An introduction to discrete holomorphic local dynamics in one complex variable

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January 2008

1. Introduction

In this survey, by *one-dimensional discrete holomorphic local dynamical system* we mean a holomorphic function $f\colon U \to \mathbb{C}$ such that $f(0) = 0$, where $U \subseteq \mathbb{C}$ is an open neighbourhood of 0; we shall also assume that $f \not\equiv \mathrm{id}_U$. We shall denote by $\mathrm{End}(\mathbb{C}, 0)$ the set of one-dimensional discrete holomorphic local dynamical systems.

Remark 1.1: Since in this survey we shall only be concerned with the one-dimensional discrete case, we shall often drop the adjectives "one-dimensional" and "discrete", and we shall call an element of $\mathrm{End}(\mathbb{C}, 0)$ simply a holomorphic local dynamical system. We shall not discuss at all continuous holomorphic local dynamical systems (e.g., holomorphic ODEs or foliations); however, replacing $\mathbb{C}$ by a complex manifold M and 0 by a point $p \in M$ we recover the general definition of discrete holomorphic local dynamical system in M at p.

Remark 1.2: Since we are mainly concerned with the behavior of f nearby 0, we shall sometimes replace f by its restriction to some suitable open neighbourhood of 0. It is possible to formalize this fact by using germs of maps and germs of sets at the origin, but for our purposes it will be enough to use a somewhat less formal approach.

To talk about the dynamics of an $f \in \mathrm{End}(\mathbb{C}, 0)$ we need to introduce the iterates of f. If f is defined on the set U, then the second iterate $f^2 = f \circ f$ is defined on $U \cap f^{-1}(U)$ only, which still is an open neighbourhood of the origin. More generally, the k-th iterate $f^k = f \circ f^{k-1}$ is defined on $U \cap f^{-1}(U) \cap \cdots \cap f^{-(k-1)}(U)$. Thus it is natural to introduce the *stable set* K_f of f by setting

$$K_f = \bigcap_{k=0}^{\infty} f^{-k}(U).$$

Clearly, $0 \in K_f$, and so the stable set is never empty (but it can happen that $K_f = \{0\}$; see the next section for an example). The stable set of f is the set of all points $z \in U$ such that the *orbit* $\{f^k(z) \mid k \in \mathbb{N}\}$ is well-defined. If $z \in U \setminus K_f$, we shall say that z (or its orbit) *escapes* from U.

The first natural question in local holomorphic dynamics then is:

(Q1) *What is the topological structure of K_f?*

For instance, when does K_f have non-empty interior? As we shall see in Section 5, holomorphic local dynamical systems such that 0 belongs to the interior of the stable set enjoy special properties.

Remark 1.3: Both the definition of stable set and Question 1 (as well as several other definitions or questions we shall meet later on) are topological in character; we might state them for local dynamical systems which are continuous only. As we shall see, however, the *answers* will strongly depend on the holomorphicity of the dynamical system.

Clearly, the stable set K_f is *completely f-invariant*, that is $f^{-1}(K_f) = K_f$ (this implies, in particular, that $f(K_f) \subseteq K_f$). Therefore the pair (K_f, f) is a discrete dynamical system in the usual sense, and so the second natural question in local holomorphic dynamics is

(Q2) *What is the dynamical structure of* (K_f, f)*?*

For instance, what is the asymptotic behavior of the orbits? Do they converge to the origin, or have they a chaotic behavior? Is there a dense orbit? Do there exist proper f*-invariant* subsets, that is sets $L \subset K_f$ such that $f(L) \subseteq L$? If they do exist, what is the dynamics on them?

To answer all these questions, the most efficient way is to replace f by a "dynamically equivalent" but simpler (e.g., linear) map g. In our context, "dynamically equivalent" means "locally conjugated"; and we have at least three kinds of conjugacy to consider.

Let $f_1: U_1 \to \mathbb{C}$ and $f_2: U_2 \to \mathbb{C}$ be two holomorphic local dynamical system. We shall say that f_1 and f_2 are *holomorphically* (respectively, *topologically*) *locally conjugated* if there are open neighbourhoods $W_1 \subseteq U_1$ and $W_2 \subseteq U_2$ of the origin, and a biholomorphism (respectively, a homeomorphism) $\varphi: W_1 \to W_2$ with $\varphi(0) = 0$ such that

$$f_1 = \varphi^{-1} \circ f_2 \circ \varphi \qquad \text{on} \qquad \varphi^{-1}\bigl(W_2 \cap f_2^{-1}(W_2)\bigr) = W_1 \cap f_1^{-1}(W_1).$$

In particular we have

$$f_1^k = \varphi^{-1} \circ f_2^k \circ \varphi \quad \text{on} \quad \varphi^{-1}\bigl(W_2 \cap \cdots \cap f_2^{-(k-1)}(W_2)\bigr) = W_1 \cap \cdots \cap f_1^{-(k-1)}(W_1),$$

for every $k \in \mathbb{N}$, and thus $K_{f_2|_{W_2}} = \varphi(K_{f_1|_{W_1}})$. So the local dynamics of f_1 is to all purposes equivalent to the local dynamics of f_2.

Whenever we have an equivalence relation in a class of objects, there are classification problems. So the third natural question in local holomorphic dynamics is

(Q3) *Find a (possibly small) class* $\mathcal{F}$ *of holomorphic local dynamical systems such that every holomorphic local dynamical system* $f \in \mathrm{End}(\mathbb{C}, O)$ *is holomorphically (respectively, topologically) locally conjugated to a (possibly) unique element of* $\mathcal{F}$*, called the* holomorphic (respectively, topological) normal form *of* f*.*

Unfortunately, the holomorphic classification is often too complicated to be practical; the family $\mathcal{F}$ of normal forms might be uncountable. A possible replacement is looking for invariants instead of normal forms:

(Q4) *Find a way to associate a (possibly small) class of (possibly computable) objects, called* invariants*, to any holomorphic local dynamical system* f *so that two holomorphically conjugated local dynamical systems have the same invariants. The class of invariants is furthermore said* complete *if two holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same invariants.*

As remarked before, up to now all the questions we asked make sense for topological local dynamical systems; the next one instead makes sense only for holomorphic local dynamical systems.

A holomorphic local dynamical system is clearly given by an element of $\mathbb{C}_0\{z\}$, the space of converging power series in z without constant terms. The space $\mathbb{C}_0\{z\}$ is a subspace of the space $\mathbb{C}_0[[z]]$ of formal power series without constant terms. An element $\Phi \in \mathbb{C}_0[[z]]$ has an inverse (with respect to composition) still belonging to $\mathbb{C}_0[[z]]$ if and only if its linear part is not zero, that is if and only if it is not divisible by z^2. We shall then say that two holomorphic local dynamical systems f_1, $f_2 \in \mathbb{C}_0\{z\}$ are *formally conjugated* if there exists an invertible $\Phi \in \mathbb{C}_0[[z]]$ such that $f_1 = \Phi^{-1} \circ f_2 \circ \Phi$ in $\mathbb{C}_0[[z]]$.

It is clear that two holomorphically locally conjugated dynamical systems are both formally and topologically locally conjugated too. On the other hand, we shall see (in Remark 4.2) examples of holomorphic local dynamical systems that are topologically locally conjugated without being neither formally nor holomorphically locally conjugated, and (in Remarks 4.2 and 5.3) examples of holomorphic local dynamical systems that are formally conjugated without being neither holomorphically nor topologically locally conjugated. So the last natural question in local holomorphic dynamics we shall deal with is

(Q5) *Find normal forms and invariants with respect to the relation of formal conjugacy for holomorphic local dynamical systems.*

In this survey we shall present some of the main results known on these questions. But before entering the main core of the paper I would like to heartily thank François Berteloot, Salvatore Coen, Santiago Diaz-Madrigal, Vincent Guedj, Giorgio Patrizio, Mohamad Pouryayevali, Jasmin Raissy, Francesca Tovena and Alekos Vidras, without whom this survey would never have been written.

2. Hyperbolic dynamics

As remarked in the previous section, an one-dimensional discrete holomorphic local dynamical system is given by a converging power series f without constant term:

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbb{C}_0\{z\}.$$

The number $a_1 = f'(0)$ is the *multiplier* of f. Since $a_1 z$ is the best linear approximation of f, it is sensible to expect that the local dynamics of f will be strongly influenced by the value of a_1. We then introduce the following definitions:

- if $|a_1| < 1$ we say that the fixed point 0 is *attracting;*
- if $a_1 = 0$ we say that the fixed point 0 is *superattracting;*
- if $|a_1| > 1$ we say that the fixed point 0 is *repelling;*
- if $|a_1| \neq 0, 1$ we say that the fixed point 0 is *hyperbolic;*
- if $a_1 \in S^1$ is a root of unity, we say that the fixed point 0 is *parabolic* (or *rationally indifferent*);
- if $a_1 \in S^1$ is not a root of unity, we say that the fixed point 0 is *elliptic* (or *irrationally indifferent*).

Remark 2.1: If $a_1 \neq 0$ then f is locally invertible, that is there exists $f^{-1} \in \mathrm{End}(\mathbb{C}, 0)$ so that $f^{-1} \circ f = f \circ f^{-1} = \mathrm{id}$ where defined. In particular, if 0 is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C}, 0)$ with non-zero multiplier, then it is a repelling fixed point for the inverse function f^{-1}.

As we shall see in a minute, the dynamics of one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point is pretty elementary; so we start with this case.

Assume first that 0 is attracting (but not superattracting) for the holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}, 0)$. Then we can write $f(z) = a_1 z + O(z^2)$, with $0 < |a_1| < 1$; hence we can find a large constant $M > 0$, a small constant $\varepsilon > 0$ and $0 < \delta < 1$ such that if $|z| < \varepsilon$ then

$$|f(z)| \leq (|a_1| + M\varepsilon)|z| \leq \delta |z|. \tag{2.1}$$

In particular, if Δ_ε is the disk of center 0 and radius ε, we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$ for $\varepsilon > 0$ small enough, and the stable set of $f|_{\Delta_\varepsilon}$ is Δ_ε itself (in particular, it contains the origin in its interior). Furthermore, since Δ_ε is f-invariant, we can apply (2.1) to $f(z)$; arguing by induction we get

$$|f^k(z)| \leq \delta^k |z| \to 0 \tag{2.2}$$

as $k \to +\infty$, and thus every orbit starting in Δ_ε is attracted by the origin, which is the reason of the name "attracting" for such a fixed point.

If instead 0 is a repelling fixed point, a similar argument (or the observation that 0 is attracting for f^{-1}) shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ reduces to the origin only: all (non-trivial) orbits escape.

It is also not difficult to find holomorphic and topological normal forms in this case, as shown in the following result, which has marked the beginning of the theory of holomorphic dynamical systems:

Theorem 2.1: (Kœnigs, 1884 [Kœ]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be an one-dimensional discrete holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let $a_1 \in \mathbb{C}^* \setminus S^1$ be its multiplier. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to its linear part $g(z) = a_1 z$. The conjugation φ is uniquely determined by the condition $\varphi'(0) = 1$.*

(ii) *Two such holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same multiplier.*

(iii) *f is topologically locally conjugated to the map $g_<(z) = z/2$ if $|a_1| < 1$, and to the map $g_>(z) = 2z$ if $|a_1| > 1$.*

Proof: Let us assume $0 < |a_1| < 1$; if $|a_1| > 1$ it will suffice to apply the same argument to f^{-1}.

(i) Choose $0 < \delta < 1$ such that $\delta^2 < |a_1| < \delta$. Writing $f(z) = a_1 z + z^2 r(z)$ for a suitable holomorphic germ r, we can find $\varepsilon > 0$ such that $|a_1| + M\varepsilon < \delta$, where $M = \max_{z \in \overline{\Delta_\varepsilon}} |r(z)|$. So we have

$$|f(z) - a_1 z| \leq M|z|^2 \tag{2.3}$$

that implies (2.1), and hence we get (2.2) for all $z \in \overline{\Delta_\varepsilon}$ and $k \in \mathbb{N}$.

Put $\varphi_k = f^k/a_1^k$; we claim that the sequence $\{\varphi_k\}$ converges to a holomorphic map $\varphi\colon \Delta_\varepsilon \to \mathbb{C}$. Indeed (2.3) and (2.2) yield

$$\begin{aligned}|\varphi_{k+1}(z) - \varphi_k(z)| &= \frac{1}{|a_1|^{k+1}}\bigl|f\bigl(f^k(z)\bigr) - a_1 f^k(z)\bigr| \\ &\le \frac{M}{|a_1|^{k+1}}|f^k(z)|^2 \le \frac{M}{|a_1|}\left(\frac{\delta^2}{|a_1|}\right)^k |z|^2\end{aligned}$$

for all $z \in \overline{\Delta_\varepsilon}$, and so the telescopic series $\sum_k(\varphi_{k+1} - \varphi_k)$ is uniformly convergent in Δ_ε to $\varphi - \varphi_0$.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, by Weierstrass' theorem we have $\varphi'(0) = 1$ and so, up to possibly shrink ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi\bigl(f(z)\bigr) = \lim_{k\to+\infty} \frac{f^k\bigl(f(z)\bigr)}{a_1^k} = a_1 \lim_{k\to+\infty} \frac{f^{k+1}(z)}{a_1^{k+1}} = a_1\varphi(z),$$

that is $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another local holomorphic function such that $\psi'(0) = 1$ and $\psi^{-1} \circ g \circ \psi = f$, it follows that $\psi \circ \varphi^{-1}(\lambda z) = \lambda \psi \circ \varphi^{-1}(z)$; comparing the power series expansion of both sides we find $\psi \circ \varphi^{-1} \equiv \mathrm{id}$, that is $\psi \equiv \varphi$, as claimed.

(ii) Since $f_1 = \varphi^{-1} \circ f_2 \circ \varphi$ implies $f_1'(0) = f_2'(0)$, the multiplier is invariant under holomorphic local conjugation, and so two one-dimensional discrete holomorphic local dynamical systems with a hyperbolic fixed point are holomorphically locally conjugated if and only if they have the same multiplier.

(iii) It suffices to build a topological conjugacy between g and $g_<$ on Δ_ε. First choose a homeomorphism χ between the annulus $\{|a_1|\varepsilon \le |z| \le \varepsilon\}$ and the annulus $\{\varepsilon/2 \le |z| \le \varepsilon\}$ which is the identity on the outer circle and given by $\chi(z) = z/2a_1$ on the inner circle. Now extend χ by induction to a homeomorphism between the annuli $\{|a_1|^k\varepsilon \le |z| \le |a_1|^{k-1}\varepsilon\}$ and $\{\varepsilon/2^k \le |z| \le \varepsilon/2^{k-1}\}$ by prescribing

$$\chi(a_1 z) = \tfrac{1}{2}\chi(z).$$

Putting finally $\chi(0) = 0$ we then get a homeomorphism χ of Δ_ε with itself such that $g = \chi^{-1} \circ g_< \circ \chi$, as required. □

Remark 2.2: The proof of Theorem 2.1.(i) relies on a standard trick used to build conjugations in dynamics. Suppose we would like to prove that f and g are conjugated, with g invertible. Set $\varphi_k = g^{-k} \circ f^k$, so that

$$\varphi_k \circ f = g \circ \varphi_{k+1}.$$

If the sequence $\{\varphi_k\}$ converges as $k \to +\infty$ to a locally invertible function φ, we automatically have $\varphi \circ f = g \circ \varphi$, and so φ is the conjugation we were looking for.

Remark 2.3: The proof of Theorem 2.1.(iii) too uses a standard dynamical trick for building topological conjugations. Let $f\colon X \to X$ be a continuos closed injective map. A *fundamental domain* for f is a closed set $D \subset X$ with non-empty interior $\mathring{D}$ such that

(i) $X = \bigcup_{k\ge 0} f^k(D)$;
(ii) $f^h(\mathring{D}) \cap f^k(\mathring{D}) = \varnothing$ for all $h \ne k$;
(iii) $f^h(D) \cap f^k(D) \ne \varnothing$ if and only if $|h - k| \le 1$.

Assume now that you have two continuous closed injective maps $f_1\colon X_1 \to X_1$ and $f_2\colon X_2 \to X_2$ with fundamental domains $D_1 \subset X_1$ and $D_2 \subset X_2$. Assume furthermore that you have a homeomorphism $\chi\colon D_1 \to D_2$ such that

$$\chi\bigl(f_1(z)\bigr) = f_2\bigl(\chi(z)\bigr) \tag{2.4}$$

for all $z \in D_1 \cap f_1^{-1}(D_1)$. Then we can extend χ to a homeomorphism between $f_1(D_1)$ and $f_2(D_2)$ by setting $\chi(z) = f_2\bigl(\chi\bigl(f_1^{-1}(z)\bigr)\bigr)$ for all $z \in f_1(D_1)$; since (2.4) holds on $D_1 \cap f_1^{-1}(D_1)$, we have obtained a homeomorphism between $D_1 \cup f_1(D_1)$ and $D_2 \cup f(D_2)$ satisfying (2.4) on $\bigl(D_1 \cup f_1(D_1)\bigr) \cap f_1^{-1}\bigl(D_1 \cup f_1(D_1)\bigr)$. Proceeding in this way we get a homeomorphism $\chi\colon X_1 \to X_2$ satisfying $\chi \circ f_1 = f_2 \circ \chi$, as desired.

Remark 2.4: Notice that $g_<(z) = \frac{1}{2}z$ and $g_>(z) = 2z$ cannot be topologically conjugated, because (for instance) the stable set of $g_<$ is open whereas the stable set of $g_>$ contains the origin only.

3. Superattracting dynamics

Let us now study the superattracting case. If 0 is a superattracting point for an $f \in \mathrm{End}(\mathbb{C}, 0)$, we can write

$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \cdots$$

with $a_r \neq 0$; the number $r \geq 2$ is the *order* of the superattracting point. An argument similar to the one described in the previous section shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ still is all of Δ_ε, and the orbits converge (faster than in the attracting case) to the origin. Furthermore, we can prove the following

Theorem 3.1: (Böttcher, 1904 [Bö]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be an one-dimensional holomorphic local dynamical system with a superattracting fixed point at the origin, and let $r \geq 2$ be its order. Then:*

(i) *f is holomorphically (and hence formally and topologically) locally conjugated to the map $g(z) = z^r$.*
(ii) *two such holomorphic local dynamical systems are holomorphically (or topologically or formally) conjugated if and only if they have the same order.*

Proof: First of all, up to a linear conjugation $z \mapsto \mu z$ with $\mu^{r-1} = a_r$ we can assume $a_r = 1$.

Now write $f(z) = z^r h_1(z)$ for some holomorphic germ h_1 with $h_1(0) = 1$. By induction, it is easy to see that we can write $f^k(z) = z^{r^k} h_k(z)$ for a suitable holomorphic germ h_k with $h_k(0) = 1$. Furthermore, the equalities $f \circ f^{k-1} = f^k = f^{k-1} \circ f$ yield

$$h_{k-1}(z)^r h_1\bigl(f^{k-1}(z)\bigr) = h_k(z) = h_1(z)^{r^{k-1}} h_{k-1}\bigl(f(z)\bigr). \tag{3.1}$$

Choose $0 < \delta < 1$. Then we can clearly find $1 > \varepsilon > 0$ such that $M\varepsilon < \delta$, where $M = \max_{z \in \overline{\Delta_\varepsilon}} |h_1(z)|$; we can also assume that $h_1(z) \neq 0$ for all $z \in \overline{\Delta_\varepsilon}$. Since

$$\forall z \in \overline{\Delta_\varepsilon} \qquad |f(z)| \leq M|z|^r < \delta |z|^{r-1},$$

we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$, as anticipated before.

We also remark that (3.1) implies that each h_k is well-defined and never vanishing on $\overline{\Delta_\varepsilon}$. So for every $k \geq 1$ we can choose a unique ψ_k holomorphic in Δ_ε such that $\psi_k(z)^{r^k} = h_k(z)$ on Δ_ε and with $\psi_k(0) = 1$.

Set $\varphi_k(z) = z\psi_k(z)$, so that $\varphi_k'(0) = 1$ and $\varphi_k(z)^{r^k} = f_k(z)$ on Δ_ε. We claim that the sequence $\{\varphi_k\}$ converges to a holomorphic function φ on Δ_ε. Indeed, we have

$$\begin{aligned}\left|\frac{\varphi_{k+1}(z)}{\varphi_k(z)}\right| &= \left|\frac{\psi_{k+1}(z)^{r^{k+1}}}{\psi_k(z)^{r^{k+1}}}\right|^{1/r^{k+1}} = \left|\frac{h_{k+1}(z)}{h_k(z)^r}\right|^{1/r^{k+1}} = \bigl|h_1\bigl(f^k(z)\bigr)\bigr|^{1/r^{k+1}} \\ &= \bigl|1 + O\bigl(|f^k(z)|\bigr)\bigr|^{1/r^{k+1}} = 1 + \frac{1}{r^{k+1}} O\bigl(|f^k(z)|\bigr) = 1 + O\left(\frac{1}{r^{k+1}}\right),\end{aligned}$$

and so the telescopic product $\prod_k (\varphi_{k+1}/\varphi_k)$ converges to φ/φ_1 uniformly in Δ_ε.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrink ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\begin{aligned}\varphi_k\bigl(f(z)\bigr)^{r^k} &= f(z)^{r^k} \psi_k\bigl(f(z)\bigr)^{r^k} = z^{r^{k+1}} h_1(z)^{r^k} h_k\bigl(f(z)\bigr) \\ &= z^{r^{k+1}} h_{k+1}(z) = \bigl[\varphi_{k+1}(z)^r\bigr]^{r^k},\end{aligned}$$

and thus $\varphi_k \circ f = [\varphi_{k+1}]^r$. Passing to the limit we get $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

Finally, (ii) follows because z^r and z^s are locally topologically (or formally) conjugated if and only if $r = s$. □

Therefore the one-dimensional local dynamics about a hyperbolic or superattracting fixed point is completely clear; let us now discuss what happens about a parabolic fixed point.

4. Parabolic dynamics

Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a (non-linear) holomorphic local dynamical system with a parabolic fixed point at the origin. Then we can write

$$f(z) = e^{2i\pi p/q}z + a_{r+1}z^{r+1} + a_{r+2}z^{r+2} + \cdots, \tag{4.1}$$

with $a_{r+1} \neq 0$, where $p/q \in \mathbb{Q} \cap [0, 1)$ is the *rotation number* of f, and the number $r+1 \geq 2$ is the *multiplicity* of f at the fixed point.

The first observation is that such a dynamical system is never locally conjugated to its linear part, not even topologically, unless it is of finite order:

Proposition 4.1: *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Then f is holomorphically (or topologically or formally) linearizable if and only if $f^q \equiv \mathrm{id}$.*

Proof: Indeed, if $\varphi^{-1} \circ f \circ \varphi(z) = e^{2\pi ip/q}z$ we get $\varphi^{-1} \circ f^q \circ \varphi \equiv \mathrm{id}$, that is $f^q \equiv \mathrm{id}$. Conversely, assume that $f^q \equiv \mathrm{id}$ and set

$$\varphi(z) = \frac{1}{q}\sum_{j=0}^{q-1}\frac{f^j(z)}{\lambda^j}.$$

Then it is easy to check that $\varphi'(0) = 1$ and $\varphi \circ f(z) = \lambda\varphi(z)$, and so f is holomorphically (and topologically and formally) linearizable. □

In particular, if the rotation number is 0 (that is the multiplier is 1, and we shall say that f is *tangent to the identity*), then f *cannot* be locally conjugated to the identity (unless it was the identity to begin with, which is not a very interesting case dynamically speaking). More precisely, the stable set of such an f is never a neighbourhood of the origin. To understand why, let us first consider a map of the form

$$f(z) = z(1 + az^r)$$

for some $a \neq 0$. Let $v \in S^1 \subset \mathbb{C}$ be such that av^r is real and positive. Then for any $c > 0$ we have

$$f(cv) = c(1 + c^r av^r)v \in \mathbb{R}^+v;$$

moreover, $|f(cv)| > |cv|$. In other words, the half-line $\mathbb{R}^+v$ is f-invariant and repelled from the origin, that is $K_f \cap \mathbb{R}^+v = \varnothing$. Conversely, if av^r is real and negative then it is easy to see that the segment $[0, |a|^{-1/r}]v$ is f-invariant and attracted by the origin. So K_f neither is a neighbourhood of the origin nor reduces to $\{0\}$.

This example suggests the following definition. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be of the form

$$f(z) = z + a_{r+1}z^{r+1} + a_{r+2}z^{r+2} + \cdots. \tag{4.2}$$

Then a unit vector $v \in S^1$ is an *attracting* (respectively, *repelling*) *direction* for f at the origin if $a_{r+1}v^r$ is real and negative (respectively, positive). Clearly, there are r equally spaced attracting directions, separated by r equally spaced repelling directions: if $a_{r+1} = |a_{r+1}|e^{i\alpha}$, then $v = e^{i\theta}$ is attracting (respectively, repelling) if and only if

$$\theta = \frac{2k+1}{r}\pi - \frac{\alpha}{r} \qquad \left(\text{respectively, } \theta = \frac{2k}{r}\pi - \frac{\alpha}{r}\right).$$

Furthermore, a repelling (attracting) direction for f is attracting (repelling) for f^{-1}, which is defined in a neighbourhood of the origin.

It turns out that to every attracting direction is associated a connected component of $K_f \setminus \{0\}$. Let $v \in S^1$ be an attracting direction for an f tangent to the identity. The *basin* centered at v is the set of points $z \in K_f \setminus \{0\}$ such that $f^k(z) \to 0$ and $f^k(z)/|f^k(z)| \to v$ (notice that, up to shrinking the domain of f, we can assume that $f(z) \neq 0$ for all $z \in K_f \setminus \{0\}$). If z belongs to the basin centered at v, we shall say that the orbit of z *tends to* 0 *tangent to* v.

A slightly more specialized (but more useful) object is the following: an *attracting petal* centered at an attracting direction v is an open simply connected f-invariant set $P \subseteq K_f \setminus \{0\}$ such that a point $z \in K_f \setminus \{0\}$ belongs to the basin centered at v if and only if its orbit intersects P. In other words, the orbit of a point tends to 0 tangent to v if and only if it is eventually contained in P. A *repelling petal* (centered at a repelling direction) is an attracting petal for the inverse of f.

The basins centered at the attracting directions are exactly the connected components of $K_f \setminus \{0\}$, as shown in the *Leau-Fatou flower theorem:*

Theorem 4.2: (Leau, 1897 [L]; Fatou, 1919-20 [F1–3]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1 \geq 2$ at 0. Let $v_1, v_3, \ldots, v_{2r-1} \in S^1$ be the r attracting directions of f at the origin, and $v_2, v_4, \ldots, v_{2r} \in S^1$ the r repelling directions. Then*

(i) *There exists for each attracting (repelling) direction v_{2j-1} (v_{2j}) an attracting (repelling) petal P_{2j-1} (P_{2j}), so that the union of these $2r$ petals together with the origin forms a neighbourhood of the origin. Furthermore, the $2r$ petals are arranged ciclically so that two petals intersect if and only if the angle between their central directions is π/r.*

(ii) *$K_f \setminus \{0\}$ is the (disjoint) union of the basins centered at the r attracting directions.*

(iii) *If B is a basin centered at an attracting direction, there exists a function $\varphi\colon B \to \mathbb{C}$ such that $\varphi \circ f(z) = \varphi(z) + 1$ for all $z \in B$. Furthermore, if P is the petal constructed in part* (i)*, then $\varphi|_P$ is a biholomorphism with an open subset of the complex plane containing a right half-plane — and so $f|_P$ is holomorphically conjugated to the translation $z \mapsto z+1$.*

Proof: Up to a linear conjugation, we can assume that $a_{r+1} = -1$, so that the attracting directions are the r-th roots of unity. For any $\delta > 0$, the set $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$ has exactly r connected components, each one symmetric with respect to a different r-th root of unity; it will turns out that, for δ small enough, these connected components are attracting petals of f, even though to get a pointed neighbourhood of the origin we shall need larger petals.

For $j = 1, 3, \ldots, 2r-1$ let $\Sigma_j \subset \mathbb{C}^*$ denote the sector centered about the attractive direction v_j and bounded by two consecutive repelling directions, that is

$$\Sigma_j = \left\{ z \in \mathbb{C}^* \;\middle|\; \frac{2j-3}{r}\pi < \arg z < \frac{2j-1}{r}\pi \right\}.$$

Notice that each Σ_j contains a unique connected component $P_{j,\delta}$ of the set $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$; moreover, $P_{j,\delta}$ is tangent at the origin to the sector centered about v_j of amplitude π/r.

The main technical trick in this proof consists in transfering the setting to a neighbourhood of infinity in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. For $j = 1, 3, \ldots, 2r-1$ the function $\psi\colon \mathbb{C}^* \to \mathbb{C}^*$ given by

$$\psi(z) = \frac{1}{rz^r};$$

is a biholomorphism between Σ_j and $\mathbb{C}^* \setminus \mathbb{R}^-$, with inverse of the form $\psi^{-1}(w) = (rw)^{-1/r}$, suitably choosing the r-th root. Furthermore, $\psi(P_{j,\delta})$ is the right half-plane $H_\delta = \{w \in \mathbb{C} \mid \mathrm{Re}\, w > 1/(2r\delta)\}$.

When $|w|$ is so large that $\psi^{-1}(w)$ belongs to the domain of definition of f, the composition $F = \psi \circ f \circ \psi^{-1}$ makes sense, and we have

$$F(w) = w + 1 + O(w^{-1/r}). \tag{4.3}$$

Thus to study the dynamics of f in a neighbourhood of the origin in Σ_j it suffices to study the dynamics of F in a neighbourhood of infinity.

The first observation is that if $\mathrm{Re}\, w$ is large enough then

$$\mathrm{Re}\, F(w) > \mathrm{Re}\, w + \frac{1}{2};$$

this implies that for δ small enough H_δ is F-invariant (and thus $P_{j,\delta}$ is f-invariant). Furthermore, by induction one has

$$\forall w \in H_\delta \qquad \mathrm{Re}\, F^k(w) > \mathrm{Re}\, w + \frac{k}{2}, \tag{4.4}$$

which implies that $F^k(w) \to \infty$ in H_δ (and $f^k(z) \to 0$ in $P_{j,\delta}$) as $k \to \infty$.

Now we claim that the argument of $w_k = F^k(w)$ tends to zero. Indeed, (4.3) yields

$$\frac{w_k}{k} = \frac{w}{k} + 1 + \frac{1}{k}\sum_{l=0}^{k-1} O(w_l^{-1/r});$$

so Cesaro's theorem on the averages of a converging sequence implies

$$\frac{w_k}{k} \to 1, \tag{4.5}$$

and thus $\arg w_k \to 0$ as $k \to \infty$. Going back to $P_{j,\delta}$, this implies that $f^k(z)/|f^k(z)| \to v_j$ for every $z \in P_{j,\delta}$. Since furthermore $P_{j,\delta}$ is centered about v_j, every orbit converging to 0 tangent to v_j must intersect $P_{j,\delta}$, and thus we have proved that $P_{j,\delta}$ is an attracting petal.

Arguing in the same way with f^{-1} we get repelling petals; unfortunately, these petals are too small to obtain a full pointed neighbourhood of the origin. In fact, as remarked before each $P_{j,\delta}$ is contained in a sector centered about v_j of amplitude π/r; therefore the repelling and attracting petals obtained in this way do not intersect but are tangent to each other. We need larger petals.

So our aim is to find an f-invariant subset $\tilde{P}_j$ of Σ_j containing $P_{j,\delta}$ and which is tangent at the origin to a sector centered about v_j of amplitude strictly greater than π/r. To do so, first of all remark that there are R, $C > 0$ such that

$$|F(w) - w - 1| \le \frac{C}{|w|^{1/r}} \tag{4.6}$$

as soon as $|w| > R$. Choose $\varepsilon \in (0,1)$ and select $\delta > 0$ so that $|w| > 1/(2r\delta)$ implies

$$|F(w) - w - 1| \le \varepsilon/2.$$

Set $M_\varepsilon = \sqrt{1+\varepsilon^2}/(2r\delta)$ and let

$$\tilde{H}_\varepsilon = \{w \in \mathbb{C} \mid \varepsilon|\operatorname{Im} w| > -\operatorname{Re} w + M_\varepsilon\} \cup H_\delta;$$

in particular $|w| > 1/(2r\delta)$ for all $w \in \tilde{H}_\varepsilon$. If $w \in \tilde{H}_\varepsilon$ we have

$$\operatorname{Re} F(w) > \operatorname{Re} w + 1 - \varepsilon/2 \qquad \text{and} \qquad |\operatorname{Im} F(w) - \operatorname{Im} w| < \varepsilon/2; \tag{4.7}$$

it is then easy to check that $F(\tilde{H}_\varepsilon) \subset \tilde{H}_\varepsilon$ and that every orbit starting in $\tilde{H}_\varepsilon$ must eventually enter H_δ. Therefore $\tilde{P}_j = \psi^{-1}(\tilde{H}_\varepsilon)$ is as required, and we have proved (i).

To prove (ii) we need a further property of $\tilde{H}_\varepsilon$. Since

$$f^{-1}(z) = z + z^{r+1} + O(z^{r+2}) \,,$$

we have

$$F^{-1}(w) = w - 1 + O(w^{-1/r});$$

up to decreasing δ we can thus assume that $|F^{-1}(w) - w + 1| < \varepsilon/2$ on $\tilde{H}_\varepsilon$. But then if $w \in \tilde{H}_\varepsilon$ we have

$$\operatorname{Re} F^{-1}(w) < \operatorname{Re} w - 1 + \frac{\varepsilon}{2}$$

and

$$\varepsilon|\operatorname{Im} F^{-1}(w)| + \operatorname{Re} F^{-1}(w) < \varepsilon|\operatorname{Im} w| + \operatorname{Re} w - \left(1 - \frac{\varepsilon(1+\varepsilon)}{2}\right);$$

this means that every inverse orbit *must* eventually leave $\tilde{H}_\varepsilon$.

Coming back to the z-plane, we have thus proved that every (forward) orbit of f must eventually leave any repelling petal. So if $z \in K_f \setminus \{O\}$, where the stable set is computed working in the neighborhood of the origin constructed in part (i), the orbit of z must eventually land in an attracting petal, and thus z belongs to a basin centered at one of the r attracting directions — and (ii) is proved.

To prove (iii), first of all notice that

$$|F'(w) - 1| \le \frac{2^{1+1/r}C}{|w|^{1+1/r}} \tag{4.8}$$

in $\tilde H_\varepsilon$. Indeed, (4.6) says that if $|w|>1/(2r\delta)$ then the function $w\mapsto F(w)-w-1$ sends the disk of center w and radius $|w|/2$ into the disk of center the origin and radius $C/(|w|/2)^{1/r}$ for some $C>0$; inequality (4.8) then follows from the Cauchy estimates on the derivative.

Now choose $w_0\in H_\delta$, and set $\tilde\varphi_k(w)=F^k(w)-F^k(w_0)$. Given $w\in\tilde H_\varepsilon$, as soon as $k\in\mathbb{N}$ is so large that $F^k(w)\in H_\delta$ we can apply Lagrange's theorem to the segment from $F^k(w_0)$ to $F^k(w)$ to get a $t_k\in[0,1]$ such that

$$\begin{aligned}\left|\frac{\tilde\varphi_{k+1}(w)}{\tilde\varphi_k(w)}-1\right| &= \left|\frac{F\bigl(F^k(w)\bigr)-F^k\bigl(F^k(w_0)\bigr)}{F^k(w)-F^k(w_0)}-1\right|\\ &=\left|F'\bigl(t_kF^k(w)+(1-t_k)F^k(w_0)\bigr)-1\right|\\ &\le \frac{2^{1+1/r}C}{\min\{\operatorname{Re}|F^k(w),\operatorname{Re}|F^k(w_0)|\}^{1+1/r}}\le\frac{C'}{k^{1+1/r}},\end{aligned}$$

where we used (4.8) and (4.7), and the constant C' is uniform on compact subsets of $\tilde H_\varepsilon$ (and it can be chosen uniform on H_δ).

As a consequence, the telescopic product $\prod_k\tilde\varphi_{k+1}/\tilde\varphi_k$ converges uniformly on compact subsets of $\tilde H_\varepsilon$ (and uniformly on H_δ), and thus the sequence $\tilde\varphi_k$ converges, uniformly on compact subsets, to a holomorphic function $\tilde\varphi\colon\tilde H_\varepsilon\to\mathbb{C}$. Since we have

$$\begin{aligned}\tilde\varphi_k\circ F(w)&=F^{k+1}(w)-F^k(w_0)=\tilde\varphi_{k+1}(w)+F\bigl(F^k(w_0)\bigr)-F^k(w_0)\\ &=\tilde\varphi_{k+1}(w)+1+O\bigl(|F^k(w_0)|^{-1/r}\bigr),\end{aligned}$$

it follows that

$$\tilde\varphi\circ F(w)=\tilde\varphi(w)+1$$

on $\tilde H_\varepsilon$. In particular, $\tilde\varphi$ is not constant; being the limit of injective functions, by Hurwitz's theorem it is injective.

We now prove that the image of $\tilde\varphi$ contains a right half-plane. First of all, we claim that

$$\lim_{\substack{|w|\to+\infty\\ w\in H_\delta}}\frac{\tilde\varphi(w)}{w}=1. \tag{4.9}$$

Indeed, choose $\eta>0$. Since the convergence of the telescopic product is uniform on H_δ, we can find $k_0\in\mathbb{N}$ such that

$$\left|\frac{\tilde\varphi(w)-\tilde\varphi_{k_0}(w)}{w-w_0}\right|<\frac{\eta}{2}$$

on H_δ. Furthermore, we have

$$\left|\frac{\tilde\varphi_{k_0}(w)}{w-w_0}-1\right|=\left|\frac{k_0+\sum_{j=0}^{k_0-1}O(|F^j(w)|^{-1/r})+w_0-F^{k_0}(w_0)}{w-w_0}\right|=O(|w|^{-1})$$

on H_δ; therefore we can find $R>0$ such that

$$\left|\frac{\tilde\varphi(w)}{w-w_0}-1\right|<\eta$$

as soon as $|w|>R$ in H_δ.

Equality (4.9) clearly implies that $(\tilde\varphi(w)-w^o)/(w-w^o)\to1$ as $|w|\to+\infty$ in H_δ for any $w^o\in\mathbb{C}$. But this means that if $\operatorname{Re}w^o$ is large enough then the difference between the variation of the argument of $\tilde\varphi-w^o$ along a suitably small closed circle around w^o and the variation of the argument of $w-w^o$ along the same circle will be less than 2π — and thus it will be zero. Then the principle of the argument implies that $\tilde\varphi-w^o$ and $w-w^o$ have the same number of zeroes inside that circle, and thus $w^o\in\tilde\varphi(H_\delta)$, as required.

So setting $\varphi=\tilde\varphi\circ\psi$, we have defined a function φ with the required properties on $\tilde P_j$. To extend it to the whole basin B it suffices to put

$$\varphi(z)=\varphi\bigl(f^k(z)\bigr)-k,$$

where $k \in \mathbb{N}$ is the first integer such that $f^k(z) \in \tilde{P}_j$. $\square$

Remark 4.1: It is possible to construct petals that cannot be contained in any sector strictly smaller than Σ_j. To do so we need an F-invariant subset $\hat{H}_\varepsilon$ of $\mathbb{C}^* \setminus \mathbb{R}^-$ containing $\tilde{H}_\varepsilon$ and containing eventually every half-line issuing from the origin (but $\mathbb{R}^-$). For $M >> 1$ and $C > 0$ large enough, replace the straight lines bounding $\tilde{H}_\varepsilon$ on the left of $\operatorname{Re} w = M_\varepsilon$ by the curves

$$|\operatorname{Im} w| = \begin{cases} C \log |\operatorname{Re} w| & \text{if } r = 1, \\ C |\operatorname{Re} w|^{1-1/r} & \text{if } r > 1. \end{cases}$$

Then it is not too difficult to check that the domain $\hat{H}_\varepsilon$ so obtained is as desired (see [CG]).

So we have a complete description of the dynamics in the neighbourhood of the origin. Actually, Camacho has pushed this argument even further, obtaining a complete topological classification of one-dimensional discrete holomorphic local dynamical systems tangent to the identity:

Theorem 4.3: (Camacho, 1978 [C]; Shcherbakov, 1982 [S]) *Let $f \in \operatorname{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1$ at the fixed point. Then f is topologically locally conjugated to the map*

$$z \mapsto z - z^{r+1}.$$

The formal classification is simple too, though different, and it can be obtained with an easy computation:

Proposition 4.4: *Let $f \in \operatorname{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1$ at the fixed point. Then f is formally conjugated to the map*

$$g(z) = z - z^{r+1} + \beta z^{2r+1}, \tag{4.10}$$

where β is a formal (and holomorphic) invariant given by

$$\beta = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)}, \tag{4.11}$$

where the integral is taken over a small positive loop γ about the origin.

Proof: A computation shows that if f is given by (4.10) then β is given by the integral (4.11). Conversely, let φ be a local biholomorphism fixing the origin, and set $F = \varphi^{-1} \circ f \circ \varphi$. Then

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)} &= \frac{1}{2\pi i} \int_{\varphi^{-1} \circ \gamma} \frac{\varphi'(w)\, dw}{\varphi(w) - f\bigl(\varphi(w)\bigr)} \\ &= \frac{1}{2\pi i} \int_{\varphi^{-1} \circ \gamma} \frac{\varphi'(w)\, dw}{\varphi(w) - \varphi\bigl(F(w)\bigr)}. \end{aligned}$$

Now, we can clearly find M, $M_1 > 0$ such that

$$\begin{aligned} &\left| \frac{1}{w - F(w)} - \frac{\varphi'(w)}{\varphi(w) - \varphi\bigl(F(w)\bigr)} \right| \\ &\qquad = \frac{1}{\bigl|\varphi(w) - \varphi\bigl(F(w)\bigr)\bigr|} \left| \frac{\varphi(w) - \varphi\bigl(F(w)\bigr)}{w - F(w)} - \varphi'(w) \right| \\ &\qquad \le M \frac{|w - F(w)|}{\bigl|\varphi(w) - \varphi\bigl(F(w)\bigr)\bigr|} \le M_1, \end{aligned}$$

in a neighbourhood of the origin, where the last inequality follows from the fact that $\varphi'(0) \neq 0$. This means that the two meromorphic functions $1/\bigl(w - F(w)\bigr)$ and $\varphi'(w)/\bigl(\varphi(w) - \varphi(\bigl(F(w)\bigr)\bigr)$ differ by a holomorphic function; so they have the same integral along any small loop surrounding the origin, and

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)} = \frac{1}{2\pi i} \int_{\varphi^{-1} \circ \gamma} \frac{dw}{w - F(w)},$$

as claimed.

To prove that f is formally conjugated to g, let us first take a local formal change of coordinates φ of the form

$$\varphi(z) = z + \mu z^d + O_{d+1} \tag{4.12}$$

with $\mu \neq 0$, and where we are writing O_{d+1} instead of $O(z^{d+1})$. It follows that $\varphi^{-1}(z) = z - \mu z^d + O_{d+1}$, $(\varphi^{-1})'(z) = 1 - d\mu z^{d-1} + O_d$ and $(\varphi^{-1})^{(j)} = O_{d-j}$ for all $j \geq 2$. Then using the Taylor expansion of φ^{-1} we get

$$\begin{aligned}
\varphi^{-1} \circ f \circ \varphi(z) &= \varphi^{-1}\left(\varphi(z) + \sum_{j \geq r+1} a_j \varphi(z)^j\right) \\
&= z + (\varphi^{-1})'\big(\varphi(z)\big) \sum_{j \geq r+1} a_j z^j (1 + \mu z^{d-1} + O_d)^j + O_{d+2r} \\
&= z + [1 - d\mu z^{d-1} + O_d] \sum_{j \geq r+1} a_j z^j (1 + j\mu z^{d-1} + O_d) + O_{d+2r} \\
&= z + a_{r+1} z^{r+1} + \cdots + a_{r+d-1} z^{r+d-1} \\
&\qquad + [a_{r+d} + (r+1-d)\mu a_{r+1}] z^{r+d} + O_{r+d+1}.
\end{aligned} \tag{4.13}$$

This means that if $d \neq r+1$ we can use a polynomial change of coordinates of the form $\varphi(z) = z + \mu z^d$ to remove the term of degree $r+d$ from the Taylor expansion of f without changing lower degree terms.

So to conjugate f to g it suffices to use a linear change of coordinates to get $a_{r+1} = -1$, and then apply a sequence of change of coordinates of the form $\varphi(z) = z + \mu z^d$ to kill all the terms in the Taylor expansion of f but the term of degree z^{2r+1}.

Finally, formula (4.13) also shows that two maps of the form (4.10) with different β cannot be formally conjugated, and we are done. □

The number β given by (4.11) is called *index* of f at the fixed point.

The holomorphic classification is much more complicated: as shown by Voronin [V] and Écalle [É1–2] in 1981, it depends on functional invariants. We shall now (very) roughly describe it; see [I], [M1–2] and [K] (and the original papers) for details.

Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity with multiplicity $r+1$ at the fixed point; up to a linear change of coordinates we can assume that $a_{r+1} = 1$. Let $P_1, \ldots, P_{2r}$ be a set of petals as in Theorem 4.2.(i), chosen so that P_{2r} is centered on the positive real semiaxis, and the others are arranged cyclically counterclockwise. Denote by H_j the biholomorphism conjugating $f|_{P_j}$ to the shift $z \mapsto z+1$ in either a right (if j is odd) or left (if j is even) half-plane given by Theorem 4.2.(iii) — applied to f^{-1} for the repelling petals. If we moreover require that

$$H_j(z) = -\frac{1}{rz^r} + \beta \log z + o(1), \tag{4.14}$$

where β is the index of f at the origin, then H_j is uniquely determined. Thus in the sets $H_j(P_j \cap P_{j+1})$ we can consider the composition $\tilde{\Phi}_j = H_{j+1} \circ H_j^{-1}$. It is easy to check that $\tilde{\Phi}_j(w+1) = \tilde{\Phi}_j(w) + 1$ for $j = 1, \ldots, 2r-1$, and thus $\psi_j = \tilde{\Phi}_j - \mathrm{id}$ is a 1-periodic holomorphic function (for $j = 2r$ we need to take $\psi_{2r} = \tilde{\Phi}_{2r} - \mathrm{id} + 2\pi i \beta$ to get a 1-periodic function). Hence each ψ_j can be extended to a suitable upper (if j is odd) or lower (if j is even) half-plane. Furthermore, it is possible to prove that the functions $\psi_1, \ldots, \psi_{2r}$ are exponentially decreasing, that is they are bounded by $\exp(-c|w|)$ as $|\operatorname{Im} w| \to +\infty$, for a suitable $c > 0$ depending on f.

Now, if we replace f by a holomorphic local conjugate $g = h^{-1} \circ f \circ h$, and denote by G_j the corresponding biholomorphisms, it turns out that

$$H_j \circ G_j^{-1} = \mathrm{id} + a$$

for a suitable $a \in \mathbb{C}$ independent of j. This suggests the introduction of an equivalence relation on the set of $2r$-uple of functions of the kind $(\psi_1, \ldots, \psi_{2r})$.

Let M_r denote the set of $2r$-uple of holomorphic 1-periodic functions $\psi = (\psi_1, \ldots, \psi_{2r})$, with ψ_j defined in a suitable upper (if j is odd) or lower (if j is even) half-plane, and exponentially decreasing when $|\operatorname{Im} w| \to +\infty$. We shall say that $\psi, \tilde{\psi} \in M_r$ are *equivalent* if there is $a \in \mathbb{C}$ such that $\tilde{\psi}_j = \psi_j \circ (\mathrm{id} + a)$ for $j = 1, \ldots, 2r$. We denote by $\mathcal{M}_r$ the set of all equivalence classes.

The procedure described above allows us to associate to any $f \in \mathrm{End}(\mathbb{C},0)$ tangent to the identity with multiplicity $r+1$ at the fixed point an element $\mu_f \in \mathcal{M}_r$, called the *sectorial invariant.* Then the holomorphic classification proved by Écalle and Voronin is

Theorem 4.5: (Écalle, 1981 [É1–2]; Voronin, 1981 [V]) *Let f, $g \in \mathrm{End}(\mathbb{C},0)$ be two holomorphic local dynamical systems tangent to the identity. Then f and g are holomorphically locally conjugated if and only if they have the same multiplicity, the same index and the same sectorial invariant. Furthermore, for any $r \geq 1$, $\beta \in \mathbb{C}$ and $\mu \in \mathcal{M}_r$ there exists $f \in \mathrm{End}(\mathbb{C},0)$ tangent to the identity with multiplicity $r+1$, index β and sectorial invariant μ.*

Remark 4.2: In particular, holomorphic local dynamical systems tangent to the identity give examples of local dynamical systems that are topologically conjugated without being neither holomorphically nor formally conjugated, and of local dynamical systems that are formally conjugated without being holomorphically conjugated.

Finally, if $f \in \mathrm{End}(\mathbb{C},0)$ satisfies $a_1 = e^{2\pi i p/q}$, then f^q is tangent to the identity. Therefore we can apply the previous results to f^q and then infer informations about the dynamics of the original f. We list here a few results; see [Mi], [Ma], [C], [É1–2] and [V] for proofs and further details.

Proposition 4.6: *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$. Then there exist $n \geq 1$ and $c \in \mathbb{C}$ such that f is formally conjugated to*

$$g(z) = \lambda z + z^{nq+1} + cz^{2nq+1}.$$

Proposition 4.7: (Camacho) *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$. Then there exist $n \geq 1$ such that f is topologically conjugated to*

$$g(z) = \lambda z + z^{nq+1}.$$

Theorem 4.8: (Leau-Fatou) *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier λ, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$. Then there exist $n \geq 1$ such that f^q has multiplicity $nq+1$, and f acts on the attracting (respectively, repelling) petals of f^q as a permutation composed by n disjoint cycles. Finally, $K_f = K_{f^q}$.*

5. Elliptic dynamics

We are left with the elliptic case:

$$f(z) = e^{2\pi i\theta} z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\}, \tag{5.1}$$

with $\theta \notin \mathbb{Q}$. It turns out that the local dynamics depends mostly on the numerical properties of θ. More precisely, for a full measure subset B of $\theta \in [0,1] \setminus \mathbb{Q}$ all holomorphic local dynamical systems of the form (5.1) are *holomorphically linearizable,* that is holomorphically locally conjugated to their (common) linear part, the irrational rotation $z \mapsto e^{2\pi i\theta} z$. Conversely, the complement $[0,1] \setminus B$ is a G_δ-dense set, and for all $\theta \in [0,1] \setminus B$ the quadratic polynomial $z \mapsto z^2 + e^{2\pi i\theta} z$ is not holomorphically linearizable. This is the gist of the results due to Cremer, Siegel, Bryuno and Yoccoz we are going to describe in this section.

The first worthwhile observation in this setting is that it is possible to give a topological characterization of holomorphically linearizable local dynamical systems:

Proposition 5.1: *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier $0 < |\lambda| \leq 1$. Then f is holomorphically linearizable if and only if it is topologically linearizable if and only if 0 is contained in the interior of the stable set of f.*

Proof: If f is holomorphically linearizable it is topologically linearizable, and if it is topologically linearizable (and $|\lambda| \le 1$) then K_f is an open neighbourhood of the origin. Assume then that 0 is contained in the interior of the stable set. If $0 < |\lambda| < 1$, we already know that f is holomorphically linearizable. If $|\lambda| = 1$, set

$$\varphi_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{f^j(z)}{\lambda^j},$$

so that $\varphi_k'(0) = 1$ and

$$\varphi_k \circ f = \lambda \varphi_k + \frac{\lambda}{k} \left(\frac{f^k}{\lambda^k} - \mathrm{id} \right). \tag{5.2}$$

The hypothesis implies that there are bounded open sets $V \subset U$ containing the origin such that $f^k(V) \subset U$ for all $k \in \mathbb{N}$. Since $|\lambda| = 1$, it follows that $\{\varphi_k\}$ is a uniformly bounded family on V, and hence, by Montel's theorem, it admits a converging subsequence. But (5.2) implies that a converging subsequence converges to a conjugation between f and the rotation $z \mapsto \lambda z$, and thus f is holomorphically linearizable. □

The second important observation is that two elliptic holomorphic local dynamical systems with the same multiplier are always formally linearizable:

Proposition 5.2: *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system of multiplier $\lambda = e^{2\pi i \theta} \in S^1$ with $\theta \notin \mathbb{Q}$. Then f is formally conjugated to its linear part.*

Proof: We shall prove that there is a unique formal power series

$$h(z) = z + h_2 z^2 + \cdots \in \mathbb{C}[[z]]$$

such that $h(\lambda z) = f\big(h(z)\big)$. Indeed we have

$$\begin{aligned} &h(\lambda z) - f\big(h(z)\big) \\ &\quad = \sum_{j \ge 2} \left\{ \big[(\lambda^j - \lambda) h_j - a_j\big] z^j - \sum_{\ell=1}^{j} \binom{j}{\ell} z^{\ell + j} \left(\sum_{k \ge 2} h_k z^{k-2} \right)^{\ell} \right\} \\ &\quad = \sum_{j \ge 2} \big[(\lambda^j - \lambda) h_j - a_j - X_j(h_2, \ldots, h_{j-1})\big] z^j, \end{aligned}$$

where X_j is a polynomial in $j-2$ variables. It follows that the coefficients of h are uniquely determined by induction using the formula

$$h_j = \frac{a_j + X_j(h_2, \ldots, h_{j-1})}{\lambda^j - \lambda}, \tag{5.3}$$

where X_j is a polynomial. In particular, h_j depends only on λ, $a_2, \ldots, a_j$. □

The formal power series linearizing f is not converging if its coefficients grow too fast. Thus (5.3) links the radius of convergence of h to the behavior of $\lambda^j - \lambda$: if the latter becomes too small, the series defining h does not converge. This is known as the *small denominators problem* in this context.

It is then natural to introduce the following quantity:

$$\Omega_\lambda(m) = \min_{1 \le k \le m} |\lambda^k - 1|,$$

for $\lambda \in S^1$ and $m \ge 1$. Clearly, λ is a root of unity if and only if $\Omega_\lambda(m) = 0$ for all m greater or equal to some $m_0 \ge 1$; furthermore,

$$\lim_{m \to +\infty} \Omega_\lambda(m) = 0$$

for all $\lambda \in S^1$.

The first one to actually prove that there are non-linearizable elliptic holomorphic local dynamical systems has been Cremer, in 1927 [Cr1]. His more general result is the following:

Theorem 5.3: (Cremer, 1938 [Cr2]) *Let $\lambda \in S^1$ be such that*

$$\limsup_{m\to+\infty} \frac{1}{m} \log \frac{1}{\Omega_\lambda(m)} = +\infty. \tag{5.4}$$

Then there exists $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ which is not holomorphically linearizable. Furthermore, the set of $\lambda \in S^1$ satisfying (5.4) *contains a G_δ-dense set.*

Proof: Choose inductively $a_j \in \{0, 1\}$ so that $|a_j + X_j| \geq 1/2$ for all $j \geq 2$, where X_j is as in (5.3). Then

$$f(z) = \lambda z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\}$$

while (5.4) implies that the radius of convergence of the formal linearization h is 0, and thus f cannot be holomorphically linearizable, as required.

Let now $S(q_0) \subset S^1$ denote the set of $\lambda = e^{2\pi i\theta} \in S^1$ such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2^{q!}} \tag{5.5}$$

for some $p/q \in \mathbb{Q}$ in lowest terms with $q \geq q_0$. Then it is not difficult to check that each $S(q_0)$ is a dense open set in S^1, and that all $\lambda \in \mathcal{S} = \bigcap_{q_0 \geq 1} S(q_0)$ satisfy (5.4). Indeed, if $\lambda = e^{2\pi i\theta} \in \mathcal{S}$ we can find $q \in \mathbb{N}$ arbitrarily large such that there is $p \in \mathbb{N}$ so that (5.5) holds. Now, it is easy to see that

$$|e^{2\pi i t} - 1| \leq 2\pi|t|$$

for all $t \in [-1/2, 1/2]$. Then let p_0 be the integer closest to $q\theta$, so that $|q\theta - p_0| \leq 1/2$. Then we have

$$|\lambda^q - 1| = |e^{2\pi i q\theta} - e^{2\pi i p_0}| = |e^{2\pi i(q\theta - p_0)} - 1| \leq 2\pi|q\theta - p_0| \leq 2\pi|q\theta - p| < \frac{4\pi}{2^{q!}}$$

for arbitrarily large q, and (5.4) follows. □

On the other hand, Siegel in 1942 gave a condition on the multiplier ensuring holomorphic linearizability:

Theorem 5.4: (Siegel, 1942 [Si]) *Let $\lambda \in S^1$ be such that there exist $\beta > 1$ and $\gamma > 0$ such that*

$$\forall m \geq 2 \qquad \frac{1}{\Omega_\lambda(m)} \leq \gamma\, m^\beta. \tag{5.6}$$

Then all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ are holomorphically linearizable. Furthermore, the set of $\lambda \in S^1$ satisfying (5.6) *for some $\beta \geq 1$ and $\gamma > 0$ is of full Lebesgue measure in S^1.*

Remark 5.1: It is interesting to notice that for generic (in a topological sense) $\lambda \in S^1$ there is a non-linearizable holomorphic local dynamical system with multiplier λ, while for almost all (in a measure-theoretic sense) $\lambda \in S^1$ every holomorphic local dynamical system with multiplier λ is holomorphically linearizable.

The original proof of Theorem 5.4 was based on the method of majorant series, that requires finding a convergent series whose coefficients are greater than the coefficients of the formal linearization. A different proof is in the spirit of the so-called Kolmogorov-Arnold-Moser (or KAM) method; see [HK]. Unfortunately, both proofs (as well as the proofs of the next two theorems) are well beyond the scope of this survey.

A bit of terminology is now useful: if $f \in \mathrm{End}(\mathbb{C}, 0)$ is elliptic, we shall say that the origin is a *Siegel point* if f is holomorphically linearizable; otherwise it is a *Cremer point.*

Theorem 5.4 suggests the existence of a number-theoretical condition on λ ensuring that the origin is a Siegel point for any holomorphic local dynamical system of multiplier λ. And indeed this is the content of the celebrated *Bryuno-Yoccoz theorem:*

Theorem 5.5: (Bryuno, 1965 [Br1–3], Yoccoz, 1988 [Y1–2]) *Let $\lambda \in S^1$. Then the following statements are equivalent:*

(i) *the origin is a Siegel point for the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$;*
(ii) *the origin is a Siegel point for all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ;*
(iii) *the number λ satisfies Bryuno's condition*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_\lambda(2^{k+1})} < +\infty. \tag{5.7}$$

Bryuno, using majorant series as in Siegel's proof of Theorem 5.4 (see also [He] and references therein) has proved that condition (iii) implies condition (ii). Yoccoz, using a more geometric approach based on conformal and quasi-conformal geometry, has proved that (i) is equivalent to (ii), and that (ii) implies (iii), that is that if λ does not satisfy (5.7) then the origin is a Cremer point for some $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ — and hence it is a Cremer point for the quadratic polynomial $f_\lambda(z)$. See also [P8, 9] for related results.

Remark 5.2: Conditions (5.4), (5.6) and (5.7) are usually expressed in a different way. Write $\lambda = e^{2\pi i\theta}$, and let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions. Then (5.7) is equivalent to

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty,$$

while (5.6) is equivalent to

$$q_{n+1} = O(q_n^\beta)\ ,$$

and (5.4) is equivalent to

$$\limsup_{k\to+\infty} \frac{1}{q_k} \log q_{k+1} = +\infty.$$

See [He], [Y2] and references therein for details.

If 0 is a Siegel point for $f \in \mathrm{End}(\mathbb{C}, 0)$, the local dynamics of f is completely clear, and simple enough. On the other hand, if 0 is a Cremer point of f, then the local dynamics of f is very complicated and not yet completely understood. Pérez-Marco (in [P2, 4–7]) and Biswas ([B]) have studied the topology and the dynamics of the stable set in this case. Some of their results are summarized in the following

Theorem 5.6: (Pérez-Marco, 1995 [P6, 7], Biswas, 2007 [B]) *Assume that* 0 *is a Cremer point for an elliptic holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}, 0)$. Then:*

(i) *The stable set K_f is compact, connected, full (i.e., $\mathbb{C} \setminus K_f$ is connected), it is not reduced to $\{0\}$, and it is not locally connected at any point distinct from the origin.*
(ii) *Any point of $K_f \setminus \{0\}$ is recurrent (that is, a limit point of its orbit).*
(iii) *There is an orbit in K_f which accumulates at the origin, but no non-trivial orbit converges to the origin.*
(iv) *The rotation number and the conformal class of K_f are a complete set of holomorphic invariants for Cremer points. In other words, two elliptic non-linearizable holomorphic local dynamical systems f and g are holomorphically locally conjugated if and only if they have the same rotation number and there is a biholomorphism of a neighbourhood of K_f with a neighbourhood of K_g.*

Remark 5.3: So, if $\lambda \in S^1$ is not a root of unity and does not satisfy Bryuno's condition (5.7), we can find f_1, $f_2 \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ such that f_1 is holomorphically linearizable while f_2 is not. Then f_1 and f_2 are formally conjugated without being neither holomorphically nor topologically locally conjugated.

See also [P1, 3] for other results on the dynamics about a Cremer point.

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