S_1' = \partial B' \cap \overline{B_1}$ (cf. Figure 2.1).

Define $h\colon \mathbf{R}^N \to \mathbf{R}$ by

$$h(x) = e^{-\alpha\|x\|^2} - e^{-\alpha\varepsilon^2},$$

where $\alpha > 0$. Then $h > 0$ on $B' \subset B$ and

$$\triangle h = e^{-\alpha\|x\|^2}(4\alpha^2\|x\|^2 - 2\alpha N).$$

In particular, if α is large enough then $\triangle h > 0$ on B'. Now set

$$v(x) = f(x) + \delta h(x).$$

If δ is small enough then $v(x) < f(x_0)$ on S_1'; moreover we have $v(x) = f(x) < f(x_0)$ for all $x \in S' \setminus \{x_0\}$. Since v is subharmonic in B', we infer that

$$\max_{x \in \overline{B}'} v(x) = f(x_0).$$

Therefore

$$\liminf_{t \to 0^+} \frac{v(x_0) - v(x_0 - t\mathbf{n})}{t} = \delta \frac{\partial h}{\partial \mathbf{n}}(x_0) + \liminf_{t \to 0^+} \frac{f(x_0) - f(x_0 - t\mathbf{n})}{t} \geq 0.$$

But $\partial h/\partial \mathbf{n}(x_0) = -2\alpha\varepsilon e^{-\alpha\varepsilon^2} < 0$, and the assertion follows, **q.e.d.**

And now, here we are:

Theorem 2.3.58: *Let D, $D' \subset\subset \mathbf{C}^n$ be strongly pseudoconvex domains, and $f\colon D \to D'$ a biholomorphism. Then f extends continuously to a homeomorphism of $\overline{D}$ with $\overline{D'}$.*

Proof: We need a preliminary observation. Let ρ be a defining function for D strictly plurisubharmonic in a neighbourhood U of ∂D. We can assume that U has C^2 boundary, is contained in a tubular neighbourhood of ∂D and that $f(U \cap D)$ is contained in a tubular neighbourhood of $\partial D'$. Then we can apply Proposition 2.3.57 to the subharmonic function $\rho \circ f^{-1}$ defined on $f(U \cap D)$ which assumes maximum on $\partial D'$, obtaining that there exists $c > 0$ such that for all $x' \in \partial D'$

$$\liminf_{t \to 0^+} \frac{\rho \circ f^{-1}(x' - t\mathbf{n}_{x'})}{-t} \geq c > 0, \tag{2.3.31}$$

where $\mathbf{n}_{x'}$ is the outer unit normal vector to $\partial D'$ at x'.

Now, (2.3.31) means that there is $\varepsilon > 0$ such that

$$\rho \circ f^{-1}(x' - t\mathbf{n}_{x'}) \leq -ct$$

for all $t \in [0, \varepsilon]$ and $x' \in \partial D'$; moreover, $t = d(x' - t\mathbf{n}_{x'}, \partial D')$, for $f(U \cap D)$ is contained in a tubular neighbourhood of $\partial D'$. Then, shrinking U if necessary, we infer

$$c\, d\bigl(f(z), \partial D'\bigr) \leq -\rho(z)$$

for all $z \in U \cap D$. Now the expansion (2.3.23) shows that $-\rho(z)$ is of the order of $d(z, \partial D)$ near ∂D (essentially because the gradient of ρ is nowhere vanishing). Therefore there exists a different constant $K > 0$ such that

$$d\bigl(f(z), \partial D'\bigr) \leq K\, d(z, \partial D) \tag{2.3.32}$$

for all $z \in U \cap D$ and thus, since $D \setminus U$ is compact, for all $z \in D$ (possibly changing K again).

Now we can show that f extends continuously to ∂D. Choose $x_0 \in \partial D$ and assume, by contradiction, that there are two sequences $\{z_\nu^1\}$, $\{z_\nu^2\} \subset D$ such that z_ν^1, $z_\nu^2 \to x_0$ as $\nu \to +\infty$ and $f(z_\nu^1) \to y^1 \in \partial D'$, $f(z_\nu^2) \to y^2 \in \partial D'$ with $y^1 \neq y^2$. By Theorem 2.3.56, we eventually have

$$k_D(z_\nu^1, z_\nu^2) \leq -\tfrac{1}{2} \sum_{j=1}^{2} \log d(z_\nu^j, \partial D) + \tfrac{1}{2} \sum_{j=1}^{2} \log\bigl(d(z_\nu^j, \partial D) + \|z_\nu^1 - z_\nu^2\|\bigr) + O(1). \tag{2.3.33}$$

On the other hand, Corollary 2.3.55 yields

$$k_{D'}\bigl(f(z_\nu^1), f(z_\nu^2)\bigr) \geq -\tfrac{1}{2} \sum_{j=1}^{2} \log d\bigl(f(z_\nu^j), \partial D\bigr) + O(1). \tag{2.3.34}$$

But $k_{D'}\bigl(f(z_\nu^1), f(z_\nu^2)\bigr) \leq k_D(z_\nu^1, z_\nu^2)$; hence (2.3.32), (2.3.33) and (2.3.34) imply

$$-\tfrac{1}{2} \sum_{j=1}^{2} \log\bigl(d(z_\nu^j, \partial D) + \|z_\nu^1 - z_\nu^2\|\bigr) \leq O(1),$$

and letting $\nu \to +\infty$ we get a contradiction.

Finally, $f\colon \overline{D} \to \overline{D'}$ is clearly a homeomorphism, because for the same reason also f^{-1} extends to a continuous map from $\overline{D'}$ to $\overline{D}$, and this obviously is the inverse of the extension of f, **q.e.d.**

2.3.6 *Localization at the boundary*

The Kobayashi distance, metric and volume form are globally defined objects, but we saw in the previous section that it is possible to study them locally near the boundary. In this section we shall further pursue this argument, showing that in a strongly pseudoconvex domain D the behavior of k_D, κ_D and $\tilde{\kappa}_D$ near a point $x_0 \in \partial D$ depends only on the local shape of ∂D near x_0, and not on the overall geometry of D. As an application, we shall describe the boundary behavior of the Kobayashi metric in strongly pseudoconvex domains.

We start with the Kobayashi volume form. Let D be a domain in $\mathbf{C}^n$; then the *Kobayashi volume element* $K_D\colon D \to \mathbf{R}^+$ is defined by

$$\forall z \in D \qquad\qquad \tilde{\kappa}_D(z) = K_D(z)\Theta,$$

where Θ is given by (2.3.11); the properties of K_D can be easily deduced from the properties of $\tilde{\kappa}_D$ described in section 2.3.3.

Our aim is to compare K_D with $K_{D\cap U}$ near $x_0 \in \partial D$, where U is any neighbourhood of x_0. Obviously, $K_D \le K_{D\cap U}$; to get some information in the reverse direction, we need a couple of preliminary facts.

Let D be a bounded domain of $\mathbf{C}^n$; a point $x \in \partial D$ is a *local peak point* for D if there is a local peak function for D at x.

Proposition 2.3.59: *Let X be a complex manifold, D a bounded domain of $\mathbf{C}^n$, $x \in \partial D$ a local peak point for D and $\{f_\nu\}$ a sequence in* $\mathrm{Hol}(X, D)$. *If there is a point $z_0 \in X$ such that $\lim\limits_{\nu\to\infty} f_\nu(z_0) = x$, then $f_\nu \to x$.*

Proof: Since D is tautly imbedded in $\mathbf{C}^n$, it suffices to show that x is the only limit point of $\{f_\nu\}$ in $\mathrm{Hol}(X, \mathbf{C}^n)$. Let $f \in \mathrm{Hol}(X, \mathbf{C}^n)$ be a limit point of $\{f_\nu\}$; obviously, $f(z_0) = x$. Let U be a neighbourhood of x in $\mathbf{C}^n$ such that there is a peak function $h \in \mathrm{Hol}(U, \mathbf{C})$ for $D \cap U$ at x. Then $h \circ f$ is defined on a neighbourhood of z_0, and has a maximum in z_0; therefore it is constant. This implies that f is constant in a neighbourhood of z_0, and hence everywhere (for X is connected). Since $f(z_0) = x$, this implies $f \equiv x$, and we are done, **q.e.d.**

Corollary 2.3.60: *Let X be a complex manifold, D a bounded domain of $\mathbf{C}^n$, $x \in \partial D$ a local peak point for D and $z_0 \in X$. Then given a neighbourhood U of x and a compact subset $K \subset X$ containing z_0 there exists a neighbourhood $V \subset U$ of x such that for all $f \in$* $\mathrm{Hol}(X, D)$

$$f(z_0) \in V \Longrightarrow f(K) \subset U.$$

Proof: Let $\{V_\nu\}$ be a fundamental system of neighbourhoods of x, with $V_{\nu+1} \subset V_\nu$. If, by contradiction, there exist a neighbourhood U of x and a compact subset $K \subset X$ containing z_0 such that for any $\nu \in \mathbf{N}$ there is $f_\nu \in \mathrm{Hol}(X, D)$ so that $f_\nu(z_0) \in V_\nu$ and $f_\nu(K) \not\subset U$, then we would have $f_\nu(z_0) \to x$ and $f_\nu \not\to x$, against Proposition 2.3.59, **q.e.d.**

Now we can prove our first localization theorem, that will be the model for others to follow:

Theorem 2.3.61: *Let D be a bounded domain of $\mathbf{C}^n$, $x \in \partial D$ a local peak point for D and U a neighbourhood of x in $\mathbf{C}^n$ such that $U \cap D$ is connected. Then*

$$\lim_{z \to x} \frac{K_D(z)}{K_{D\cap U}(z)} = 1.$$

Proof: Since $K_D \le K_{D\cap U}$, we clearly have

$$\limsup_{z\to x} \frac{K_D(z)}{K_{D\cap U}(z)} \le 1.$$

To estimate the $\liminf$, let $\{z_\nu\} \subset D \cap U$ be a sequence converging to x, and fix $\varepsilon > 0$. Let $f_\nu \in \mathrm{Hol}(B^n, D)$ be such that $f_\nu(0) = z_\nu$ and

$$|\det d(f_\nu)_0|^{-2} \le (1+\varepsilon) K_D(z_\nu).$$

Define $f_\nu^\varepsilon \in \mathrm{Hol}(B^n, D)$ by $f_\nu^\varepsilon(w) = f_\nu\big((1-\varepsilon)w\big)$. By Corollary 2.3.60, $f_\nu^\varepsilon(B^n) \subset U \cap D$ for all ν large enough. Therefore

$$K_{D\cap U}(z_\nu) \le (1-\varepsilon)^{-2n} |\det d(f_\nu)_0|^{-2} \le \frac{1+\varepsilon}{(1-\varepsilon)^{2n}} K_D(z_\nu),$$

and thus

$$\liminf_{\nu\to\infty} \frac{K_D(z_\nu)}{K_{D\cap U}(z_\nu)} \ge \frac{(1-\varepsilon)^{2n}}{1+\varepsilon}.$$

But this is true for any sequence converging to x and for any $\varepsilon > 0$; hence

$$\liminf_{z\to x} \frac{K_D(z)}{K_{D\cap U}(z)} \ge 1,$$

and the assertion follows, **q.e.d.**

We remark that this theorem can be applied to strongly pseudoconvex points, by Lemma 2.1.12 and Proposition 2.1.13.

Now we move on to the localization theorem for the Kobayashi metric. The main step is contained in

Lemma 2.3.62: *Let X be a hyperbolic manifold, and D, U open domains in X. Define $\delta\colon D \cap U \to \mathbf{R}^+$ by*

$$\forall z \in D \cap U \qquad \delta(z) = \delta_D(z, D \setminus U) = \inf_{w \in D\setminus U} \delta_D(z, w),$$

where δ_D is the function introduced in (2.3.2). *Then*

$$\kappa_{D\cap U}(z;v) \le \operatorname{cotanh}\bigl(\delta(z)\bigr)\kappa_D(z;v) \tag{2.3.35}$$

for every $z\in D\cap U$ and $v\in T_zX$.

Proof: Note that $\operatorname{cotanh}\delta_D(z,w)=\sup\{|\zeta|^{-1}\mid \exists\varphi\in\operatorname{Hol}(\Delta,X):\varphi(0)=z,\varphi(\zeta)=w\}$. Since cotanh is a decreasing function, this implies

$$\begin{aligned}\operatorname{cotanh}\bigl(\delta(z)\bigr) &= \sup_{w\in D\setminus U}\operatorname{cotanh}\delta_D(z,w)\\ &= \sup\bigl\{|\zeta|^{-1}\bigm| \exists\varphi\in\operatorname{Hol}(\Delta,X):\varphi(0)=z,\varphi(\zeta)\in D\setminus U\bigr\}.\end{aligned}$$

Now fix $\varepsilon>0$ and take $\varphi\in\operatorname{Hol}(\Delta,X)$ and $\xi\in\mathbf{C}$ such that $\varphi(0)=z$, $d\varphi_0(\xi)=v$ and $|\xi|<(1+\varepsilon)\kappa_D(z;v)$. If $r^{-1}>\operatorname{cotanh}\bigl(\delta(z)\bigr)$ then we have $\varphi(\Delta_r)\subset D\cap U$, and thus

$$\kappa_{D\cap U}(z;v)\le\frac{|\xi|}{r}<\frac{1+\varepsilon}{r}\kappa_D(z;v).$$

Being r and $\varepsilon>0$ arbitrary, (2.3.35) follows, **q.e.d.**

Then we have

Theorem 2.3.63: (i) *Let $D\subset\subset\mathbf{C}^n$ be a complete hyperbolic domain, $x\in\partial D$ and U a neighbourhood of x in $\mathbf{C}^n$ such that $U\cap D$ is connected. Then*

$$\forall v\in\mathbf{C}^n\setminus\{0\}\qquad \lim_{z\to x}\frac{\kappa_D(z;v)}{\kappa_{D\cap U}(z;v)}=1,$$

uniformly in $v\in\mathbf{C}^n\setminus\{0\}$.
(ii) *Let $D\subset\subset\mathbf{C}^n$ be a strongly pseudoconvex domain, and U a neighbourhood of* 0 *in $\mathbf{C}^n$ such that $U_x\cap D$ is connected for all $x\in\partial D$, where $U_x=x+U$. Then*

$$\forall v\in\mathbf{C}^n\setminus\{0\}\qquad \lim_{z\to x}\frac{\kappa_D(z;v)}{\kappa_{D\cap U_x}(z;v)}=1,$$

uniformly in $v\in\mathbf{C}^n\setminus\{0\}$ and in $x\in\partial D$.

Proof: As usual, $\kappa_D(z;v)\le\kappa_{D\cap U}(z;v)$. Conversely, Lemma 2.3.62 yields

$$\frac{\kappa_{D\cap U}(z;v)}{\kappa_D(z;v)}\le\operatorname{cotanh}\bigl(\delta_D(z,D\setminus U)\bigr)\le\operatorname{cotanh}\bigl(k_D(z,D\setminus U)\bigr)\longrightarrow 1$$

as $z\to x$, because cotanh is a decreasing function and D is complete hyperbolic, and the proof of (i) is complete.

To prove (ii), it suffices to show that $\operatorname{cotanh}\bigl(k_D(z,D\setminus U_x)\bigr)\to 1$ as $z\to x$ uniformly in $x\in\partial D$. Choose $\delta>0$ such that $B(0,2\delta)\subset U$, and let ε_0, $\varepsilon_1\in(0,\delta)$ and $c\in\mathbf{R}$ be given by Theorem 2.3.54. Then $B(x,2\varepsilon_1)\subset U_x$ for all $x\in\partial D$ and

$$\operatorname{cotanh}\bigl(k_D(z,D\setminus U_x)\bigr)\le\operatorname{cotanh}\bigl(k_D(z,D\setminus B(x,2\varepsilon_1))\bigr)\le\operatorname{cotanh}\bigl(-\tfrac12\log d(z,\partial D)+c\bigr),$$

and we are done, **q.e.d.**

The reason of the splitting in two parts of the previous statement is that we shall need the precise uniform assertion for strongly pseudoconvex domains in our study of the boundary behavior of the Kobayashi metric.

The next step is the localization theorem for the Kobayashi distance. The idea is to integrate Theorem 2.3.63; to control the integration paths, we need the following

Lemma 2.3.64: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain. Take $x_0 \in \partial D$, and two sequences $\{z_\nu\}$, $\{w_\nu\} \subset D$ converging to x_0. Let $\{y_\nu\} \subset D$ be a third sequence such that there is a constant $C > 0$ so that*

$$\forall \nu \in \mathbf{N} \qquad k_D(z_\nu, y_\nu) + k_D(y_\nu, w_\nu) \leq k_D(z_\nu, w_\nu) + C.$$

Then $y_\nu \to x_0$ as $\nu \to +\infty$.

Proof: Assume, by contradiction, that $\{y_\nu\}$ does not converge to x_0; then, up to a subsequence, we can assume $y_\nu \to y_0 \in \overline{D}$, $y_0 \neq x_0$. If $y_0 \in D$, Theorem 2.3.52 yields

$$\begin{aligned}-\tfrac{1}{2}\log d(z_\nu, \partial D) - \tfrac{1}{2}\log d(w_\nu, \partial D) + 2c_2 &\leq k_D(z_\nu, y_0) + k_D(w_\nu, y_0)\\ &\leq k_D(z_\nu, y_\nu) + k_D(w_\nu, y_\nu) + 2k_D(y_\nu, y_0) \leq k_D(z_\nu, w_\nu) + C_1\end{aligned}$$

for a suitable constant $C_1 > 0$, since the sequence $\{k_D(y_\nu, y_0)\}$ is bounded, and this is impossible, by Theorem 2.3.56.

So $y_0 \in \partial D$; but then Corollary 2.3.55 yields

$$\begin{aligned}-\tfrac{1}{2}\log d(z_\nu, \partial D) - \tfrac{1}{2}\log d(w_\nu, \partial D) &- \log d(y_\nu, \partial D) + 2K\\ &\leq k_D(z_\nu, y_\nu) + k_D(y_\nu, w_\nu) < k_D(z_\nu, w_\nu) + C,\end{aligned}$$

and again this is impossible by Theorem 2.3.56, **q.e.d.**

Then

Theorem 2.3.65: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain. Take $x_0 \in \partial D$, and let U be a neighbourhood of x_0 in $\mathbf{C}^n$ such that $D \cap U$ is connected. Then*

$$\lim_{\substack{z,w\to x_0\\ z\neq w}} \frac{k_D(z,w)}{k_{D\cap U}(z,w)} = 1.$$

Proof: Clearly,

$$\limsup_{\substack{z,w\to x_0\\ z\neq w}} \frac{k_D(z,w)}{k_{D\cap U}(z,w)} \leq 1.$$

To estimate the lim inf, let $\{z_\nu\}$, $\{w_\nu\} \subset D \cap U$ be two sequences converging to x_0, with $z_\nu \neq w_\nu$, and fix $\varepsilon > 0$. By Theorem 2.3.63 there is a neighbourhood $V \subset U$ of x_0 such that for every $z \in V \cap D$ and $v \in \mathbf{C}^n$ we have

$$\kappa_{D\cap U}(z; v) \leq (1+\varepsilon)\kappa_D(z; v). \tag{2.3.36}$$

Set $\varepsilon_\nu = \min\{\varepsilon, k_D(z_\nu, w_\nu)^{-1}\}$, and choose a curve $\sigma_\nu\colon [0,1] \to D$ connecting z_ν to w_ν such that

$$\int_0^1 \kappa_D\bigl(\sigma_\nu(t); \dot\sigma_\nu(t)\bigr)\, dt \le (1+\varepsilon_\nu) k_D(z_\nu, w_\nu);$$

σ_ν exists by Theorem 2.3.32. We claim that $\sigma_\nu([0,1]) \subset V \cap D$ for every ν sufficiently large. Indeed, otherwise we can find a subsequence $\{\sigma_{\nu_j}\}$ and points $t_j \in [0,1]$ such that $y_j = \sigma_{\nu_j}(t_j) \notin V$ for all $j \in \mathbf{N}$. Now

$$k_D(z_{\nu_j}, y_j) + k_D(y_j, w_{\nu_j}) \le \int_0^1 \kappa_D\bigl(\sigma_{\nu_j}(t); \dot\sigma_{\nu_j}(t)\bigr)\, dt \le k_D(z_{\nu_j}, w_{\nu_j}) + 1,$$

and so, by Lemma 2.3.64, $y_j \to x_0$, contradiction.

Hence $\sigma_\nu([0,1]) \subset V \cap D$ eventually, and thus

$$\begin{aligned} k_{D\cap U}(z_\nu, w_\nu) &\le \int_0^1 \kappa_{D\cap U}\bigl(\sigma_\nu(t); \dot\sigma_\nu(t)\bigr)\, dt \\ &\le (1+\varepsilon) \int_0^1 \kappa_D\bigl(\sigma_\nu(t); \dot\sigma_\nu(t)\bigr)\, dt \le (1+\varepsilon)^2 k_D(z_\nu, w_\nu), \end{aligned}$$

by (2.3.36). Therefore, being both ε and the sequences $\{z_\nu\}$, $\{w_\nu\}$ arbitrary,

$$\liminf_{\substack{z,w\to x_0\\ z\ne w}} \frac{k_D(z,w)}{k_{D\cap U}(z,w)} \ge 1,$$

and the assertion follows, **q.e.d.**

We are left with the study of the boundary behavior of the Kobayashi metric in strongly pseudoconvex domains. The idea is to replace D locally by simpler domains, and then to invoke the localization Theorem 2.3.63. So we begin introducing a special class of domains.

Let $H\colon \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$ be a positive definite hermitian product on $\mathbf{C}^n$; then the *analytic ellipsoid* associated to H is the domain

$$E = \{z \in \mathbf{C}^n \mid \rho_E(z) = -z_1 - \bar z_1 + H(z,z) < 0\}.$$

Note that $L_{\rho_E, z} = H$ for every $z \in \mathbf{C}^n$; so ρ_E is everywhere strictly plurisubharmonic.

Lemma 2.3.66: *Let E be the analytic ellipsoid associated to the positive definite hermitian product H. Then*

$$\forall z \in E \, \forall v \in \mathbf{C}^n \quad \big(\kappa_E(z;v)\big)^2 = \frac{H(v,v)}{-\rho_E(z)} + \left|\frac{H(v,z)-v_1}{-\rho_E(z)}\right|^2. \tag{2.3.37}$$

Proof: The idea is to find a biholomorphism $\Psi\colon E \to B^n$ and then to compute $\kappa_E(z;v)$ as $\kappa_{B^n}\big(\Psi(z); d\Psi_z(v)\big)$. Let $A = (a_{hk})$ be the positive definite hermitian matrix representing H; up to a unitary transformation leaving $e_1 = (1,0,\dots,0)$ fixed (and thus (2.3.37) invariant) we can assume $a_{hk} = 0$ for $h \neq k$, h, $k = 2,\dots,n$. Since A is positive definite, this implies that $a_{22},\dots,a_{nn} > 0$ and

$$\det A = a_{22}\dots a_{nn}\left[a_{11} - \sum_{j=2}^n \frac{|a_{1j}|^2}{a_{jj}}\right] > 0.$$

Then define $\Psi\colon E \to \mathbf{C}^n$ by

$$\begin{cases} \Psi_1(z) = a_0 z_1 - 1, \\ \Psi_j(z) = (a_0 a_{jj})^{1/2}(z_j + a_{1j}z_1/a_{jj}) \quad \text{for } j = 2,\dots,n, \end{cases}$$

where $a_0 = a_{11} - \sum\limits_{j=2}^n |a_{1j}|^2/a_{jj} > 0$. Since $\|\Psi(z)\|^2 = 1 + a_0\rho_E(z)$, it is clear that Ψ is a biholomorphism between E and B^n. Finally, being

$$\begin{cases} \big(d\Psi_z(v)\big)_1 = a_0 v_1, \\ \big(d\Psi_z(v)\big)_j = (a_0 a_{jj})^{1/2}(v_j + a_{1j}v_1/a_{jj}) \quad \text{for } j = 2,\dots,n, \end{cases}$$

for every $z \in E$ and $v \in \mathbf{C}^n$, it easily follows that $\big[\kappa_{B^n}\big(\Psi(z); d\Psi_z(v)\big)\big]^2$ is given by (2.3.37), **q.e.d.**

Given positive constants $c_2 > c_1 > 0$, we denote by $\mathcal{E}(c_1,c_2)$ the set of analytic ellipsoids associated to positive definite hermitian products H such that

$$\forall v \in \mathbf{C}^n \qquad c_1\|v\|^2 \le H(v,v) \le c_2\|v\|^2.$$

Since we introduced analytic ellipsoids as a tool for our study of the boundary behavior of the Kobayashi metric, we are clearly interested in the behavior of κ_E near $0 \in \partial E$. For the moment, we restrict ourselves to non-tangential behavior, that is we restrict z to approach 0 within the cone

$$\Lambda_\alpha = \{z \in \mathbf{C}^n \mid \operatorname{Re} z_1 > \alpha\|z\|\} \tag{2.3.38}$$

for some $\alpha > 0$; note that $-e_1$ is exactly the outer unit normal vector to ∂E in 0 for any analytic ellipsoid E.

To state the next lemma, we introduce a new notation. Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain, and ρ a defining function for D. Then we can find a tubular neighbourhood U_ε of ∂D such that $\operatorname{grad}\rho$ is nowhere vanishing in U_ε; in particular, we can extend differentiably the outer unit normal vector field $\mathbf{n}$ to U_ε. Take $v \in \mathbf{C}^n$ and $z \in U_\varepsilon$; the *normal part* $v_N(z)$ of v at z is given by $v_N(z) = (v,\mathbf{n}_z)\mathbf{n}_z$, and the *tangential part* $v_T(z)$ of v at z is given by $v_T(z) = v - v_N(z)$.

Lemma 2.3.67: *Fix $0 < c_1 < c_2$, $\alpha > 0$ and let $E \in \mathcal{E}(c_1, c_2)$ be an analytic ellipsoid. Then*

$$\forall v \in \mathbf{C}^n \qquad \lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \kappa_E(z; v)\operatorname{Re} z_1 = \tfrac{1}{2}|v_1|, \tag{2.3.39}$$

uniformly in $E \in \mathcal{E}(c_1, c_2)$ and v of unit length. Furthermore,

$$\forall v \in \mathbf{C}^n \qquad \lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \bigl[\kappa_E\bigl(z; v_T(z)\bigr)\bigr]^2 \operatorname{Re} z_1 = \tfrac{1}{2} L_{E,0}\bigl(v_T(0), v_T(0)\bigr), \tag{2.3.40}$$

again uniformly in $E \in \mathcal{E}(c_1, c_2)$ and v of unit length.

Proof: First of all

$$\forall z \in \Lambda_\alpha \cap E \qquad \frac{L_{E,0}(z, z)}{\operatorname{Re} z_1} < \alpha c_2 \|z\|;$$

therefore

$$\lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \frac{\operatorname{Re} z_1}{-\rho_E(z)} = \frac{1}{2},$$

uniformly in $E \in \mathcal{E}(c_1, c_2)$, and (2.3.39) follows. To prove (2.3.40), we must show that

$$\lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \frac{\bigl|L_{E,0}\bigl(v_T(z), z\bigr) - \bigl(v_T(z)\bigr)_1\bigr|^2}{-\rho_E(z)} = 0,$$

uniformly in E and v. But indeed for every $z \in \Lambda_\alpha \cap E$ and $v \in \mathbf{C}^n$ we have

$$\begin{gathered}\bigl|L_{E,0}\bigl(v_T(z), z\bigr)\bigr| \le c_2\|v\|\,\|z\|,\\ -\rho_E(z) \ge (2\alpha - c_2\|z\|)\|z\|,\end{gathered}$$

and

$$\bigl|\bigl(v_T(z)\bigr)_1\bigr| \le c_3\|v\|\,\|z\| + o(\|z\|),$$

uniformly in E and v, for some constant $c_3 > 0$ independent of v, and we are done, **q.e.d.**

Using Lemma 2.3.66 we can also see what happens varying both z and v together:

Lemma 2.3.68: *Choose $c_2 > c_1 > 0$, let $E \in \mathcal{E}(c_1, c_2)$ be an analytic ellipsoid, U a neighbourhood of 0, and $v: U \cap E \to \mathbf{C}^n$ a continuous map. Then*
(i) *if $\|v(z)\| = O(\|z\|)$ and $|v_1(z)| = o(\|z\|)$ as $z \to 0$, then for every $\alpha > 0$*

$$\lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \kappa_E\bigl(z; v(z)\bigr) = 0,$$

uniformly in E;
(ii) *if $\|v(z)\| = O(\|z\|)$ as $z \to 0$, then $\kappa_E\bigl(z; v(z)\bigr)$ is bounded in $\Lambda_\alpha \cap E \cap U$ for every $\alpha > 0$, uniformly in E.*

Proof: This follows remarking that $-\rho_E(z) = O(\|z\|)$ in Λ_α, uniformly in $E \in \mathcal{E}(c_1, c_2)$, and using (2.3.37), **q.e.d.**

The passage from analytic ellipsoids to strongly pseudoconvex domains is accomplished using the following technical lemma:

Lemma 2.3.69: *Let $\mathcal{M}$ be a compact set of symmetric $n \times n$ complex matrices, and let $\mathcal{F} \subset \mathrm{Hol}(\mathbf{C}^n, \mathbf{C}^n)$ denote the set of maps of the form*

$$\Phi(z) = z - \left[\frac{1}{2}\sum_{h,k=1}^{n} a_{hk}\, z_h z_k\right] e_1$$

with $(a_{hk}) \in \mathcal{M}$. Then
(i) *there exists $\varepsilon_0 > 0$ such that every $\Phi \in \mathcal{F}$ is a biholomorphism in $B(0, \varepsilon_0)$;*
(ii) *fix $c_2 > c_1 > 0$ and $\alpha > 0$. Choose $E \in \mathcal{E}(c_1, c_2)$, $\Phi \in \mathcal{F}$ and set $D' = E \cap \Phi\big(B(0, \varepsilon_0)\big)$ and $D = \Phi^{-1}(D')$. Then*

$$\forall v \in \mathbf{C}^n \qquad \lim_{\substack{z \to 0 \\ z \in \Lambda_\alpha}} \kappa_D(z; v) \operatorname{Re} z_1 = \tfrac{1}{2}|v_1| \tag{2.3.41}$$

and

$$\forall v \in \mathbf{C}^n \qquad \lim_{\substack{z \to 0 \\ z \in \Lambda_\alpha}} \big[\kappa_D\big(z; v_T(z)\big)\big]^2 \operatorname{Re} z_1 = \tfrac{1}{2} L_{E,0}\big(v_T(0), v_T(0)\big), \tag{2.3.42}$$

uniformly in $E \in \mathcal{E}(c_1, c_2)$, $\Phi \in \mathcal{F}$ and $v \in \mathbf{C}^n$ of unit length.

Proof: (i) The assertion follows immediately from the compactness of $\mathcal{M}$ and the formula

$$\det d\Phi_z = 1 - \sum_{h=1}^{n} a_{1h} z_h.$$

(ii) The first step is transfer the limits from D to D'. First of all, we must show that if $z \to 0$ within Λ_α, then there is $\alpha' > 0$ such that $\Phi(z) \to 0$ within $\Lambda_{\alpha'}$ for all $\Phi \in \mathcal{F}$. Indeed, the compactness of $\mathcal{M}$ provides us with a constant $C > 0$ such that $|\operatorname{Re} \Phi_1(z)| \geq |\operatorname{Re} z_1| - C\|z\|^2$ and $\|\Phi(z)\| \leq \|z\| + C\|z\|^2$ for all $\Phi \in \mathcal{F}$; hence

$$\frac{\operatorname{Re} \Phi_1(z)}{\|\Phi(z)\|} \geq \frac{\alpha - C\|z\|}{1 + C\|z\|}$$

for all $\Phi \in \mathcal{F}$ and $z \in \Lambda_\alpha$. So $\Phi(z) \in \Lambda_{\alpha'}$ for every $\alpha' < \alpha$ and z close enough to 0, uniformly in $\Phi \in \mathcal{F}$, as claimed.

Next, we must replace $\operatorname{Re} z_1$ by $\operatorname{Re} \Phi_1(z)$; but indeed

$$\forall z \in \Lambda_\alpha \qquad \left| \frac{\operatorname{Re} \Phi_1(z)}{\operatorname{Re} z_1} - 1 \right| \leq C\alpha\|z\|,$$

and so

$$\lim_{\substack{z \to 0 \\ z \in \Lambda_\alpha}} \frac{\operatorname{Re} \Phi_1(z)}{\operatorname{Re} z_1} = 1,$$

uniformly in $\Phi\in\mathcal{F}$. In other words, then, (2.3.41) is equivalent to

$$\lim_{\substack{z\to0\\ z\in\Lambda_\alpha}} \kappa_{D'}\bigl(\Phi(z);d\Phi_z(v)\bigr)\operatorname{Re}\Phi_1(z)=\tfrac12|v_1|,$$

and (2.3.42) is equivalent to

$$\lim_{\substack{z\to0\\ z\in\Lambda_\alpha}} \Bigl[\kappa_{D'}\bigl(\Phi(z);d\Phi_z(v_T(z))\bigr)\Bigr]^2\operatorname{Re}\Phi_1(z)=\tfrac12 L_{E,0}\bigl(v_T(0),v_T(0)\bigr).$$

On the other hand, Lemma 2.3.67 and Theorem 2.3.63.(ii) yield

$$\lim_{\substack{w\to0\\ w\in\Lambda_{\alpha'}}} \kappa_{D'}(w;v)\operatorname{Re}w_1=\tfrac12|v_1|$$

and

$$\lim_{\substack{w\to0\\ w\in\Lambda_{\alpha'}}} \bigl[\kappa_{D'}\bigl(w;v'_T(w)\bigr)\bigr]^2\operatorname{Re}w_1=\tfrac12 L_{E,0}\bigl(v_T(0),v_T(0)\bigr),$$

uniformly in E and v, where $v'_T(w)$ is the tangential part of v at $w\in E$. Therefore we must show

$$\lim_{\substack{z\to0\\ z\in\Lambda_\alpha}} \operatorname{Re}\Phi_1(z)\Bigl[\kappa_{D'}\bigl(\Phi(z);v\bigr)-\kappa_{D'}\bigl(\Phi(z);d\Phi_z(v)\bigr)\Bigr]=0$$

and

$$\lim_{\substack{z\to0\\ z\in\Lambda_\alpha}} \operatorname{Re}\Phi_1(z)\Bigl\{\bigl[\kappa_{D'}\bigl(\Phi(z);d\Phi_z(v_T(z))\bigr)\bigr]^2-\bigl[\kappa_{D'}\bigl(\Phi(z);v'_T(\Phi(z))\bigr)\bigr]^2\Bigr\}=0,$$

uniformly in Φ, E and v.

Now, by Theorem 2.3.63, for every $\eta>1$ we can find $\delta>0$ such that for every v_1, $v_2\in\mathbf{C}^n$ and $w\in B(0,\delta)\cap D'$ we have

$$\kappa_{D'}(w;v_1+v_2)\le\eta\kappa_E(w;v_1+v_2)\le\eta\bigl(\kappa_E(w;v_1)+\kappa_E(w;v_2)\bigr)\le\eta\bigl(\kappa_{D'}(w;v_1)+\kappa_{D'}(w;v_2)\bigr);$$

therefore it suffices to show (applying once again Theorem 2.3.63) that

$$\lim_{\substack{z\to0\\ z\in\Lambda_\alpha}} \operatorname{Re}\Phi_1(z)\kappa_E\bigl(\Phi(z);v-d\Phi_z(v)\bigr)=0 \tag{2.3.43}$$

and

$$\begin{aligned}\lim_{\substack{z\to0\\ z\in\Lambda_\alpha}} \operatorname{Re}\Phi_1(z)\Bigl\{&\kappa_E\bigl(\Phi(z);\,d\Phi_z(v_T(z))-v'_T(\Phi(z))\bigr)\cdot\\ &\cdot\Bigl[\kappa_{D'}\bigl(\Phi(z);d\Phi_z(v_T(z))\bigr)+\kappa_{D'}\bigl(\Phi(z);v'_T(\Phi(z))\bigr)\Bigr]\Bigr\}=0,\end{aligned} \tag{2.3.44}$$

uniformly in Φ, E and v. But now a computation shows that $v-d\Phi_z(v)=O(\|z\|)$ and $d\Phi_z\bigl(v_T(z)\bigr)-v'_T\bigl(\Phi(z)\bigr)=O(\|z\|)$, uniformly in Φ, E and v; furthermore, as $z\to0$ both $d\Phi_z\bigl(v_T(z)\bigr)$ and $v'_T\bigl(\Phi(z)\bigr)$ tend to $v_T(0)=v'_T(0)$, uniformly in the usual quantities. Then Lemma 2.3.68 yields immediately (2.3.43), and (2.3.44) follows remarking that, by (the uniform statement in) Lemma 2.3.67 and Theorem 2.3.63,

$$\lim_{\substack{z\to0\\ z\in\Lambda_\alpha}} \bigl[\operatorname{Re}\Phi_1(z)\bigr]^{1/2}\bigl[\kappa_{D'}\bigl(\Phi(z);d\Phi_z(v_T(z))\bigr)+\kappa_{D'}\bigl(\Phi(z);v'_T(\Phi(z))\bigr)\bigr]$$

exists finite, with the usual uniformities, and applying again Lemma 2.3.68, **q.e.d.**

And now we can finally prove:

Theorem 2.3.70: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain, and $x_0 \in \partial D$. Then*

$$\forall v \in \mathbf{C}^n \qquad \lim_{z\to x_0} \kappa_D(z;v)d(z,\partial D) = \tfrac{1}{2}\|v_N(x_0)\|,$$

and

$$\forall v \in \mathbf{C}^n \quad \lim_{z\to x_0} \bigl[\kappa_D\bigl(z;v_T(z)\bigr)\bigr]^2 d(z,\partial D) = \tfrac{1}{2}L_{D,x_0}\bigl(v_T(x_0),v_T(x_0)\bigr),$$

uniformly in $x_0 \in \partial D$ and $v \in \mathbf{C}^n$ of unit length.

Proof: Let ρ be a defining function for D strictly plurisubharmonic in a neighbourhood of ∂D, and set $\gamma(x) = \|\operatorname{grad}\rho(x)\|$ for all $x \in \partial D$. Up to an affine isometry of $\mathbf{C}^n$, we can assume $x_0 = 0$ and $\gamma(x_0)\mathbf{n}_{x_0} = \operatorname{grad}\rho(x_0) = -\gamma(x_0)e_1$. Therefore $v_N(x_0) = v_1$ for all $v \in \mathbf{C}^n$ and ρ becomes

$$\rho(z) = \gamma(x_0)\left\{2\operatorname{Re}\left[-z_1 + \frac{1}{2\gamma(x_0)}\sum_{h,k=1}^n \frac{\partial^2\rho}{\partial z_h\partial z_k}(0)z_hz_k\right] + L_{D,x_0}(z,z)\right\} + o(\|z\|^2),$$

where, by a slight abuse of notation, we are still denoting by L_{D,x_0} the extension of the Levi form of D to all $\mathbf{C}^n$ obtained by setting $L_{D,x_0} = \gamma(x_0)^{-1}L_{\rho,x_0}$.

Now, when x_0 ranges over ∂D, the set $\mathcal{F}$ of the maps $\Phi\colon \mathbf{C}^n \to \mathbf{C}^n$ defined in (2.1.2) satisfies the hypotheses of Lemma 2.3.69; therefore we can find a neighbourhood P (independent of x_0) of the origin such that every $\Phi \in \mathcal{F}$ is a biholomorphism in P and

$$\forall w \in \Phi(P) \quad \rho\circ\Phi^{-1}(w) = \gamma(x_0)\bigl[-w_1 - \bar{w}_1 + L_{D,x_0}(w,w)\bigr] + o(\|w\|^2). \tag{2.3.45}$$

Choose $\varepsilon_0 > 0$ be smaller than the eigenvalues of L_{D,x_0} for all $x_0 \in \partial D$, and define the analytic ellipsoids $E_{\pm\varepsilon}$ for $\varepsilon < \varepsilon_0$ by

$$E_{\pm\varepsilon} = \bigl\{w \in \mathbf{C}^n \bigm| -w_1 - \bar{w}_1 + L_{D,x_0}(w,w) \mp \varepsilon\|w\|^2 < 0\bigr\};$$

note that there are $c_2 > c_1 > 0$ independent of x_0 such that $E_{\pm\varepsilon} \in \mathcal{E}(c_1,c_2)$ for all $\varepsilon < \varepsilon_0$.

Now set $D_{\pm\varepsilon} = \Phi^{-1}\bigl(E_{\pm\varepsilon}\cap\Phi(P)\bigr)$; by (2.3.45) we can suppose, shrinking uniformly P if necessary, that

$$D_{-\varepsilon} \subset D \cap P \subset D_\varepsilon. \tag{2.3.46}$$

Fix $\alpha > 0$. Since $d(z,\partial D) = \operatorname{Re} z_1 + O(\|z\|^2)$ as $z \to 0$, uniformly in x_0, it is clear that

$$\lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \frac{\operatorname{Re} z_1}{d(z,\partial D)} = 1,$$

uniformly in $x_0 \in \partial D$; therefore it remains to show that

$$\lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \kappa_D(z;v)\operatorname{Re} z_1 = \tfrac{1}{2}|v_1| \tag{2.3.47}$$

and

$$\lim_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \bigl[\kappa_D\bigl(z;v_T(z)\bigr)\bigr]^2 \operatorname{Re} z_1 = \tfrac{1}{2}L_{D,0}\bigl(v_T(0),v_T(0)\bigr), \tag{2.3.48}$$

uniformly in $x_0\in\partial D$ and v of unit length; indeed, the uniformness in x_0 will allow us to delete the restriction $z\in\Lambda_\alpha$ in both limits.

The first inclusion in (2.3.46) together with Lemma 2.3.69 yields

$$\limsup_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \kappa_{D\cap P}(z;v)\operatorname{Re} z_1 \le \tfrac{1}{2}|v_1|$$

and

$$\limsup_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \bigl[\kappa_{D\cap P}\bigl(z;v_T(z)\bigr)\bigr]^2 \operatorname{Re} z_1 \le \tfrac{1}{2}\bigl[L_{D,0}\bigl(v_T(0),v_T(0)\bigr)+\varepsilon\|v_T(0)\|^2\bigr],$$

because if we denote by $v_T^{-\varepsilon}(z)$ the tangential part of v at $z\in D_{-\varepsilon}$, then it is not difficult to check that $\|v_T^{-\varepsilon}(z)-v_T(z)\| = O(\|z\|)$, uniformly in x_0 and v, and we can apply Lemma 2.3.68 in $D_{-\varepsilon}$.

Analogously, the second inclusion in (2.3.46) together with Lemma 2.3.69 yields

$$\liminf_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \kappa_{D\cap P}(z;v)\operatorname{Re} z_1 \ge \tfrac{1}{2}|v_1|$$

and

$$\liminf_{\substack{z\to 0\\ z\in\Lambda_\alpha}} \bigl[\kappa_{D\cap P}\bigl(z;v_T(z)\bigr)\bigr]^2 \operatorname{Re} z_1 \ge \tfrac{1}{2}\bigl[L_{D,0}\bigl(v_T(0),v_T(0)\bigr)-\varepsilon\|v_T(0)\|^2\bigr].$$

Therefore, since by Theorem 2.3.63 we can replace $\kappa_{D\cap P}$ by κ_D everywhere, (2.3.47) immediately follows, and (2.3.48) is obtained by letting $\varepsilon\to 0$. Finally, the uniform statement follows from the analogous statements in Lemmas 2.3.68, 2.3.69 and Theorem 2.3.63.(ii), **q.e.d.**

Notes

As already remarked, this chapter is only an introduction to the theory of invariant objects on complex manifolds. After Kobayashi's construction of k_X and κ_X in 1967, there has been a flourishing of alternative definitions and related concepts: we only mention the metrics and distances introduced by Hahn [1981], Klimek [1985], Azukawa [1986] and Demailly [1987], the general approach of Harris [1979], and the intermediate dimensional invariant measures introduced by Eisenman [1970] and thoroughly studied in Graham and Wu [1985b] and Venturini [1985, 1987]. Furthermore, in our approach we focused on strongly pseudoconvex domains, because of the applications we have in mind; to get a more complete picture (though not containing all the results we presented), the interested reader may consult chapter 2.6 (of course), Reiffen [1963], Kobayashi [1970, 1976], Lang [1987] and Franzoni and Vesentini [1980]. This latter book also deals with invariant metrics and distances in domains of infinite dimensional complex Banach spaces.

The Carathéodory distance was introduced by Carathéodory [1926, 1927, 1928]. He was mainly interested in bounded domains in $\mathbf{C}^2$, where he could prove, using a normal family argument, that his distance was finite and non-degenerate. The Carathéodory distance remained a sort of curiosity for almost fifty years — in spite of some sporadic applications (like H. Cartan [1936], where it is used to study the automorphism group of a product of two domains) and of Reiffen's work (summarized in Reiffen [1963]) — till the publication of Kobayashi [1967a, b] where the Kobayashi distance and metric are defined. By the way, it should be remarked that, in general, the one-disk function δ_X is not a distance, not even for bounded domains of $\mathbf{C}^n$; an example is in Lempert [1981].

Explicit computations of c_X are in Simha [1975] and Jarnicki and Pflug [1988].

In connection with Proposition 2.3.8, Royden [1971] and Jarnicki and Pflug [1989] have shown that the Kobayashi (Carathéodory) distance on a product manifold is obtained by taking the maximum of the Kobayashi (Carathéodory) distance of the coordinates. An analogous statement holds for Kobayashi and Carathéodory metric; see Proposition 2.3.27 and Jarnicki and Pflug [1989].

The definition of hyperbolic manifold is already present in Kobayashi [1967a, b]. At that time, there was a rival notion around: a complex manifold X is called *tight* if there is a distance d on X inducing the manifold topology such that $\mathrm{Hol}(\Delta, X)$ is equicontinuous with respect to d (Wu [1967] and Barth [1970]). Shortly later, Kiernan [1970] proved that a complex manifold is tight iff it is hyperbolic, and after the proof of Proposition 2.3.10 (Barth [1972]), the notion of tight manifold disappeared from the literature. By the way, it should be remarked some of the statements of section 2.1.2 (like Lemma 2.1.20 and Theorem 2.1.21) hold for hyperbolic manifolds too; cf. Kobayashi [1967a, 1970].

In general it is not known whether the Carathéodory distance induces the manifold topology (assuming it is non-degenerate, of course); the answer is affirmative if the closed Carathéodory balls are compact (Sibony [1975]). Note that this condition is stronger than the completeness of c_X, for Lemma 2.3.15 does not hold for the Carathéodory distance (an example is in Franzoni and Vesentini [1980]).

Propositions 2.3.11, 2.3.13 and 2.3.20 are in Kobayashi [1967a], where it is also shown that every (complete) hermitian manifold with holomorphic sectional curvature bounded above by a negative constant is (complete) hyperbolic. Brody [1978] has characterized compact hyperbolic manifolds: a compact complex manifold X is hyperbolic iff there are no nonconstant holomorphic maps $f\colon \mathbf{C} \to X$.

Theorem 2.3.14 is due to Kiernan [1970]. Proposition 2.3.17 is in Kobayashi [1967a], and it is typical of inner distances; see Rinow [1961] for definition and properties of inner distances, and Kobayashi [1973] for a direct proof of the fact that k_X is an inner distance. In general, c_X is not inner; see Barth [1977].

Theorem 2.3.18 is again due to Kiernan [1970] (but part of the proof comes from Wu [1967]). In connection with Proposition 2.3.19, Nakajima [1985] has proved that every homogeneous hyperbolic manifold is biholomorphic to a bounded homogeneous domain.

The Carathéodory metric was introduced by Carathéodory [1928], and the Kobayashi metric by Kobayashi [1967a, b], but the theory of invariant metrics really started only with Royden [1971], who proved Theorems 2.3.29, 2.3.32, and Propositions 2.3.33 and 2.3.34. The complete proof of Lemma 2.3.28 is in Royden [1974]; Siu [1976] contains a more general statement. It is possible to prove directly that κ_X is measurable (and then Theorem 2.3.32) without using Lemma 2.3.28: see Venturini [1988a], who also suggested the proof we presented of Theorem 2.3.32.

The situation for the Carathéodory metric is quite different. In fact, γ_X is always locally Lipschitz, but c_X is not the integrated form of γ_X; to be precise, the integrated form of the Carathéodory metric is the inner distance induced by the Carathéodory distance. However, γ_X is, in a very precise sense, the derivative of c_X, whereas κ_X is not the derivative of k_X. For proofs and even more on this matter, consult Reiffen [1963], Harris [1979] and Venturini [1988a].

The Carathéodory and Kobayashi pseudovolume forms were introduced in Eisenman [1970]. It can be shown that $\tilde{\gamma}_X$ is always continuous; see Eisenman [1970]. Theorem 2.3.40 is from Pelles [1975] (formerly Eisenman), while Theorem 2.3.41 is due to Yau [1975]. For a proof of Osgood's theorem see, for instance, Range [1986]. A direct proof of the fact that a hyperbolic manifold is measure hyperbolic is in Kobayashi [1970].

Theorem 2.3.43 was first proved by Wong [1977] for complete hyperbolic domains in $\mathbf{C}^n$, and later generalized to bounded domains in $\mathbf{C}^n$ (Rosay [1979]) and to hyperbolic manifolds (Dektyarev [1981]). The version presented here is due to Graham and Wu [1985a], and it will be used in chapter 2.5 to prove another characterization of B^n, the original theorem of Wong [1977].

In Greene and Krantz [1984] it is shown that the invariant distances, metrics and volume forms depend continuously on the domain.

We should remark that a good deal of the material presented in sections 2.3.1, 2.3.2 and 2.3.3 can be carried over without sostantial changes to complex analytic spaces (see, e.g., Kobayashi [1976] and Lang [1987]). For instance, if X is a complex analytic space, we can define the invariant pseudovolume forms on the regular part of X; since the singular part is a subspace of codimension at least 1, it does not influence the measures associated to the invariant pseudovolume forms. In particular, Theorem 2.3.41 holds for compact hyperbolic complex spaces too, and we shall need this general form in the next chapter.

Propositions 2.3.44 and 2.3.46 are due to Lempert [1981], where the Kobayashi distance in convex domains is thoroughly investigated; cf. also chapter 2.6. Patrizio [1984] has proved that if D is a strongly convex C^3 domain, then $k_D(z_0, \cdot)$ is a convex function for every $z_0 \in D$. Proposition 2.3.45 is due to Harris [1979], who also proved that a convex domain in $\mathbf{C}^n$ is hyperbolic iff it is biholomorphic to a bounded domain. Our proof is taken from Barth [1980], where it is also proved that a convex domain in $\mathbf{C}^n$ is hyperbolic iff it contains no complex affine lines.

Theorems 2.3.47 and 2.3.48 were first proved by Kohn [1963, 1964] (see also Graham [1975]). Later on, several new proofs have been developed; for a general account of the present state of the theory consult, for instance, Krantz [1982] or Range [1986].

Theorem 2.3.49 is in Graham [1975]; see also Fornaess and Krantz [1979]. Theorems 2.3.51 and 2.3.52 were first obtained for the Carathéodory distance by Vormoor [1973], with a different proof; our approach is taken from Abate [1986]. Corollary 2.3.53 is due to Graham [1975]. Corollary 2.3.55 was proved by Vormoor [1973] (see also Fadlalla [1983]) for the Carathéodory distance, in a completely different way; cf. also Abate [1988a]. The approach presented here, via Theorem 2.3.54, is suggested by Forstneric and Rosay [1987], though our proof of Theorem 2.3.54 is different. Theorem 2.3.56 is from Forstneric and Rosay [1987] too; Vormoor [1973] proved something similar for the Carathéodory distance only. It should be remarked that all these estimates are local in character; cf. Forstneric and Rosay [1987]. Other estimates of the Carathéodory and Kobayashi distances in pseudoconvex domains have been obtained by Range [1978] and Catlin [1989].

Proposition 2.3.57 is due to Hopf [1952] and Oleinik [1952].

Fefferman [1974] proved, by means of a very deep study of the Bergmann metric, that every biholomorphism between strongly pseudoconvex smooth domains extends smoothly to the boundary. Later, his proof was considerably simplified; see, e.g., Krantz [1982]. Theorem 2.3.58 is taken from Forstneric and Rosay [1987], and it was inspired by Vormoor [1973]. It can be proved that the extension is actually Hölder-continuous of exponent $1/2$; cf. for instance Henkin [1973]. A very detailed study of the regularity of the extension under various smoothness assumptions on the boundary is Lempert [1986].

The localization Theorem 2.3.61 is due to Wong [1977], and it is another step toward his characterization of B^n that we shall discuss in chapter 2.5. Lemma 2.3.62 is stated in Royden [1971]; the proof as well as Theorem 2.3.63 are in Graham [1975]. Theorem 2.3.65 is due to Venturini [1988b]. Finally, the discussion culminating in Theorem 2.3.70 is taken from Graham [1975]; see also Henkin [1973].